Order statistics of Matrix-Geometric distributions



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Outline

- 1. Motivation: Maximum and minimum of two independent phase-type distributions.
- 2. The Maximum of three independent Matrix-geometric distributions.
- 3. Generalization: The r-th order statistics of n independent Matrix-geometric distributions.

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The Maximum and Minimum

Let's consider two Markov chains:

 $\left\{X_n^1\right\}_{n\in\mathbb{N}}\quad\text{and}\quad\left\{X_n^2\right\}_{n\in\mathbb{N}}.$

The space of states are given by E_1 and E_2 , respectively. In both of them it is suppose that the states are transient, except one, which is absorbing.

Let α_1 and α_2 , be the initial distributions of the corresponding Markov chains.

Let

$$\Lambda_1 = \begin{pmatrix} S_1 & \mathbf{s}_1 \\ \mathbf{0} & 1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} S_2 & \mathbf{s}_2 \\ \mathbf{0} & 1 \end{pmatrix},$$

$$\mathbf{s}_i = \mathbf{e} - \mathbf{S}_i \mathbf{e}, i = 1, 2.$$

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Let

 $Y_1 \sim \mathsf{DPH}(\alpha_1, S_1) \quad \text{and} \quad Y_2 \sim \mathsf{DPH}(\alpha_2, S_2),$

which are independent.

$$Y_{(1)} = \min(Y_1, Y_2),$$

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Denote

$$Y_{(1)} = \min(Y_1, Y_2),$$

 $Y_{(2)} = \max(Y_1, Y_2),$

the first and the second order statistics.

Multivariable Markov chain

Consider the multivariable Markov chain

$$\{X_n\} = \left(X_n^1, X_n^2\right), \quad n \in \mathbb{N}.$$

Suppose that $E_1 = \{1, 2, 3\}$ and $E_2 = \{1, 2, 3\}$ are the space of states of the corresponding Markov chains.

Consequently, the space of state of the multivariable Markov chain is:

 $\left\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(1,3),(2,3),(3,3)\right\}.$

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Order statistics for independent Matrix-geometric distributions

Order statistics of two independent discrete phase-type distributions



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Multivariable Markov chain



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Multivariable Markov chain



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Probabilistic interpretation

 $\bullet\,$ The initial distribution of the multivariable Markov chain $\{X_n\}$ is given by

 $(\alpha_1\otimes\alpha_2,\mathbf{0}).$

• The transition probability matrix is

($oldsymbol{S}_1\otimesoldsymbol{S}_2$	$\mathbf{s}_1\otimes oldsymbol{S}_2$	$oldsymbol{S}_1\otimes \mathbf{s}_2$	$\mathbf{s}_1 \otimes \mathbf{s}_2$)
	0	$oldsymbol{S}_2$	0	\mathbf{s}_2
	0	0	$oldsymbol{S}_1$	\mathbf{s}_1
				1

• Sub-transition probability matrix for the minimum $Y_{(1)}$:

$$A_{(1)} = S_1 \otimes S_2.$$

$$m{A}_{(2)} = \left(egin{array}{cccc} m{S}_1 \otimes m{S}_2 & m{s}_1 \otimes m{S}_2 & m{S}_1 \otimes m{s}_2 \ 0 & m{S}_2 & m{0} \ m{0} & m{O} & m{S}_1 \end{array}
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Representation for the minimum $Y_{(1)}$

Denote $\overline{\alpha_2} = \alpha_1 \otimes \alpha_2$. Let $m \in \mathbb{N}$. Then the product

 $\overline{lpha_2} oldsymbol{A}^m_{(1)} oldsymbol{e}$

is the survival function of $Y_{(1)}$.

$$\overline{\alpha_2} \mathbf{A}_{(1)}^m \mathbf{e} = (\alpha_1 \otimes \alpha_2) (\mathbf{S}_1 \otimes \mathbf{S}_2)^m \mathbf{e}$$

$$= (\alpha_1 \mathbf{S}_1^m \mathbf{e}) (\alpha_2 \mathbf{S}_2^m \mathbf{e})$$

$$= \mathbb{P} (Y_1 > m) \mathbb{P} (Y_2 > m)$$

$$= \mathbb{P} (Y_1 > m, Y_2 > m)$$

$$= \mathbb{P} (Y_{(1)} > m).$$

Consequently, by the Kronecker product properties and independence of the variables, we conclude that

$$Y_{(1)} \sim DPH\left(\overline{\alpha_2}, \boldsymbol{A}_{(1)}\right).$$

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Representation for the maximum $Y_{(2)}$

Recall

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We are going to verify that the product

 $(\overline{\alpha_2}, \mathbf{0}) \mathbf{A}_{(2)}^m \mathbf{e}, \quad m \in \mathbb{N},$

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Let us denote

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Observe that

$$A^m_{(2)} = \begin{pmatrix} A^m_{(1)} & B_{(1,m)} \\ 0 & C^m_{(1)} \end{pmatrix}, \quad m \ge 2,$$

and

$$(\overline{\alpha_2}, \mathbf{0}) \mathbf{A}_{(2)}^m \mathbf{e} = \overline{\alpha_2} \mathbf{A}_{(1)}^m \mathbf{e} + \overline{\alpha_2} \mathbf{B}_{(1,m)} \mathbf{e}$$

Since we already have that

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It can be proved by induction that

$$\overline{\alpha_2} \boldsymbol{B}_{(1,m)} \mathbf{e} = \mathbb{P} \left(Y_{(1)} \leq m, Y_{(2)} > m \right), \quad \text{for all} \quad m \in \mathbb{N}.$$

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Then,

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$$= \mathbb{P} \left(Y_{(1)} > m \right) + \mathbb{P} \left(Y_{(1)} \le m, Y_{(2)} > m \right)$$

$$= \mathbb{P} \left(Y_{(2)} > m \right).$$

Therefore,

$$\left(\left(\overline{lpha_{2}}, \mathbf{0}
ight), \boldsymbol{A}_{(2)}
ight)$$

is a representation for the distribution of $Y_{(2)}$.

Representation for the third order statistic

Representation for the maximum

Let Y_1, Y_2 and Y_3 be three independent Matrix-geometric distributed random variables with the following representations.

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Order statistics for independent Matrix-geometric distributions

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$$\boldsymbol{A}_{(3)} = \begin{pmatrix} s_1 \otimes s_2 \otimes s_3 & s_1 \otimes s_1 \otimes s_2 \otimes s_3 & s_1 \otimes s_1 \otimes s_2 \otimes s_1 \otimes s_1 \otimes s_2 & s_1 \otimes s$$

We are going to prove that

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Representation for the third order statistic

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$$\boldsymbol{A}_{(3)} = \begin{pmatrix} s_1 \otimes s_2 \otimes s_3 & 0 \\ 0 & S_2 \otimes S_3 & 0 & 0 & s_2 \otimes S_3 & S_2 \otimes s_3 & 0 \\ 0 & 0 & S_1 \otimes S_3 & 0 & S_1 \otimes S_3 & 0 & S_1 \otimes s_2 \\ 0 & 0 & 0 & 0 & S_1 \otimes S_2 & 0 & s_1 \otimes S_2 & s_1 \otimes s_1 \otimes$$

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Let us write the matrix $A_{(3)}$ as follows.

$$\left(egin{array}{ccc} m{A}_{(1)} & m{B}_{(1)} & m{B}_{(2)} \ m{0} & m{C}_{(1)} & m{C}_{(1,2)} \ m{0} & m{0} & m{C}_{(2)} \end{array}
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where $oldsymbol{A}_{(1)}, oldsymbol{B}_{(1)}$ and $oldsymbol{C}_{(1)}$ is as in the second order statistic,

$$C_{(1,2)}=\left(egin{array}{ccc} \mathbf{s}_2\otimes oldsymbol{S}_3&oldsymbol{S}_2\otimes oldsymbol{s}_3&oldsymbol{0}\ \mathbf{s}_1\otimes oldsymbol{S}_3&oldsymbol{0}&oldsymbol{S}_1\otimes oldsymbol{s}_3\ oldsymbol{0}&oldsymbol{s}_1\otimes oldsymbol{S}_2&oldsymbol{S}_1\otimes oldsymbol{s}_2\end{array}
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Since we are going to calculate

$$(\overline{\alpha_3}, \mathbf{0}) \mathbf{A}^m_{(3)} \mathbf{e},$$

we need to obtain an expression for $A^m_{(3)}$.

Observe that

$$egin{array}{rcl} A^m_{(3)} &=& \left(egin{array}{ccc} A^m_{(1)} & B_{(1,m)} & B_{(1,2,m)} \ m 0 & C^m_{(1)} & C_{(1,2,m)} \ m 0 & m 0 & C^m_{(2)} \end{array}
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Then,

$$(\overline{\alpha_3},\mathbf{0}) \mathbf{A}_{(3)}^m \mathbf{e} = \overline{\alpha_3} \mathbf{A}_{(1)}^m \mathbf{e} + \overline{\alpha_3} \mathbf{B}_{(1,m)} \mathbf{e} + \overline{\alpha_3} \mathbf{B}_{(1,2,m)} \mathbf{e},$$

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$$\overline{\alpha_3} \boldsymbol{A}_{(1)}^m \mathbf{e} + \overline{\alpha_3} \boldsymbol{B}_{(1,m)} \mathbf{e} = \mathbb{P}\left(Y_{(2)} > m\right)$$

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We can prove by induction, as in the second order statistics, that

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Consequently,

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Therefore,

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Representation for the r-th order statistics

Let Y_1, Y_2, \ldots, Y_n be independents Matrix-geometric distributed random variables, with representation

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Let $2 \le r \le n$. The r-th order statistics of Y_1, Y_2, \ldots, Y_n has a Matrix-geometric representation given by

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• B_j block of all the combinations with i exits, $1 \le j \le r-1$.

- C_(i) is a block diagonal matrix which consists on all the combinations formed with i − 1 elements of S₁, S₂,..., S_n, and depends of the block exit given by B_(i−1).
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Questions??

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