Martingale decomposition for large queue asymptotics

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Motivation and purpose

Large queues are a key feature in queueing networks. We are interested in how they are influenced by randomness in arrivals and services, but this is hard to analyze.

- To overcome it, we employ **asymptotic analyses** for their stationary joint queue length distributions.
  $\Rightarrow$ Their tail asymptotics (Large deviations).
  $\Rightarrow$ Their weak limits in heavy traffic (Approximation).
- We aim to refine a **tool to solve both problems**.
  $\Rightarrow$ This talk focuses on the tail asymptotic problem in a d-**node generalized Jackson network** as its application.

Stochastic process for the problems to be solved

Stochastic process for describing those problems is preferable to be general but simple.

- One candidate for this is a joint queue length process supplemented by remaining arrival and service times, which can be considered as a **PDMP** (Piecewise Deterministic Markov Process) due to Davis [2].
- Another is discretization of the state space, which is typically used in **MAM** (matrix analytic method).

We will take the **PDMP for our analysis** because it has a simple sample path, but a basic idea is also applicable to MAM.

Piecewise deterministic Markov process (PDMP)

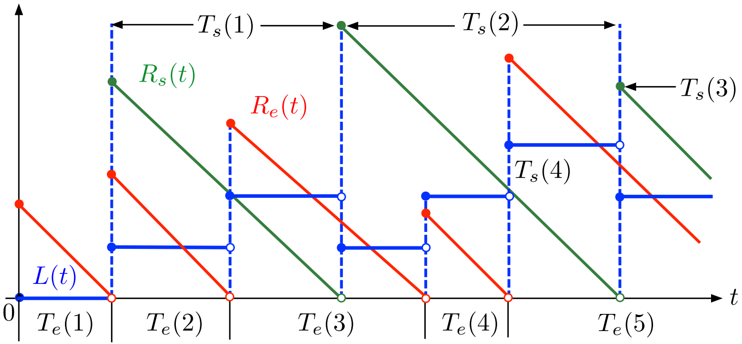


Figure: A sample path of PDMP $X(t) \equiv (L(t), R_e(t), R_s(t))$ for $GI/G/1$ queue

A sample path of a PDMP is composed of two parts, deterministic and continuous sections and discontinuous changes due to expiring of remaining times.

Analytic tools: test function and filtration

$X(t)$ has a multidimensional state space S, and is right continuous in t. We need two tools.

- A function $f : S \to \mathbb{R}$, called a **test function**.
- An increasing family of σ-field $\{\mathcal{F}_t; t \geq 0\}$, called a **filtration**, such that $\{X(t)\}$ is a $\mathcal{F}_t$-Markov process.

How they can be used ?

- Let $C^1(S)$ be the set of all functions from S to $\mathbb{R}$ that are continuously differentiable.
- For $f \in C^1(S)$, derive an equation for the **time evolution of sample function** $f(X(\cdot))$, then decompose it as a predictable process plus a martingale with respect to $\{\mathcal{F}_t; t \geq 0\}$, which is called a **special semi-martingale** in Jacod and Shriyaev [4].

Example: $X(t) \equiv (L(t), R_e(t), R_s(t))$ for $GIG/1$ queue

Let $t_{e,n}$ be the n-th arrival instant of customers, and Let $t_{s,n}$ be its service completion instant. Define counting processes as

$$N_e(t) = \sum_{n=1}^{\infty} 1(t_{e,i} \le t), \qquad N_s(t) = \sum_{n=1}^{\infty} 1(t_{s,i} \le t),$$

$$N(t) = N_e(t) + N_s(t), \qquad t_i = \inf\{t > t_{i-1}; \Delta N(t) > 0\},$$

for $t \ge 0$, then we have, for $f \in C^1(S)$,

$$f(X(t)) = f(X(0)) + \int_0^t \mathcal{A}f(X(u))du + \int_0^t \Delta f(X(u))dN(u), \quad (1)$$

where $\Delta f(X(u)) = f(X(u)) - f(X(u-))$ and

$$\mathcal{A}f(X(t)) = -\frac{\partial}{\partial R_e(t)} f(X(t)) - \frac{\partial}{\partial R_s(t)} f(X(t)) 1(L(t) \ge 1). \quad (2)$$

$\Rightarrow$ (1) is also valid for PDMP, while (2) is specific for the $GI/G/1$.

Martingale decomposition (Davis [2])

Define a jump kernel Q as $Qf(X(t_i-)) = \mathbb{E}(f(X(t_i))|X(t_i-))$, then

$$M(t) \equiv \int_0^t (f(X(u)) - Qf(X(u-)))dN(u)$$

is **$\mathcal{F}_t$-martingale**, that is,

$$\mathbb{E}(M(t)|\mathcal{F}_s) = M(s), \qquad 0 \le s < t,$$

if $\mathbb{E}(|M(t)|) < \infty$. Hence, the time evolution (1) yields **martingale decomposition** under appropriate conditions on f and N.

$$f(X(t)) = f(X(0)) + \int_0^t \mathcal{A}f(X(u))du + \int_0^t (Qf(X(u-)) - f(X(u-)))dN(u) + M(t). \qquad (3)$$

The second integration is hard to evaluate because of N !

A conditions for a simpler martingale decomposition

Let $\Lambda = \{\boldsymbol{x} \in S; \exists i \geq 1, \boldsymbol{x} = X(t_i-)\}$. If

$$Qf(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Lambda, \tag{4}$$

for a test function f, which is referred to as a **terminal condition**, then the martingale decomposition (3) is simplified to

$$f(X(t)) = f(X(0)) + \int_0^t \mathcal{A}f(X(u))du + M(t). \tag{5}$$

- (5) can be considered as a Dynkin's formula for a Markov process with generator $\mathcal{A}$.
- PDMP is hard to analyze because of the terminal condition (4).

$\Rightarrow$ It is important to find a class of **good test functions**.

Exponential change of measure for asymptotic analysis

Suppose the martingale decomposition (5) is obtained. Let

$$Y(t) = \frac{1}{f(X(0))} \exp\Big(- \int_0^t \frac{\mathcal{A}f(X(u))}{f(X(u))} du \Big),$$

$$Y \cdot M(t) = 1 + \int_0^t Y(u) dM(u),$$

then $Y \cdot M(t)$ is an $\mathcal{F}_t$-martingale obtained as

$$E^f(t) \equiv Y \cdot M(t) = \frac{f(X(t))}{f(X(0))} \exp\Big(- \int_0^t \frac{\mathcal{A}f(X(u))}{f(X(u))} du \Big), \qquad (6)$$

which is positive, and $\mathbb{E}(E^f(t)) = \mathbb{E}(E^f(0)) = 1$. Hence,

$$\widetilde{\mathbb{P}}(A) = \int_A E^f(t) d\mathbb{P}, \qquad A \in \mathcal{F}_t \qquad (7)$$

is a probability measure. Under $\widetilde{\mathbb{P}}$, $X(t)$ may have diffrent asymptotics as $t \to \infty$, which is useful to see them under $\mathbb{P}$.

Return to the $GI/G/1$ queue: good test functions

Let $X(t) \equiv (L(t), R_e(t), R_s(t))$ be a PDP for $GI/G/1$ queue.
Choose the followng test function f with parameters $\theta, \eta, \zeta \in \mathbb{R}$.

$$f(z, y) = e^{\theta \mathbf{z} \vee \mathbf{1} + \eta y_e + \zeta y_s}, \qquad (z, y_e, y_s) \in S, \tag{8}$$

where $a \vee b = \max(a, b)$. Let T_s, T_e be random variables subject to the interarrival and service time distributions F_e, F_s, then

$$Qf(z, y) = \mathbb{E}(f(z + 1, T_e, y_s)), \qquad (z, 0, y_s) \in \Lambda,$$
$$Qf(z, y) = \mathbb{E}(f(z - 1, y_e, T_s 1(z \geq 2))), \qquad (z, y_e, 0) \in \Lambda.$$

Hence, f satisfies the **terminal condition** (4) if and only if

$$e^{\theta} \widehat{F}_e(\eta) = 1, \qquad e^{-\theta} \widehat{F}_s(\zeta) = 1, \tag{9}$$

where $\widehat{F}_e(\eta) = \mathbb{E}(e^{\eta T_e})$, $\widehat{F}_s(\zeta) = \mathbb{E}(e^{\zeta T_s})$. We denote these η, ζ by $\eta(\theta), \zeta(\theta)$, and denote f with them by f_θ.

Time evolution for the the exponential test function (8)

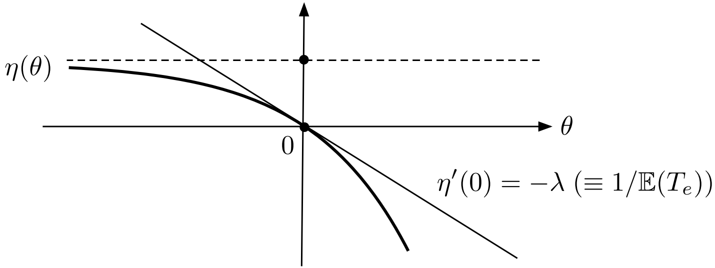
Substituting $f = f_\theta$ and $\eta = \eta(\theta), \zeta = \zeta(\theta)$ into the martingale decomposition (5), we have

$$
\begin{aligned}
e^{\theta L(t) \vee 1 + \eta(\theta) R_e(t) + \zeta(\theta) R_s(t)} - e^{\theta L(0) \vee 1 + \eta(\theta) R_e(0) + \zeta(\theta) R_s(0)} \\
= - \int_0^t (\eta(\theta) + \zeta(\theta)) e^{\theta L(u) \vee 1 + \eta(\theta) R_e(u) + \zeta(\theta) R_s(u)} du \\
+ \int_0^t \zeta(\theta) 1(L(u) = 0) e^{\theta + \eta(\theta) R_e(u) + \zeta(\theta) R_s(u)} du \\
+ M(t).
\end{aligned}
$$

This gives a concrete expression for the $\mathcal{F}_t$-martingale $M(t)$ in terms of the process $\{X(t); t \geq 0\}$.

The shape of function $\eta(\theta) = \widehat{F}_e^{-1}(e^{-\theta})$

$\eta(\theta)$ is concave and decreasing in θ.



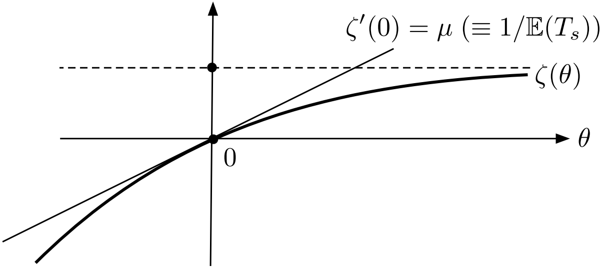
From Theorem 1 of Glynn & Whitt [3], we have

$$\eta(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(e^{\theta N_e(t)}\right).$$

Hence, $-\eta(\theta)$ is the rate function of large deviations of $N_e(t)$.

The shape of function $\zeta(\theta) = \widehat{F}_s^{-1}(e^{\theta})$

$\zeta(\theta)$ is concave and increasing in θ.



Similar to $\eta(\theta)$, we have

$$\zeta(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big(e^{-\theta N_s(t)}\big).$$

A problem is that $\eta(\theta), \zeta(\theta)$ may not be well defined for some θ, we need truncation of T_e, T_s in such cases.

Change of measure for the $GI/G/1$ queue

Let $\tau_n^+ = \inf\{t > 0; L(t) \geq n\}$ and $\tau_0^- = \inf\{t > \tau_0^+; L(t) = 0\}$, and let $\mathbb{E}_0$ represents the expectation when $L(0) = 0$ and $X(0)$ is subject to the stationary distribution of $X(t)$. Then, we have

$$\mathbb{P}(L > n) = \frac{1}{\mathbb{E}_0(\tau_0^- - \tau_0^+)} \mathbb{E}_0\Big(\int_{\tau_0^+}^{\tau_0^-} 1(L(u) > n) du \Big). \tag{10}$$

From (2) and (5), it follows that

$$E^f(t) = \frac{e^{\theta L(t) + \eta(\theta) R_e(t) + \zeta(\theta) R_s(t)}}{e^{\theta L(0) + \eta(\theta) R_e(0) + \zeta(\theta) R_s(0)}} e^{(\eta(\theta)+\zeta(\theta))t - \zeta(\theta) \int_0^t 1(L(u)=0) du}.$$

Using this, we define a new measure $\widetilde{\mathbb{P}}_0$ by (7), then

$$\mathbb{E}_0(Z(\tau_n^+ -)) = \widetilde{\mathbb{E}}_0(e^{-(n-1)\theta - \zeta(\theta) R_s(\tau_n^+ -)} e^{-(\eta(\theta)+\zeta(\theta))\tau_n^+} Z(\tau_n^+ -)) \tag{11}$$

for $Z(t) = \mathbb{E}_0\big(\int_t^{\tau_0^-} 1(L(u) > n) du 1(t < \tau_0^-) | \mathcal{F}_t \big)$.

Tail asymptotic for a queue in the $GI/G/1$ queue

Let $\gamma(\theta) = -(\eta(\theta) + \zeta(\theta))$, and assume that

$$1 < \gamma(\theta_0) < \infty, \qquad \text{for some } \theta_0 > 0. \tag{12}$$

Noting that $\gamma(\theta)$ is convex and $\gamma'(0) = \lambda - \mu < 0$, define α as

$$\alpha = \sup\{\theta \geq 0; \gamma(\theta) \leq 0\}.$$

Then, we can prove that (10) and (11) imply, for some $c > 0$,

$$\lim_{n \to \infty} e^{\alpha n} \mathbb{P}(L > n) = \frac{c}{\mathbb{E}_0(\tau_0^- - \tau_0^+)}. \tag{13}$$

When (12) fails, we need to truncate T_s as $T_s \wedge v$ for $v > 0$ and replace $\zeta(\theta)$ by $\zeta(v, \theta)$, where $a \wedge v = \min(a, b)$, then let v to infinity. This truncation is shown to work in Miyazawa [7].

Effect of the truncation $T_s \wedge v$ for $\zeta(v, \theta)$

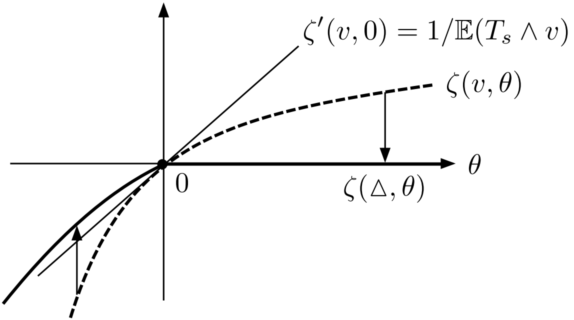
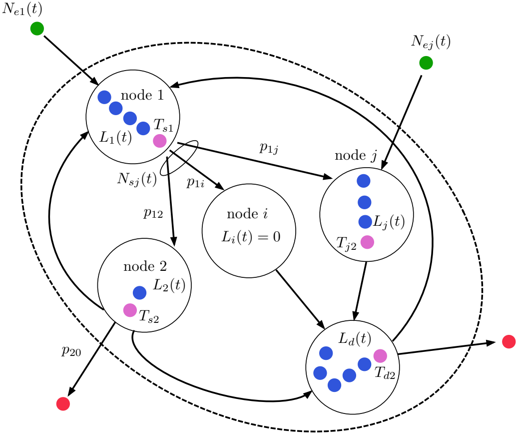


Figure: This is the case that T_s has a heavy tail distribution

$\zeta(v, \theta)$ for $v > 0$ and $\zeta(\triangle, \theta)$ are determined by

$$e^{-\theta}\mathbb{E}(e^{\zeta(v,\theta)(T_s \wedge v)}) = 1, \qquad \zeta(\triangle, \theta) = \lim_{v \to \infty} \zeta(v, \theta).$$

d-node generalized Jackson network (GJN)



$J \equiv \{1, 2, \ldots, d\}$: the set of all nodes, p_{ij}, $i, j \in J$ are routing probabilities. $N_{e,j}(t)$ is a renewal process, $T_{s,j}$ is a random variable subject to the service time distribution $F_{s,j}$ at node j.

Assumptions on the generalized Jackson network (GJN)

We consider a d-station generalized Jackson network (GJN), for which we assume:

(a) Stations has single servers and Markovian routing.

(b) Exogenous arrivals are subject to renewal processes, and service times at each node are $i.i.d.$.

(c) The stability condition, $\rho_i < 1$ for $i = 1, 2, \ldots, d$, is assumed, where ρ_i is the traffic intensity at station i.

(d) **Phase type distributions** for arrivals and services.

Assumption (d) may be unnecessarily strong, but we currently need it (except for $d = 2$) to confirm that the distribution of the remaining service time of a customer being served at node i, denoted by $R_{s,i}(t)$, is well behaved under the stationary distribution.

Test functions for the terminal condition

Our strategy to get the tail asympptotics is basically the same as it for the $GI/G/1$ queue. However, there are problems to be resolved because the queue length $\boldsymbol{L}(t)$ is multidimasional.

Let $\boldsymbol{X}(t) \equiv (\boldsymbol{L}(t), \boldsymbol{R}_e(t), \boldsymbol{R}_s(t))$ be a PDMP for the GJN, and let

$$f_{\boldsymbol{\theta}}(\boldsymbol{z}, \boldsymbol{y}_e, \boldsymbol{y}_s) = e^{\langle \boldsymbol{\theta}, \boldsymbol{z} \vee \mathbf{1} \rangle + \langle \boldsymbol{\eta}(\boldsymbol{\theta}), \boldsymbol{y}_e \rangle + \langle \boldsymbol{\zeta}(\boldsymbol{\theta}), \boldsymbol{y}_s \rangle}, \quad (\boldsymbol{z}, \boldsymbol{y}_e, \boldsymbol{y}_s) \in S, \boldsymbol{\theta} \in \mathbb{R}^d,$$

where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is the inner product of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$. This $f_{\boldsymbol{\theta}}$ satisfies the **terminal condition** (4), if $\boldsymbol{\eta}(\boldsymbol{\theta}) \equiv \{\eta_i(\theta_i); i \in J\}$ and $\boldsymbol{\zeta}(\boldsymbol{\theta}) \equiv \{\zeta_i(\boldsymbol{\theta}); i \in J\}$ are given by

$$e^{\theta_i} \widehat{F}_{e,i}(\eta_i(\theta_i)) = 1, \qquad t_i(\boldsymbol{\theta}) \widehat{F}_{s,i}(\zeta_i(\boldsymbol{\theta})) = 1, \tag{14}$$

where $\widehat{F}_{e,i}(\eta) = \mathbb{E}(e^{\eta T_{e,i}})$, $\widehat{F}_{s,i}(\zeta) = \mathbb{E}(e^{\zeta T_{s,i}})$, and

$$t_i(\boldsymbol{\theta}) = e^{-\theta_i} \left(\sum_{i \in J} p_{ij} e^{\theta_j} + p_{i0} \right).$$

Interpretations of functions $\boldsymbol{\eta}(\boldsymbol{\theta})$ and $\boldsymbol{\zeta}(\boldsymbol{\theta})$

Let $N_{e,i}(t)$ be the counting process for exogenous arriving customers at node i (if no arrival, $N_{e,i}(t) \equiv 0$), then, similar to the $GI/G/1$,

$$\eta_i(\theta_i) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{\theta_i N_{e,i}(t)}), \quad \theta_i \in \mathbb{R}, i \in J.$$

Let $N_{s,i}(t)$ be the renewal process generated by the service times at node i and $\Psi_{i,j}(n)$ be the number of routing from node i to j among n departures., then, for $\boldsymbol{\theta} \in \mathbb{R}^d$,

$$\zeta_i(\boldsymbol{\theta}) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{-\theta_i N_{s,i}(t)+\sum_{j\in J} \theta_j \Phi_{i,j}(N_{s,i}(t))}), \quad i \in J.$$

In the view of these facts, we introduce convex functions:

$$\gamma_{e,i}(\theta_i) = -\eta_i(\theta_i), \qquad \gamma_{d,i}(\boldsymbol{\theta}) = -\zeta_i(\boldsymbol{\theta}),$$

and denote their vectors by $\boldsymbol{\gamma}_e(\boldsymbol{\theta})$ and $\boldsymbol{\gamma}_d(\boldsymbol{\theta})$.

The tail decay rates for the stable GJN

The following rate functions and geometric objects (convex sets) will be useful to get the tail decay rates.

$$\gamma^{(+)}(\boldsymbol{\theta}) = \sum_{i\in J}(\gamma_{e,i}(\theta_i) + \gamma_{d,i}(\boldsymbol{\theta})), \quad \gamma^{(i)}(\boldsymbol{\theta}) = \gamma^{(+)}(\boldsymbol{\theta}) - \gamma_{d,i}(\boldsymbol{\theta}),$$

$$\overline{\Gamma}^{+} = \{\boldsymbol{\theta} \in \mathbb{R}^2; \exists \boldsymbol{\theta}', \gamma^{(+)}(\boldsymbol{\theta}') \le 0, \boldsymbol{\theta} < \boldsymbol{\theta}'\},$$

$$\Gamma^0_{K+} = \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma^{(+)}(\boldsymbol{\theta}) = 0, \gamma^{(i)}(\boldsymbol{\theta}) < 0, i \in K\}, \qquad K \subset J.$$

For $d = 2$, Miyazawa [6] derives the tail decay rate in direction $\boldsymbol{c}$:

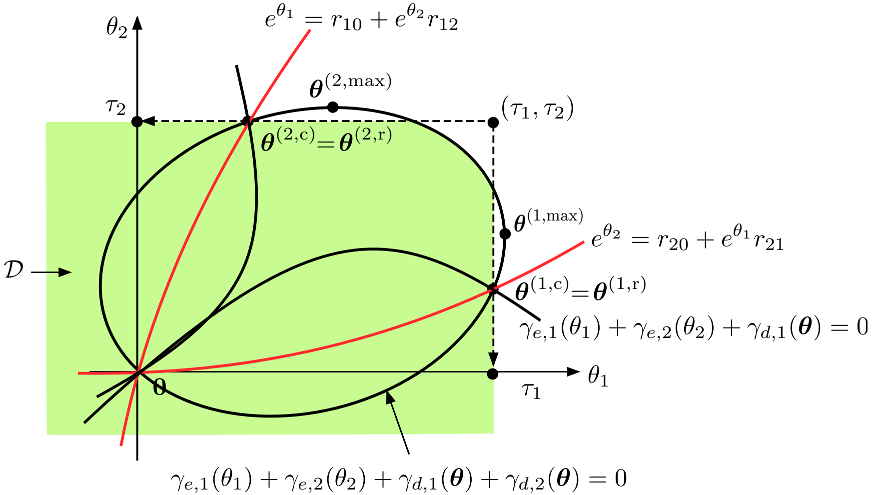
$$\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}(c_1 L_1 + c_2 L_2 > x) = \alpha_{\boldsymbol{c}},$$

where (L_1, L_2) is the stationary queue length vector, and

$$\alpha_{\boldsymbol{c}} = \sup\{u \ge 0; u\boldsymbol{c} \in \overline{\Gamma}_{+}, \theta_i < \tau_i, i = 1, 2\}, \qquad (15)$$

for the solution $\boldsymbol{\tau}$ of $\tau_i = \sup\{\theta_i \ge 0; \boldsymbol{\theta} \in \Gamma_{\{i\}+}, \theta_{3-i} < \tau_{3-i}\}$.

Geometric objects for the tail decay rates for $d = 2$



Procedure to derive the decay rates: steps 1, 2, 3

1. For the cycle formula, we choose boundary sets:

$$S_K = \{(\boldsymbol{z}, \boldsymbol{y}_e, \boldsymbol{y}_s) \in S; z_i = 0, \forall i \in K\}, \qquad K \subset J.$$

2. Derive an exponential martingale $E^{f_{\boldsymbol{\theta}}}(t)$ for change of measure with the initial distribution ν_K on S_K.

$$E^{f_{\boldsymbol{\theta}}}(t) = \frac{e^{\langle \boldsymbol{\theta}, \boldsymbol{L}(t) \vee \boldsymbol{1}\rangle - \langle \gamma_e(\boldsymbol{\theta}), \boldsymbol{R}_e(t)\rangle - \langle \boldsymbol{\gamma}_d(\theta), \boldsymbol{R}_s(t)\rangle}}{e^{\langle \boldsymbol{\theta}, \boldsymbol{L}(0) \vee \boldsymbol{1}\rangle - \langle \gamma_e(\boldsymbol{\theta}), \boldsymbol{R}_e(0)\rangle - \langle \boldsymbol{\gamma}_d(\theta), \boldsymbol{R}_s(0)\rangle}} \times e^{-\gamma^{(+)}(\boldsymbol{\theta})t + \sum_{i \in J} \gamma_{d,i}(\boldsymbol{\theta}) \int_0^t 1(L_i(u)=0)du},$$

$$\widetilde{\mathbb{P}}_{\nu_K}(A) = \mathbb{E}_{\nu_K}(E^{f_{\boldsymbol{\theta}}}(t) 1_A), \qquad A \in \mathcal{F}_t.$$

3. We get back $\mathbb{P}_{\nu_K}$ from $\widetilde{\mathbb{P}}_{\nu_K}$ by

$$\mathbb{P}_{\nu_K}(A) = \widetilde{\mathbb{E}}_{\nu_K}((E^{f_{\boldsymbol{\theta}}}(t)^{-1} A), \qquad A \in \mathcal{F}_t.$$

Procedure to derive the decay rates: further steps

4. $\tau^+_{S_K} = \inf\{t > 0; \boldsymbol{L}(t) \notin S_k\}$, $\tau^-_{S_K} = \inf\{t > \tau^+_{S_K}; \boldsymbol{L}(t) \in S_k\}$,
$$\mathbb{P}_{\nu_K}(\boldsymbol{L} \ge x\boldsymbol{c}) = c_K \mathbb{E}_{\nu_K}\Big(\int_{\tau^+_{S_K}}^{\tau^-_{S_K}} 1(\boldsymbol{L}(u) \ge x\boldsymbol{c}) du \Big).$$
5. Let $\tau^+_{x\boldsymbol{c}} = \inf\{t > 0; \boldsymbol{L}(t) \ge x\boldsymbol{c}\}$,
$Z(\tau^+_{x\boldsymbol{c}}-) = \mathbb{E}_{\nu_K}\Big(\int_{\tau^+_{x\boldsymbol{c}}}^{\tau^-_{S_K}} 1(\boldsymbol{L}(u) \ge x\boldsymbol{c}) du \Big| \mathcal{F}_{\tau^+_{x\boldsymbol{c}}-} \Big)$, then
$$\mathbb{P}(\boldsymbol{L} \ge x\boldsymbol{c}) = c_K \widetilde{\mathbb{E}}_{\nu_K}((E^{f_{\boldsymbol{\theta}}}(\tau^+_{x\boldsymbol{c}}-))^{-1} Z(\tau^+_{x\boldsymbol{c}}-) 1(\tau^+_{x\boldsymbol{c}} < \tau^-_{S_K})),$$
where
$$(E^{f_{\boldsymbol{\theta}}}(\tau^+_{x\boldsymbol{c}}-))^{-1} = \frac{e^{\langle \boldsymbol{\theta}, \boldsymbol{L}(0) \vee \boldsymbol{1} \rangle - \langle \gamma_e(\boldsymbol{\theta}), \boldsymbol{R}_e(0) \rangle - \langle \boldsymbol{\gamma}_d(\theta), \boldsymbol{R}_s(0) \rangle}}{e^{\langle \boldsymbol{\theta}, \boldsymbol{L}(\tau^+_{x\boldsymbol{c}}-) \vee \boldsymbol{1} \rangle - \langle \gamma_e(\boldsymbol{\theta}), \boldsymbol{R}_e(\tau^+_{x\boldsymbol{c}}-) \rangle - \langle \boldsymbol{\gamma}_d(\theta), \boldsymbol{R}_s(\tau^+_{x\boldsymbol{c}}-) \rangle}} \times e^{\gamma^{(+)}(\boldsymbol{\theta}) \tau^+_{x\boldsymbol{c}} - \sum_{i \in J} \gamma_{d,i}(\boldsymbol{\theta}) \int_0^{\tau^+_{x\boldsymbol{c}}} 1(L_i(u)=0) du}.$$
6. Find conditions for $e^{x\langle \boldsymbol{c}, \boldsymbol{\theta} \rangle} \mathbb{P}(\boldsymbol{L} \ge x\boldsymbol{c})$ to be bounded for $x \to \infty$.

Lemma 1

For the stable GJN, assume that the interarrival and service time distributions are of phase type. Let $\boldsymbol{L}$ be a random vector subject to the stationary distribution of $L(t)$, and let $\varphi_i(\boldsymbol{\theta}) = \mathbb{E}(e^{\langle \boldsymbol{\theta}, \boldsymbol{L} \rangle} 1(L_j = 0))$ for $\boldsymbol{\theta} \in \mathbb{R}^d, i \in J$, then we have, for $K \subset J$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\boldsymbol{L} \geq n\boldsymbol{c}) \leq -\sup\{\langle \boldsymbol{c}, \boldsymbol{\theta} \rangle; \boldsymbol{\theta} \in \Gamma^0_{K+}, \varphi_i(\boldsymbol{\theta}) < \infty, \forall i \notin K\}, \tag{16}$$

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\boldsymbol{L} \geq n\boldsymbol{c}) \geq -\inf\{\langle \boldsymbol{c}, \boldsymbol{\theta} \rangle; \boldsymbol{\theta} \in \Gamma^0_{K-}, \varphi_i(\boldsymbol{\theta}) < \infty, \forall i \notin K\}. \tag{17}$$

Concluding remarks

- For $d = 2$, we can get all the decay rate for directions $\boldsymbol{c} \geq 0$. **This refines the decay rates**, which are only known for the marginal distributions obtained in [5].
- **For $d \geq 3$, how the decay rates can be derived** ?
  $\Rightarrow$ This is a challenging problem even for the corresponding SRBM (semi-martingale reflecting Brownian motion).
- **The phase-type assumption can be removed** ?
  $\Rightarrow$ Yes, it could be. As in the extended abstract, we may apply truncation arguments on $T_{e,i}, T_{s,i}$ for this, but there still remains hard problems to be resolved.
- Do the decay rates converge to those of the corresponding **SRBM in heavy traffic** ?
  $\Rightarrow$ Yes, it is true for $d = 2$. We also conjecture it for $d \geq 3$.

References I

Anton Braverman, J.G. Dai, and Masakiyo Miyazawa.
Heavy traffic approximation for the stationary distribution of a generalized jackson network: the BAR approach.
submitted for publication, August 2015.

M. H. A. Davis.
Piecewise deterministic Markov processes: a general class of non-diffusion stochastic models.
Journal of Royal Statist. Soc. series B, 46:353–388, 1984.

P. Glynn and W. Whitt.
Logarithmic asymptotics for steady-state tail probabilities in a single-server queue.
Journal of Applied Probability, 31:131–156, 1994.

References II

Jean Jacod and Albert N. Shiryaev.
Limit Theorems for stochastic processes.
Springer, Berlin, second edition, 2003.

Masakiyo Miyazawa.
Diffusion approximation for stationary analysis of queues and their networks: a review.
Journal of the Operations Research Society of Japan, 58(1):104–148, 2015.

Masakiyo Miyazawa.
A superharmonic vector for a nonnegative matrix with QBD block structure and its application to a Markov modulated two dimensional reflecting process.
Queueing Systems, 81:1–48, August 2015.

References III

Masakiyo Miyazawa.
A unified approach for large queue asymptotics in a heterogeneous multiserver queue.
Advances in Applied Probability (to appear), March 2017.