A numerical framework for computing the limiting distribution of a stochastic fluid-fluid

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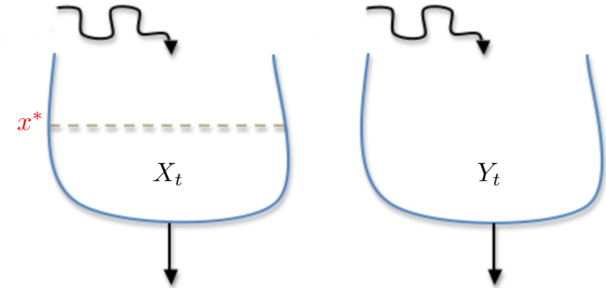
MAM9, Budapest, Tuesday 28 June, 2016

Two-buffer ON-OFF model

Input: R_1 when ON R_2 when ON



Output: $O_1(t)$ $O_2(t) = \mathcal{C} - O_1(t)$

$$
\begin{aligned}
O_1(t) &= \mathcal{C} \quad \text{for } X_t > x^* \\
&= c_1 < \mathcal{C} \quad \text{for } 0 < X_t \leq x^* \\
&= 0 \quad \text{for } X_t = 0
\end{aligned}
$$

Stochastic fluid-fluid process $\{X_t, Y_t, \varphi_t\}_{t\geq 0}$

- φ_t is a Markov chain on a finite state space $\mathcal{S}$ with generator T
- $X_t \in [0, \infty)$ is the first fluid:

$$X_t := X_0 + \int_0^t c_{\varphi_s}\, \mathrm{d}s.$$

- $Y_t \in [0, \infty)$ is the second fluid:

$$Y_t := Y_0 + \int_0^t r_{\varphi_s}(X_s)\, \mathrm{d}s.$$

Y_t is a Markov process on $\mathcal{S} \times [0, \infty)$.

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  Y_t is a Markov process on $\mathcal{S} \times [0, \infty)$.

- For each $i \in \mathcal{S}$, consider a partition of the state space $\mathcal{F} := [0, \infty)$ of $\{X_t\}$ such that

$$\mathcal{F}_i^+ := \{u \in \mathcal{F} : r_i(u) > 0\}$$
$$\mathcal{F}_i^- := \{u \in \mathcal{F} : r_i(u) < 0\}$$
$$\mathcal{F}_i^0 := \{u \in \mathcal{F} : r_i(u) = 0\}.$$

Operator matrices $\mathbb{V}(t)$, $\mathbb{B}$, $\mathbb{U}(y,s)$, and $\mathbb{D}(s)$

$$\mu_i^\ell \mathbb{V}_{ij}^{\ell m}(t)(\mathcal{A}) := \int_{x\in\mathcal{F}_i^\ell} \mathrm{d}\mu_i^\ell(x)\mathbb{P}[\varphi_t = j, X_t \in \mathcal{A} | \varphi_0 = i, X_0 = x].$$

- $\mathbb{V}(t) := [\mathbb{V}_{ij}^{\ell m}(t)]$, and let $\mathbb{B}$ be its generator: $\mathbb{V}(t) = \mathrm{e}^{\mathbb{B}t}$ for $t \geq 0$.
- $\mathbb{U}(y,s)$ is the matrix of operators recording the LST of the first time that the total in-out fluid amount of $\{Y_t\}$ reaches level y, and we can write

$$\mathbb{U}(y,s) = \mathrm{e}^{\mathbb{D}(s)y}.$$

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Lemma (Bean and O'Reilly (2014))

For $y \geq 0$, $s \in \mathbb{C}$ *with Re*$(s) \geq 0$, *and* $\ell, m \in \{+,-\}$,

$$\mathbb{D}^{\ell m}(s) = [\mathbb{R}^\ell(\mathbb{B}^{\ell m} - s\mathbb{I} + \mathbb{B}^{\ell 0}(s\mathbb{I} - \mathbb{B}^{00})^{-1}\mathbb{B}^{0m})],$$

where $\mathbb{R}^\ell := \mathrm{diag}(\mathbb{R}_i^\ell)_{i\in\mathcal{S}_\ell}, \quad \mu_i^\ell \mathbb{R}_i^\ell(\mathcal{A}) := \int_{x\in\mathcal{A}\cap\mathcal{F}_i^\ell} \frac{1}{r_i(x)}\,\mathrm{d}\mu_i^\ell(x).$

Operator matrix $\Psi(s)$

- $\Psi(s) = [\Psi_{ij}(s)]$ is the matrix of operators recording the LST of the time for $\{Y_t\}$ to return, for the first time, to the initial level of zero.

Theorem (Bean and O'Reilly, 2014)

For Re$(s) \geq 0$*,*

$$\mathbb{D}^{+-}(s) + \Psi(s)\mathbb{D}^{-+}(s)\Psi(s) + \mathbb{D}^{++}(s)\Psi(s) + \Psi(s)\mathbb{D}^{--}(s) = 0.$$

Furthermore, if s *is real then* $\Psi(s)$ *is the minimal nonnegative solution.*

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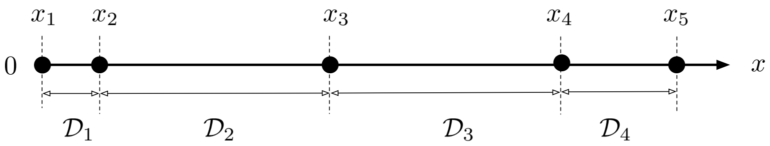
We construct a Discontinuous Galerkin method to approximate the operators $\mathbb{B}$ and $\Psi(0)$ — key components of the stationary distribution of a stochastic fluid-fluid process $\{X_t, Y_t, \varphi_t\}$.

Discontinuous Galerkin: approximate solutions to PDEs

- $\{x_1 := 0, x_2, \ldots, x_K := \mathcal{I}\}$: K nodal points on an interval $[0, \mathcal{I}]$
- $\{\mathcal{D}_1, \ldots, \mathcal{D}_{K-1}\}$: the sequence of corresponding meshes



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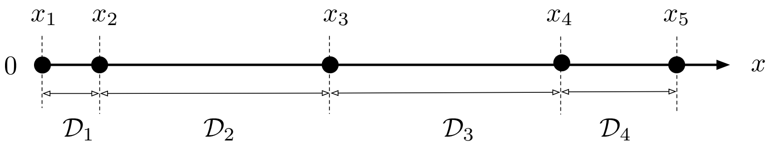

(a) Within each mesh, there is a finite-element approximation, which
- chooses appropriate piecewise polynomial basis functions, and then
- projects the PDEs onto the space constructed by these functions

$\rightarrow$ a new system of equations $\equiv$ a weak form of the original PDEs.

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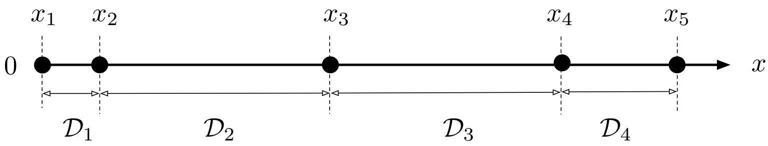

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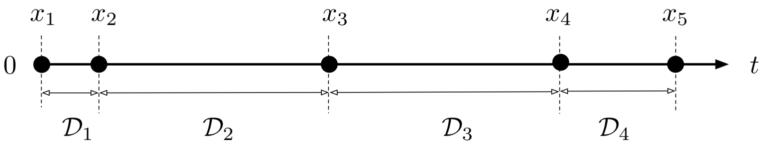
(b) Between adjacent meshes, there is a flux operator moving probability from one mesh to another.

Approximate density $f_i(x,t) = \frac{\partial}{\partial x}\mathbb{P}\left[X_t \le x, \varphi_t = i\right]$



- For each mesh k, we choose N_k polynomial basis functions $\phi_n^k : \mathcal{D}_k \mapsto [0,\infty)$, which spans our approximation space

$$V_K := \oplus_{k=1}^{K-1}\left\{\phi_1^k, \ldots, \phi_{N_k}^k\right\}.$$

- A function $u_i(\cdot,\cdot) \in V_K$ has the form:

$$u_i(x,t) = \sum_{k=1}^{K-1}\sum_{n=1}^{N_k} \alpha_{i,n}^k(t)\phi_n^k(x) \quad \text{for } x \in [0,\mathcal{I}] \text{ and } t \ge 0,$$

for some coefficient functions $\alpha_{i,n}^k(t)$.

Approximate density $f_i(x,t) = \frac{\partial}{\partial x}\mathbb{P}\left[X_t \le x, \varphi_t = i\right]$

$$\frac{\partial}{\partial t} f_i(x,t) \ = \sum_{j\in\mathcal{S}} f_j(x,t) T_{ji} - c_i \frac{\partial}{\partial x} f_i(x,t). \tag{1}$$

Theorem (Bean, N., O'Reilly, Sunkara (2016))

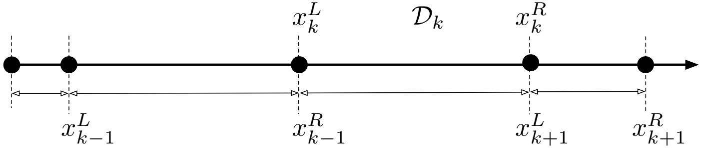
The weak formulation of (1) *is the system of ODEs*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}_i(t) = \sum_{j\in\mathcal{S}} \boldsymbol{\alpha}_j(t) T_{ji} + c_i \boldsymbol{\alpha}_i(t)(G + F_i)M^{-1} \quad \text{for } i \in \mathcal{S}.$$

$$M := \begin{bmatrix} M^{(1)} & & \\ & \ddots & \\ & & M^{(K-1)} \end{bmatrix}, \quad G := \begin{bmatrix} G^{(1)} & & \\ & \ddots & \\ & & G^{(K-1)} \end{bmatrix},$$

$$[M^{(k)}]_{mn} := \int_{\mathcal{D}_k} \phi_m^k(x)\phi_n^k(x)\,\mathrm{d}x, \quad [G^{(k)}]_{mn} := \int_{\mathcal{D}_k} \phi_m^k(x)\left[\frac{\partial}{\partial x}\phi_n^k(x)\right]\mathrm{d}x.$$

Flux operator $f_i^*(x,t)$ and the matrix F_i



- When $c_i < 0$, the flux direction is $\longleftarrow$

$$f_i^*(x_k^L,t) := u_i(x_k^L,t), \quad f_i^*(x_k^R,t) := \eta_{k+1,k} u_i(x_{k+1}^L,t).$$

- When $c_i > 0$, the flux direction is $\longrightarrow$

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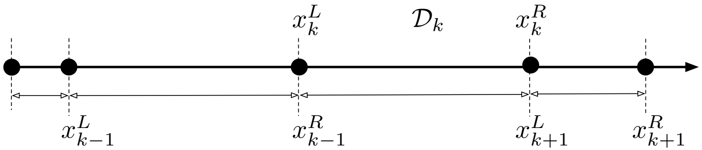

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- The matrix F_i is such that $[\boldsymbol{\alpha}_i(t) F_i]_{k,m} = \left[f_i^*(x,t)\phi_m^k(x) \right]_{x_k^L}^{x_k^R}$.

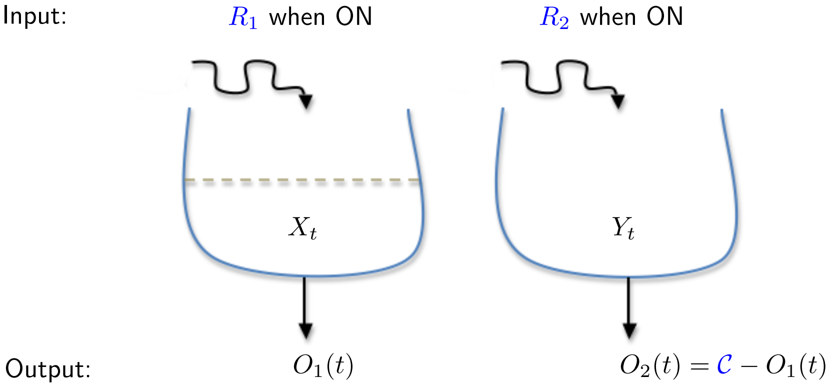
DG infinitesimal operator $\mathcal{Q}_i$ and DG approximations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\alpha}_i(t) = \sum_{j\in\mathcal{S}} \boldsymbol{\alpha}_j(t) T_{ji} + \boldsymbol{\alpha}_i(t) \underbrace{c_i(G+F_i)M^{-1}}_{\mathcal{Q}_i} \quad \text{for } i \in \mathcal{S}.$$

- Using $\mathcal{Q}_i$, we can construct DG approximations $\mathcal{B}$ and $\mathcal{D}(s)$ for the generators $\mathbb{B}$ and $\mathbb{D}(s)$, respectively.
- An approximation $\psi(s)$ of operator $\Psi(s)$ is a solution to

$$\mathcal{D}^{+-}(s) + \psi(s)\mathcal{D}^{-+}(s)\psi(s) + \mathcal{D}^{++}(s)\psi(s) + \psi(s)\mathcal{D}^{--}(s) = 0.$$

Numerical experiments: two-buffer ON-OFF model



Numerical experiments

1. Run Monte Carlo simulations to verify the DG approximation $\psi(s)$.
2. Using $\psi(s)$, evaluate the limiting joint density of $\{X_t, \varphi_t\}$ and then compare it against the analytical density.
3. Sensitivity analysis: vary the parameters related to $\{Y_t\}$.
4. Consider different choices for the level of spatial discretisation and the degree of polynomial basis functions, with respect to the order of convergence in approximation errors.

Monte Carlo simulations: cumulative distributions

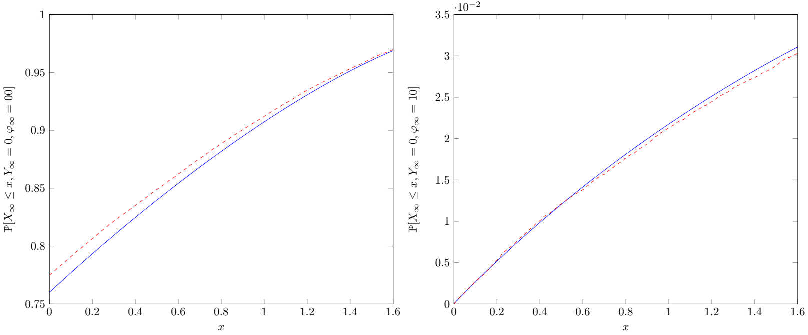


Figure 1: OFF and ON cumulative distributions of $\{X_t\}$ at the time $\{Y_t\}$ returns to level 0

- solid blue — piecewise linear DG approximation
- the dashed red — empirical cumulative distribution

Monte Carlo simulations: density functions

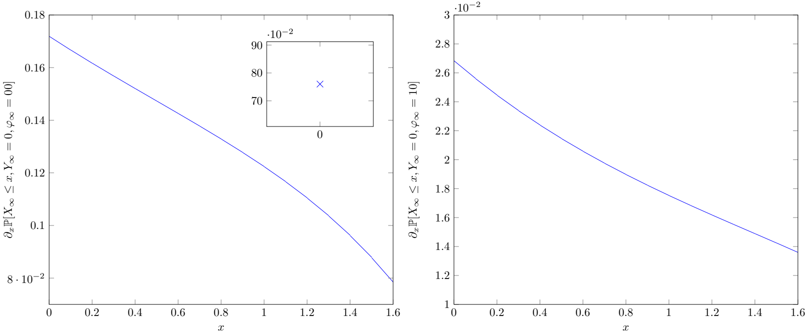


Figure 2: OFF and ON densities of $\{X_t\}$ at the time $\{Y_t\}$ returns to 0

- solid blue — piecewise linear DG approximation

DG approximations versus analytic density for $\{X_t\}$

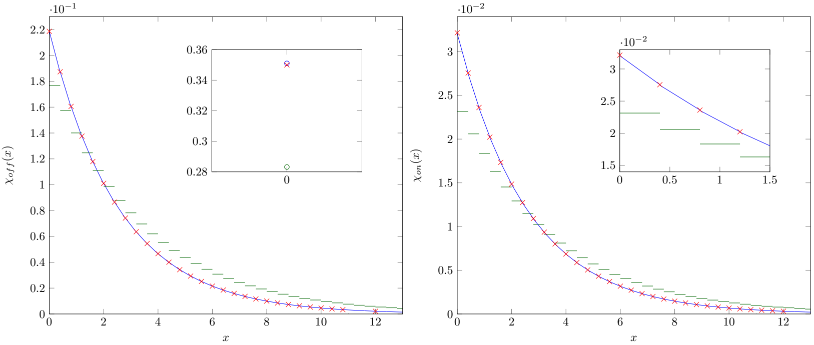


Figure 3: Approximations of the OFF and ON stationary densities of $\{X_t\}$

- red crosses — analytical solution
- green horizontal — piecewise constant approximation
- solid blue — piecewise linear approximation

All Discontinuous Galerkin approximations were evaluated using $\psi(s)$.

Varying rates for Buffer 2

$$\chi^0(x) := \lim_{t\to\infty} \frac{\partial}{\partial x} \mathbb{P}\left[X_t \le x, Y_t = 0\right]$$
$$\chi^+(x) := \lim_{t\to\infty} \frac{\partial}{\partial x} \mathbb{P}\left[X_t \le x, Y_t > 0\right]$$

- Y_t switches OFF with rate α_2, and switches on with rate β_2.

	$\alpha_2 = 11,\ \beta_2 = 1$	$\alpha_2 = 16,\ \beta_2 = 1$	$\alpha_2 = 22,\ \beta_2 = 1$
$\int_{[0,\mathcal{I}]} \widehat{\chi}^0(x)\,\mathrm{d}x$	$\approx$ 0.0	0.184	0.312
$\int_{[0,\mathcal{I}]} \widehat{\chi}^+(x)\,\mathrm{d}x$	$\approx$ 1.0	0.816	0.688

Approximation errors

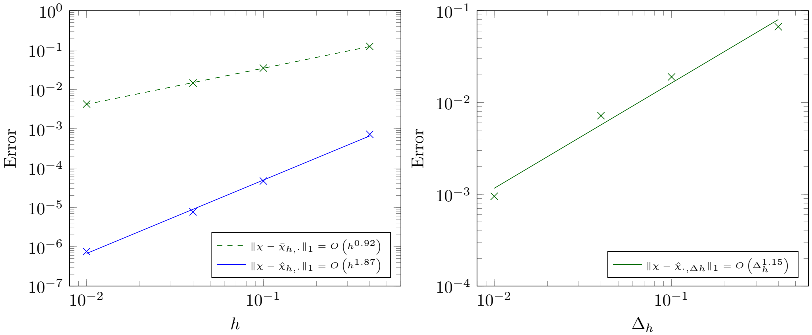


Figure 4a. Approximation error as a function of regular mesh size h

- dashed green — piecewise constant approximation
- solid blue — piecewise linear approximation

Figure 4b. Approximation error as a function of boundary mesh size Δ_h

In a nutshell

- We have a Discontinuous Galerkin method for approximating the stationary distribution of a stochastic fluid-fluid $\{X_t, Y_t, \varphi_t\}$.
- To explore: proofs of convergence, of convergence rates, of smoothness of approximations.

Case 1. When $i \neq j$, each $N_k \times N_k$ sub-block $\left[\mathcal{B}_{ij}^{\ell m}\right]_{kk}$ is given by

$$\left[\mathcal{B}_{ij}^{\ell m}\right]_{kk} := T_{ij} I_{N_k} \mathbb{1}_{\{k \in \gamma_{i\ell} \cap \gamma_{jm}\}} \quad \text{for } k = 1, \ldots, K-1.$$

Case 2. When $i = j$ and $\ell \neq m$,

$$\left[\mathcal{B}_{ii}^{\ell m}\right]_{k,k+1} := \mathcal{Q}_{k,k+1}^{i} \mathbb{1}_{\{k \in \gamma_{i\ell}, k+1 \in \gamma_{jm}\}} \quad \text{for } c_i > 0, k = 1, \ldots, K-2,$$
$$\left[\mathcal{B}_{ii}^{\ell m}\right]_{k,k-1} := \mathcal{Q}_{k,k-1}^{i} \mathbb{1}_{\{k \in \gamma_{i\ell}, k-1 \in \gamma_{jm}\}} \quad \text{for } c_i < 0, k = 2, \ldots, K-1.$$

Case 3. When $i = j$ and $\ell = m$,

$$\left[\mathcal{B}_{ii}^{\ell\ell}\right]_{kk} := \left(T_{ii} I_{N_k} + \mathcal{Q}_{kk}^{i}\right) \mathbb{1}_{\{k \in \gamma_{i\ell}\}} \quad \text{for } k = 1, \ldots, K-1,$$

where $\gamma_{i\ell}$ is the set of meshes included in $\mathcal{F}_i^{\ell}$, $i \in \mathcal{S}$, $\ell \in \{+, -, 0\}$.

DG approximations $\mathcal{R}$, $\mathcal{D}$, ψ

- An approximation $\mathcal{R}_i^\ell$ of operator $\mathbb{R}_i^\ell$ is a block-diagonal matrix with

$$\left[\mathcal{R}_i^\ell\right]_{kk} := \operatorname{diag}\left[\left(\frac{1}{\left|\rho_{i,(k,n)}\right|}\right)_{n=1,\ldots,N_k}\right]\mathbb{1}_{\left\{k\in\gamma_i^\ell\right\}},$$

where $\quad \rho_{i,(k,n)} := \int_{\mathcal{D}_k} r_i(x)\phi_n^k(x)\,\mathrm{d}x \quad$ for $\varphi_t = i, X_t = x \in \mathcal{D}_k$.

- An approximation $\mathcal{D}^{\ell m}(s)$ of operator $\mathbb{D}^{\ell m}(s)$ is

$$\mathcal{D}^{\ell m}(s) := \mathcal{R}^\ell\left(\mathcal{B}^{\ell m} - sI + \mathcal{B}^{\ell 0}(sI - \mathcal{B}^{00})^{-1}\mathcal{B}^{0m}\right)$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and for $\ell, m \in \{+,-\}$.

- An approximation $\psi(s)$ of operator $\Psi(s)$ is a solution to

$$\mathcal{D}^{+-}(s) + \psi(s)\mathcal{D}^{-+}(s)\psi(s) + \mathcal{D}^{++}(s)\psi(s) + \psi(s)\mathcal{D}^{--}(s) = 0.$$