Order statistics of Matrix-Geometric distributions

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Outline

1. Motivation: Maximum and minimum of two independent phase-type distributions.

2. The Maximum of three independent Matrix-geometric distributions.

3. Generalization: The r-th order statistics of n independent Matrix-geometric distributions.

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The Maximum and Minimum

Let's consider two Markov chains:

$$\left\{X_n^1\right\}_{n\in\mathbb{N}} \quad \text{and} \quad \left\{X_n^2\right\}_{n\in\mathbb{N}}.$$

The space of states are given by E_1 and E_2, respectively. In both of them it is suppose that the states are transient, except one, which is absorbing.

Let α_1 and α_2, be the initial distributions of the corresponding Markov chains.

Let

$$\Lambda_1 = \begin{pmatrix} \boldsymbol{S}_1 & \mathbf{s}_1 \\ \mathbf{0} & 1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \boldsymbol{S}_2 & \mathbf{s}_2 \\ \mathbf{0} & 1 \end{pmatrix},$$

be the transition probability matrices of the corresponding Markov chains, where

$$\mathbf{s}_i = \mathbf{e} - \boldsymbol{S}_i\mathbf{e}, i = 1, 2.$$

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Let

$$Y_1 \sim \text{DPH}(\alpha_1, \boldsymbol{S}_1) \quad \text{and} \quad Y_2 \sim \text{DPH}(\alpha_2, \boldsymbol{S}_2),$$

which are independent.

Denote

$$Y_{(1)} = \text{mín}(Y_1, Y_2),$$

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Multivariable Markov chain

Consider the multivariable Markov chain

$$\{X_n\} = \left(X_n^1, X_n^2\right), \quad n \in \mathbb{N}.$$

Suppose that $E_1 = \{1,2,3\}$ and $E_2 = \{1,2,3\}$ are the space of states of the corresponding Markov chains.

Consequently, the space of state of the multivariable Markov chain is:

$$\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(1,3),(2,3),(3,3)\}.$$

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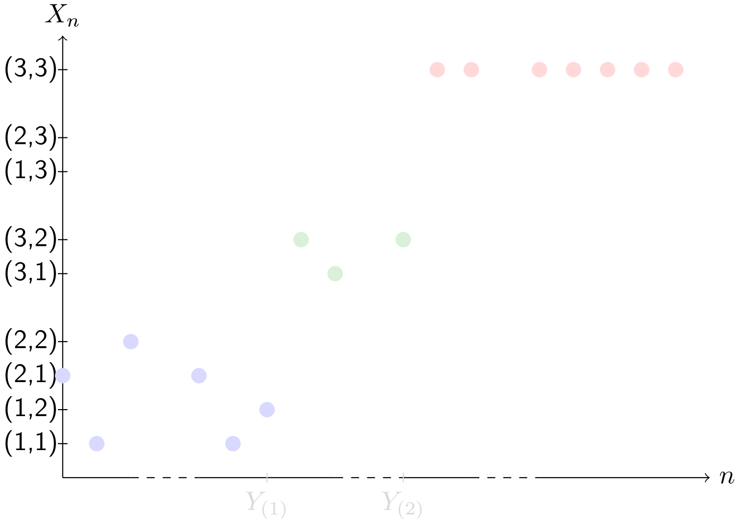

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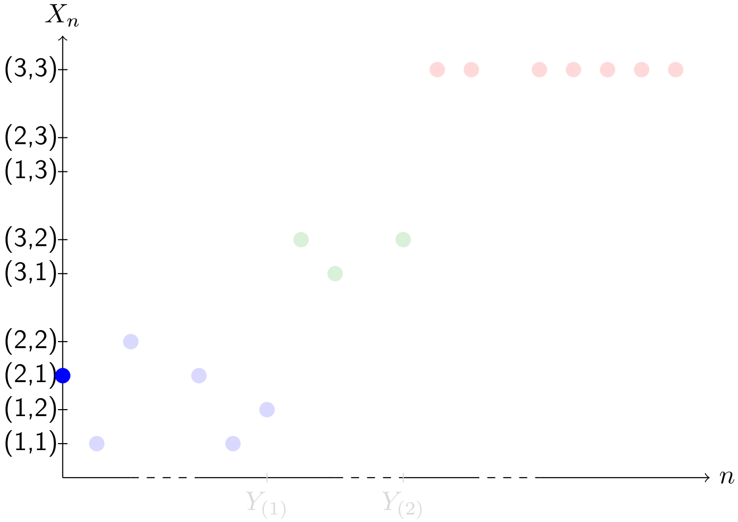

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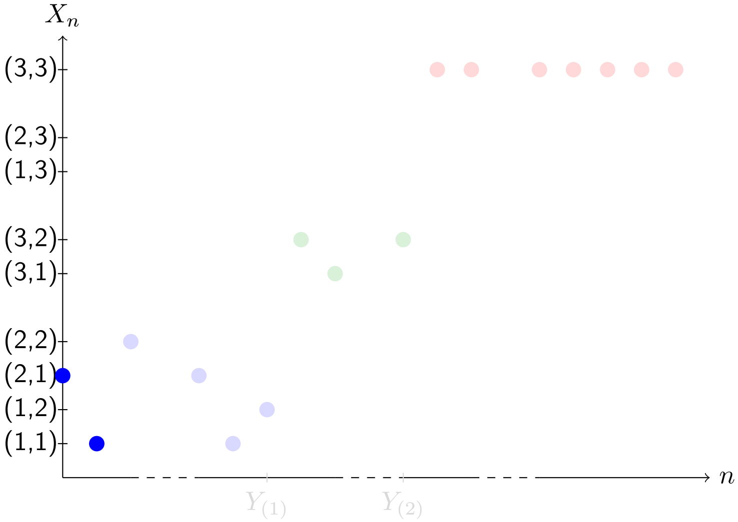

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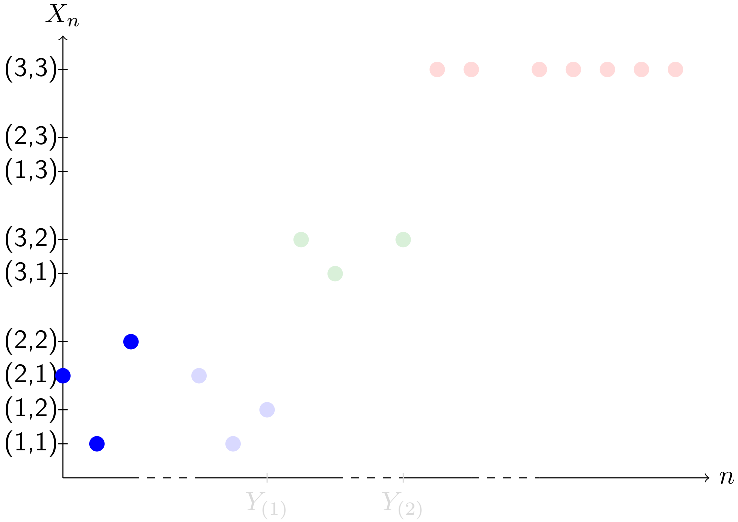

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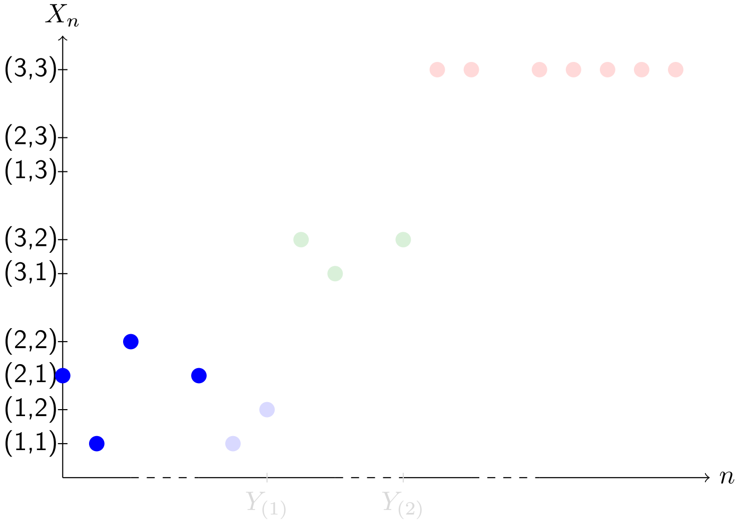

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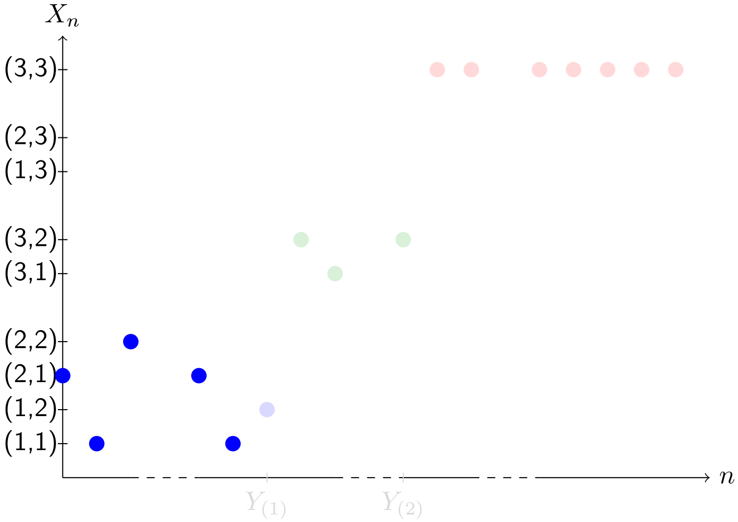

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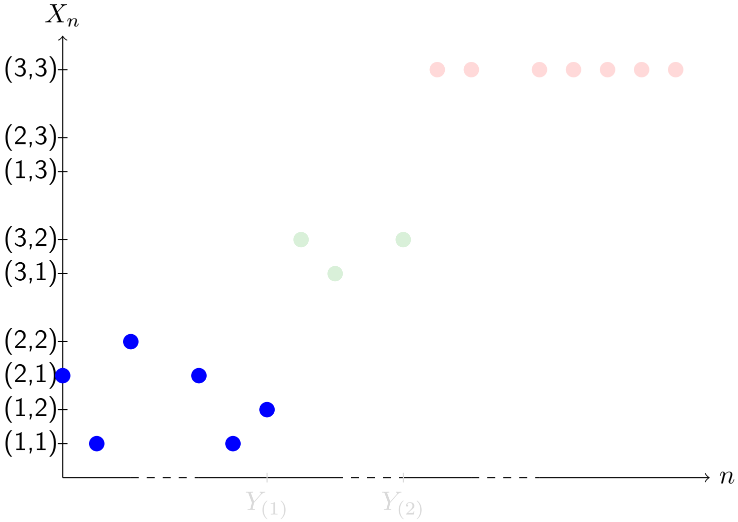

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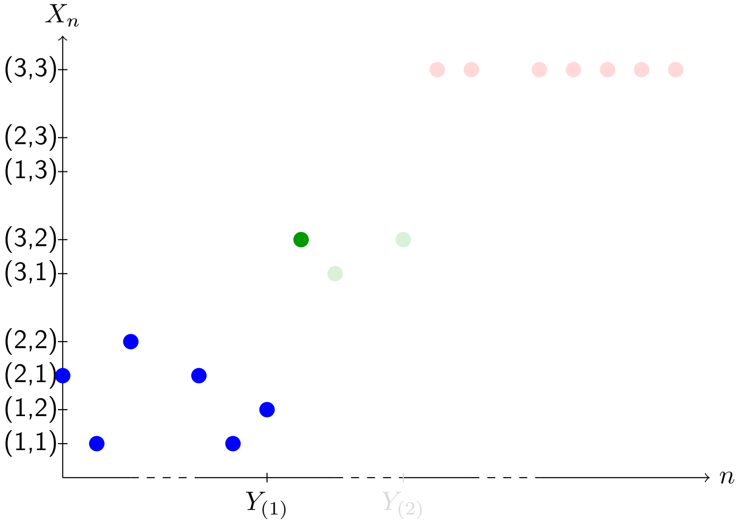

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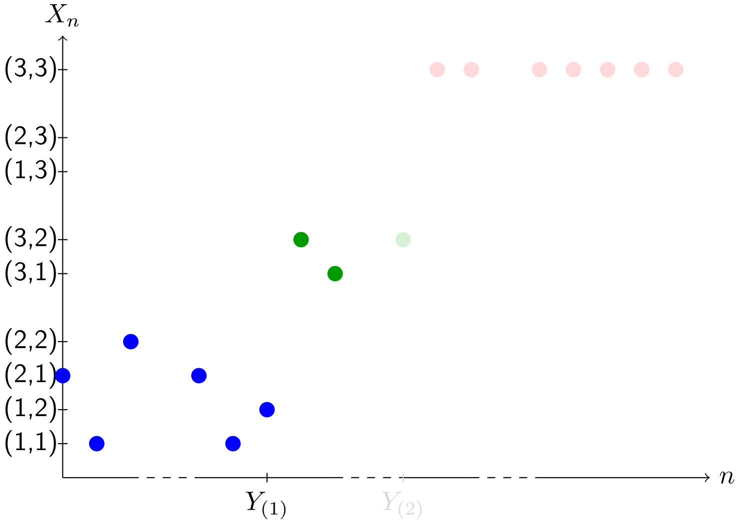

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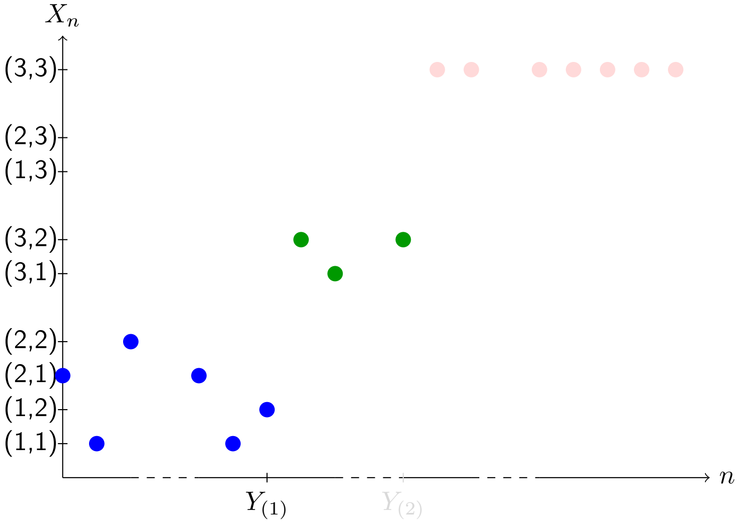

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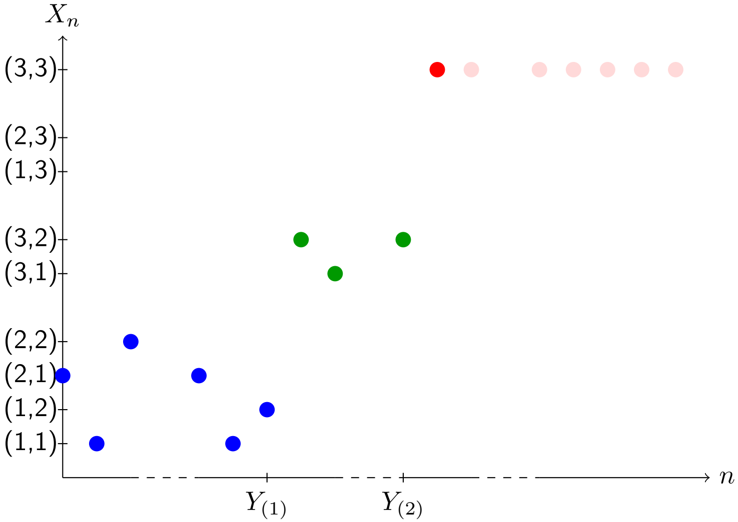

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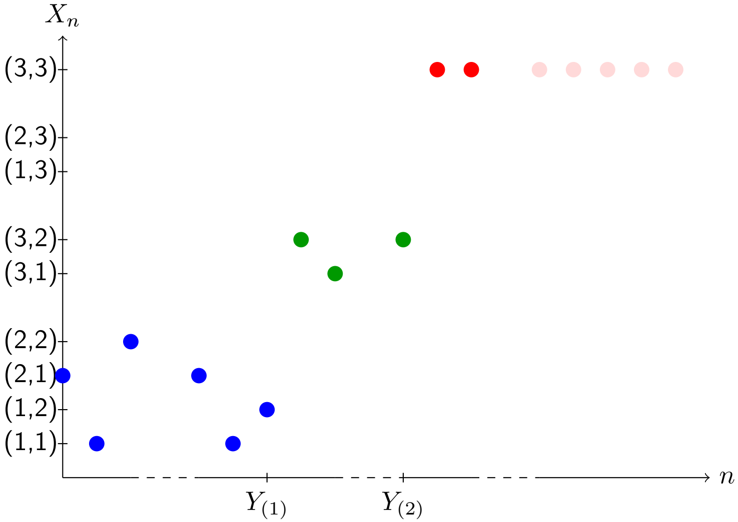

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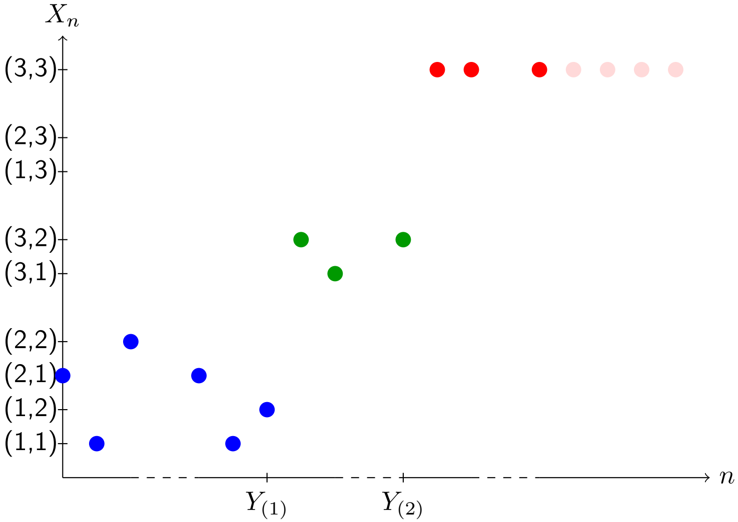

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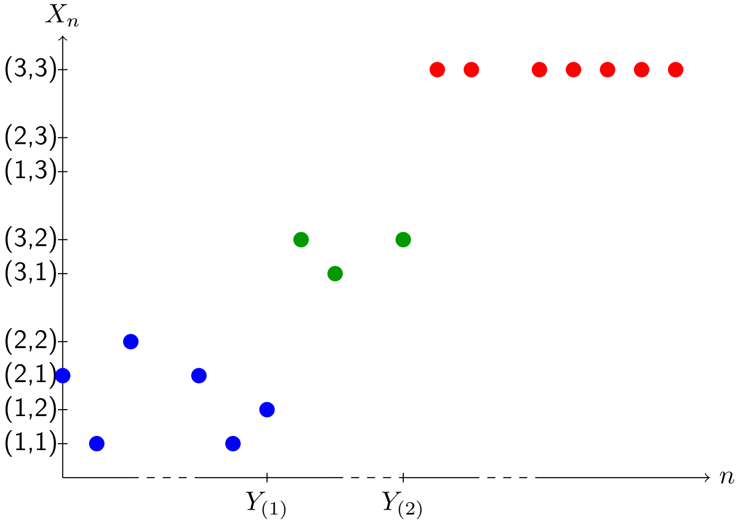

Probabilistic interpretation

- The initial distribution of the multivariable Markov chain $\{X_n\}$ is given by

$$(\alpha_1 \otimes \alpha_2, \mathbf{0}).$$

- The transition probability matrix is

$$\begin{pmatrix} \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 & \mathbf{s}_1 \otimes \mathbf{s}_2 \\ \mathbf{0} & \boldsymbol{S}_2 & \mathbf{0} & \mathbf{s}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 & \mathbf{s}_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- Sub-transition probability matrix for the minimum $Y_{(1)}$:

$$\boldsymbol{A}_{(1)} = \boldsymbol{S}_1 \otimes \boldsymbol{S}_2.$$

- Sub-transition probability matrix for the maximum $Y_{(2)}$:

$$\boldsymbol{A}_{(2)} = \begin{pmatrix} \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \\ \mathbf{0} & \boldsymbol{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \end{pmatrix}.$$

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Representation for the minimum $Y_{(1)}$

Denote $\overline{\alpha_2} = \alpha_1 \otimes \alpha_2$.

Let $m \in \mathbb{N}$. Then the product

$$\overline{\alpha_2} \boldsymbol{A}_{(1)}^m \mathbf{e}$$

is the survival function of $Y_{(1)}$.

$$\begin{aligned}
\overline{\alpha_2} \boldsymbol{A}_{(1)}^m \mathbf{e} &= (\alpha_1 \otimes \alpha_2)(\boldsymbol{S}_1 \otimes \boldsymbol{S}_2)^m \mathbf{e} \\
&= (\alpha_1 \boldsymbol{S}_1^m \mathbf{e})(\alpha_2 \boldsymbol{S}_2^m \mathbf{e}) \\
&= \mathbb{P}(Y_1 > m)\, \mathbb{P}(Y_2 > m) \\
&= \mathbb{P}(Y_1 > m, Y_2 > m) \\
&= \mathbb{P}\left(Y_{(1)} > m\right).
\end{aligned}$$

Consequently, by the Kronecker product properties and independence of the variables, we conclude that

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Recall

$$\boldsymbol{A}_{(2)} = \begin{pmatrix} \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \\ \mathbf{0} & \boldsymbol{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \end{pmatrix}.$$

We are going to verify that the product

$$(\overline{\alpha_2}, \mathbf{0})\, \boldsymbol{A}_{(2)}^m \mathbf{e}, \quad m \in \mathbb{N},$$

is the survival function of $Y_{(2)}$.

- Let us denote

$$\boldsymbol{B}_{(1)} = (\mathbf{s}_1 \otimes \boldsymbol{S}_2 \quad \boldsymbol{S}_1 \otimes \mathbf{s}_2) \quad \text{and} \quad \boldsymbol{C}_{(1)} = \begin{pmatrix} \boldsymbol{S}_2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}_1 \end{pmatrix}.$$

- Then, we have that

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Observe that

$$\boldsymbol{A}_{(2)}^{m} = \begin{pmatrix} \boldsymbol{A}_{(1)}^{m} & \boldsymbol{B}_{(1,m)} \\ \boldsymbol{0} & \boldsymbol{C}_{(1)}^{m} \end{pmatrix}, \quad m \geq 2,$$

and

$$\left(\overline{\alpha_2}, \boldsymbol{0}\right) \boldsymbol{A}_{(2)}^{m} \mathbf{e} = \overline{\alpha_2} \boldsymbol{A}_{(1)}^{m} \mathbf{e} + \overline{\alpha_2} \boldsymbol{B}_{(1,m)} \mathbf{e}.$$

Since we already have that

$$\overline{\alpha_2} \boldsymbol{A}_{(1)}^{m} \mathbf{e} = \mathbb{P}\left(Y_{(1)} > m\right).$$

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One way to calculate that is by the following product:

$$\boldsymbol{A}_{(2)}^{m} = \boldsymbol{A}_{(2)}^{m-1}\boldsymbol{A}_{(2)} = \begin{pmatrix} \boldsymbol{A}_{(1)}^{m-1} & \boldsymbol{B}_{(1,m-1)} \\ \boldsymbol{0} & \boldsymbol{C}_{(1)}^{m-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{(1)} & \boldsymbol{B}_{(1)} \\ \boldsymbol{0} & \boldsymbol{C}_{(1)} \end{pmatrix}.$$

$$\boldsymbol{B}_{(1,m)} = \boldsymbol{A}_{(1)}^{m-1}\boldsymbol{B}_{(1)} + \boldsymbol{B}_{(1,m-1)}\boldsymbol{C}_{(1)}.$$

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It can be proved by induction that

$$\overline{\alpha_2}\boldsymbol{B}_{(1,m)}\mathbf{e} = \mathbb{P}\left(Y_{(1)} \leq m, Y_{(2)} > m\right), \quad \text{for all} \quad m \in \mathbb{N}.$$

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$$\boldsymbol{A}_{(2)}^{m} = \boldsymbol{A}_{(2)}^{m-1}\boldsymbol{A}_{(2)} = \begin{pmatrix} \boldsymbol{A}_{(1)}^{m-1} & \boldsymbol{B}_{(1,m-1)} \\ \boldsymbol{0} & \boldsymbol{C}_{(1)}^{m-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{(1)} & \boldsymbol{B}_{(1)} \\ \boldsymbol{0} & \boldsymbol{C}_{(1)} \end{pmatrix}.$$

$$\boldsymbol{B}_{(1,m)} = \boldsymbol{A}_{(1)}^{m-1}\boldsymbol{B}_{(1)} + \boldsymbol{B}_{(1,m-1)}\boldsymbol{C}_{(1)}.$$

.

It can be proved by induction that

$$\overline{\alpha_2}\boldsymbol{B}_{(1,m)}\mathbf{e} = \mathbb{P}\left(Y_{(1)} \leq m, Y_{(2)} > m\right), \quad \text{for all} \quad m \in \mathbb{N}.$$

Then,

$$\begin{aligned}\left(\overline{\alpha_2}, \mathbf{0}\right) \boldsymbol{A}_{(2)}^{m}\mathbf{e} &= \overline{\alpha_2}\boldsymbol{A}_{(1)}^{m}\mathbf{e} + \overline{\alpha_2}\boldsymbol{B}_{(1,m)}\mathbf{e} \\ &= \mathbb{P}\left(Y_{(1)} > m\right) + \mathbb{P}\left(Y_{(1)} \leq m, Y_{(2)} > m\right) \\ &= \mathbb{P}\left(Y_{(2)} > m\right).\end{aligned}$$

Therefore,

$$\left(\left(\overline{\alpha_2}, \mathbf{0}\right), \boldsymbol{A}_{(2)}\right)$$

is a representation for the distribution of $Y_{(2)}$.

Representation for the maximum

Let Y_1, Y_2 and Y_3 be three independent Matrix-geometric distributed random variables with the following representations.

$$Y_i \sim MG\left(\alpha_i, \boldsymbol{S}_i, \mathbf{s}_i\right), \quad \mathbf{s}_i = \mathbf{e} - \boldsymbol{S}_i\mathbf{e}, \quad i = 1, 2, 3.$$

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$$\boldsymbol{A}_{(3)} = \begin{pmatrix} \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 \otimes \boldsymbol{S}_3 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \mathbf{s}_1 \otimes \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \otimes \mathbf{s}_3 \\ \mathbf{0} & \boldsymbol{S}_2 \otimes \boldsymbol{S}_3 & \mathbf{0} & \mathbf{0} & \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \otimes \boldsymbol{S}_3 & \mathbf{0} & \mathbf{s}_1 \otimes \boldsymbol{S}_3 & \mathbf{0} & \boldsymbol{S}_1 \otimes \mathbf{s}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 & \mathbf{0} & \mathbf{s}_1 \otimes \boldsymbol{S}_2 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \end{pmatrix}.$$

We are going to prove that

$$\left(\left(\overline{\alpha_3}, \mathbf{0} \right), \boldsymbol{A}_{(3)}, \mathbf{a}_{(3)} \right), \quad \mathbf{a}_{(3)} = \mathbf{e} - \boldsymbol{A}_{(3)} \mathbf{e}.$$

is a representation for the maximum $Y_{(3)}$.

Representation for the maximum

$$\boldsymbol{A}_{(3)} = \begin{pmatrix} \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 \otimes \boldsymbol{S}_3 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \mathbf{s}_1 \otimes \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \otimes \mathbf{s}_3 \\ \mathbf{0} & \boldsymbol{S}_2 \otimes \boldsymbol{S}_3 & \mathbf{0} & \mathbf{0} & \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \otimes \boldsymbol{S}_3 & \mathbf{0} & \mathbf{s}_1 \otimes \boldsymbol{S}_3 & \mathbf{0} & \boldsymbol{S}_1 \otimes \mathbf{s}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 & \mathbf{0} & \mathbf{s}_1 \otimes \boldsymbol{S}_2 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \end{pmatrix}.$$

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Let us write the matrix $\boldsymbol{A}_{(3)}$ as follows.

$$\begin{pmatrix} \boldsymbol{A}_{(1)} & \boldsymbol{B}_{(1)} & \boldsymbol{B}_{(2)} \\ \mathbf{0} & \boldsymbol{C}_{(1)} & \boldsymbol{C}_{(1,2)} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{C}_{(2)} \end{pmatrix},$$

where $\boldsymbol{A}_{(1)}, \boldsymbol{B}_{(1)}$ and $\boldsymbol{C}_{(1)}$ is as in the second order statistic,

$$\boldsymbol{B}_{(2)} = \begin{pmatrix} \mathbf{s}_1 \otimes \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \mathbf{s}_1 \otimes \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \otimes \mathbf{s}_3 \end{pmatrix},$$

$$\boldsymbol{C}_{(1,2)} = \begin{pmatrix} \mathbf{s}_2 \otimes \boldsymbol{S}_3 & \boldsymbol{S}_2 \otimes \mathbf{s}_3 & \mathbf{0} \\ \mathbf{s}_1 \otimes \boldsymbol{S}_3 & \mathbf{0} & \boldsymbol{S}_1 \otimes \mathbf{s}_3 \\ \mathbf{0} & \mathbf{s}_1 \otimes \boldsymbol{S}_2 & \boldsymbol{S}_1 \otimes \mathbf{s}_2 \end{pmatrix}$$

and

$$\boldsymbol{C}_{(2)} = \begin{pmatrix} \boldsymbol{S}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{S}_1 \end{pmatrix}.$$

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Since we are going to calculate

$$\left(\overline{\alpha_3}, \mathbf{0}\right) \boldsymbol{A}_{(3)}^{m} \mathbf{e},$$

we need to obtain an expression for $\boldsymbol{A}_{(3)}^{m}$.

Observe that

$$\boldsymbol{A}_{(3)}^{m} = \begin{pmatrix} \boldsymbol{A}_{(1)}^{m} & \boldsymbol{B}_{(1,m)} & \boldsymbol{B}_{(1,2,m)} \\ \mathbf{0} & \boldsymbol{C}_{(1)}^{m} & \boldsymbol{C}_{(1,2,m)} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{C}_{(2)}^{m} \end{pmatrix}.$$

Then,

$$\left(\overline{\alpha_3}, \mathbf{0}\right) \boldsymbol{A}_{(3)}^{m} \mathbf{e} = \overline{\alpha_3} \boldsymbol{A}_{(1)}^{m} \mathbf{e} + \overline{\alpha_3} \boldsymbol{B}_{(1,m)} \mathbf{e} + \overline{\alpha_3} \boldsymbol{B}_{(1,2,m)} \mathbf{e},$$

where

$$\overline{\alpha_3} \boldsymbol{A}_{(1)}^{m} \mathbf{e} + \overline{\alpha_3} \boldsymbol{B}_{(1,m)} \mathbf{e} = \mathbb{P}\left(Y_{(2)} > m\right)$$

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We can prove by induction, as in the second order statistics, that

$$\overline{\alpha_3}\boldsymbol{B}_{(1,2,m)}\mathbf{e} = \mathbb{P}\left(Y_{(1)} \leq m, Y_{(2)} \leq m, Y_{(3)} > m\right).$$

Consequently,

$$\begin{aligned} & \left(\overline{\alpha_3}, \mathbf{0}\right) \boldsymbol{A}_{(3)}^{m}\mathbf{e} \\ = \; & \overline{\alpha_3}\boldsymbol{A}_{(1)}^{m}\mathbf{e} + \overline{\alpha_3}\boldsymbol{B}_{(1,m)}\mathbf{e} + \overline{\alpha_3}\boldsymbol{B}_{(1,2,m)}\mathbf{e} \\ = \; & \mathbb{P}\left(Y_{(2)} > m\right) + \mathbb{P}\left(Y_{(1)} \leq m, Y_{(2)} \leq m, Y_{(3)} > m\right) \\ = \; & \mathbb{P}\left(Y_{(3)} > m\right). \end{aligned}$$

Therefore,

$$\left(\overline{\alpha_3}, \boldsymbol{A}_{(3)}, \mathbf{a}_{(3)}\right), \quad \text{where} \quad \mathbf{a}_{(3)} = \mathbf{e} - \boldsymbol{A}_{(3)}\mathbf{e},$$

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Representation for the r-th order statistics

Let $Y_1, Y_2, \ldots, Y_n$ be independents Matrix-geometric distributed random variables, with representation

$$(\alpha_1, \boldsymbol{S}_i, \mathbf{s}_i)\,, \quad \text{where} \quad \mathbf{s}_i = \mathbf{e} - \boldsymbol{S}_i\mathbf{e}, \quad i = 1, \ldots, n.$$

Denote $\overline{\alpha_n} = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$.

For $r = 1$. $Y_{(1)} \sim MG\left(\overline{\alpha_n}, \boldsymbol{A}_{(1)}, \mathbf{a}_{(1)}\right)$, where

$$\boldsymbol{A}_{(1)} = \boldsymbol{S}_1 \otimes \boldsymbol{S}_2 \otimes \cdots \otimes \boldsymbol{S}_n \quad \text{and} \quad \mathbf{a}_{(1)} = \mathbf{e} - \boldsymbol{A}_{(1)}\mathbf{e}.$$

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Let $2 \leq r \leq n$. The r-th order statistics of $Y_1, Y_2, \ldots, Y_n$ has a Matrix-geometric representation given by

$$((\overline{\alpha_r}, \mathbf{0}), \boldsymbol{A}_{(r)}, \mathbf{a}_{(r)}), \quad \mathbf{a}_{(r)} = \mathbf{e} - \boldsymbol{A}_{(r)}\mathbf{e}.$$

where

$$\boldsymbol{A}_{(r)} = \begin{pmatrix} \boldsymbol{A}_{(1)} & \boldsymbol{B}_{(1)} & \boldsymbol{B}_{(2)} & \cdots & \boldsymbol{B}_{(r-1)} \\ \mathbf{0} & \boldsymbol{C}_{(1)} & \boldsymbol{C}_{(1,2)} & \cdots & \boldsymbol{C}_{(1,r-1)} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{C}_{(2)} & \cdots & \boldsymbol{C}_{(2,r-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{C}_{(r-1)} \end{pmatrix},$$

- $\boldsymbol{B}_j$ block of all the combinations with i exits, $1 \leq j \leq r-1$.
- $\boldsymbol{C}_{(i)}$ is a block diagonal matrix which consists on all the combinations formed with $i-1$ elements of $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n$, and depends of the block exit given by $\boldsymbol{B}_{(i-1)}$.
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- $\boldsymbol{B}_j$ block of all the combinations with i exits, $1 \leq j \leq r-1$.
- $\boldsymbol{C}_{(i)}$ is a block diagonal matrix which consists on all the combinations formed with $i-1$ elements of $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n$, and depends of the block exit given by $\boldsymbol{B}_{(i-1)}$.
- $\boldsymbol{C}_{(j,i)}$ is a exit block corresponding to the block matrices $\boldsymbol{C}_j$ and $\boldsymbol{B}_{(i)}$.

Let $2 \leq r \leq n$. The r-th order statistics of $Y_1, Y_2, \ldots, Y_n$ has a Matrix-geometric representation given by

$$((\overline{\alpha_r}, \mathbf{0}), \boldsymbol{A}_{(r)}, \mathbf{a}_{(r)}), \quad \mathbf{a}_{(r)} = \mathbf{e} - \boldsymbol{A}_{(r)}\mathbf{e}.$$

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