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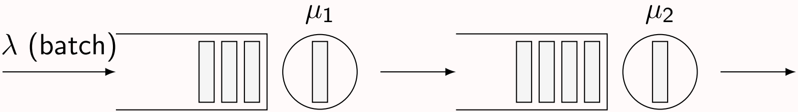
Asymptotic error bounds for truncated buffer approximations of a 2-node tandem queue

MAM-9
Budapest, June 29, 2016

Tandem network: $M^X/M/1 \to \bullet/M/1$



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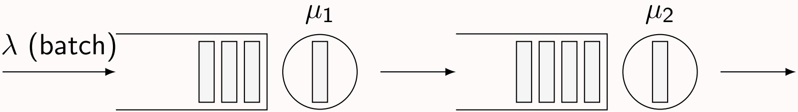
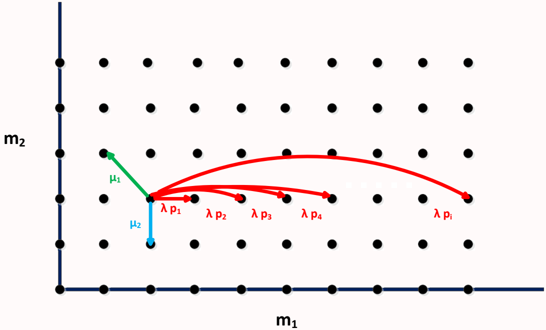


- B r.v. for the batch sizes; $\mathbb{E}B = \sum_{i=1}^{\infty} ip_i < \infty$.
- Assumption: $\lambda \mathbb{E}B/\mu_i < 1$, $i = 1, 2$
- Uniformisation: $\lambda + \mu_1 + \mu_2 = 1$
- X_n and Y_n queue lengths (including service) at the nth jump epoch, s.t. $(X_n, Y_n) \in \mathbb{N}^2$

Transition diagram of the QBD



Infinitesimal generator: $$\boldsymbol{Q} = \begin{bmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Matrix-analytic methods – MAM

For an irreducible and positive recurrent Markov chain, there exists a unique $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi}\mathbf{e} = 1$.

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The stationary distribution

If we partition $\boldsymbol{\pi}$ by level (1st coordinate) to the sub-vectors $\boldsymbol{\pi}_n$, $n \geq 0$, then

$$\boldsymbol{\pi}_0\mathbf{B} + \boldsymbol{\pi}_1\mathbf{A}_2 = \mathbf{0},$$
$$\boldsymbol{\pi}_{n-1}\mathbf{A}_0 + \boldsymbol{\pi}_n\mathbf{A}_1 + \boldsymbol{\pi}_{n+1}\mathbf{A}_2 = \mathbf{0}, \qquad n \geq 1,$$
$$\sum_{n\geq 0} \boldsymbol{\pi}_n\mathbf{e} = 1.$$

where each $\boldsymbol{\pi}_n$ is $(N+1)$-dimensional.

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$$\begin{aligned} \boldsymbol{\pi}_0\mathbf{B} + \boldsymbol{\pi}_1\mathbf{A}_2 &= \mathbf{0}, \\ \boldsymbol{\pi}_{n-1}\mathbf{A}_0 + \boldsymbol{\pi}_n\mathbf{A}_1 + \boldsymbol{\pi}_{n+1}\mathbf{A}_2 &= \mathbf{0}, \qquad n \geq 1, \\ \sum_{n\geq 0} \boldsymbol{\pi}_n\mathbf{e} &= 1. \end{aligned}$$

where each $\boldsymbol{\pi}_n$ is $(N+1)$-dimensional.

Requirement: finite number of phases (2nd coordinate)

Evaluation of (X_∞, Y_∞)

- $(X_0, Y_0) = (0,0)$ initial state
- $T_{(0,0)} = \inf\{n \geq 1 : X_n = Y_n = 0 \mid X_0 = Y_0 = 0\}$, return time to the origin or cycle length

$$\mathbb{P}\big(X_\infty \geq x, Y_\infty \geq y\big) = \frac{1}{\mathbb{E}\,T_{(0,0)}} \mathbb{E}\left[\sum_{n=1}^{T_{(0,0)}} \mathbb{1}\left(X_n \geq x, Y_n \geq y\right)\right]$$

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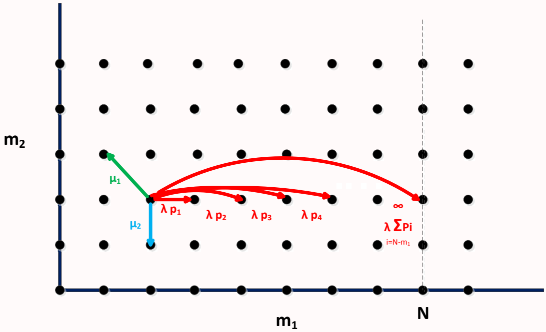
$$+ \frac{1}{\mathbb{E}\,T_{(0,0)}}\underbrace{\mathbb{E}\left[\sum_{n=1}^{T_{(0,0)}} \mathbb{1}\left(X_n \geq x, Y_n \geq y\right) \cdot \mathbb{1}\left(\max_{1\leq l \leq T_{(0,0)}} X_l \geq N\right)\right]}_{=\mathbb{II}}.$$

Truncation of the state space



$$\mathbb{I} = \mathbb{E}\left[\sum_{n=1}^{T_{(0,0)}^{(N)}} \mathbb{1}\left(X_n^{(N)} \geq x, Y_n^{(N)} \geq y\right) \cdot \mathbb{1}\left(\max_{1 \leq l \leq T_{(0,0)}^{(N)}} X_l^{(N)} < N \right) \right]$$

$$\leq \mathbb{E}\left[\sum_{n=1}^{T_{(0,0)}^{(N)}} \mathbb{1}\left(X_n^{(N)} \geq x, Y_n^{(N)} \geq y\right) \right] = \mathbb{E}\, T_{(0,0)}^{(N)} \mathbb{P}\left(X_\infty^{(N)} \geq x, Y_\infty^{(N)} \geq y\right).$$

Exceeding the truncation level

$$
\begin{aligned}
\mathbb{III} &= \mathbb{E}\left[\sum_{n=1}^{T_{(0,0)}} \mathbb{1}\left(X_n \geq x, Y_n \geq y\right) \cdot \mathbb{1}\left(\max_{1 \leq l \leq T_{(0,0)}} X_l \geq N\right)\right] \\
&\leq \mathbb{E}\left[T_{(0,0)} \cdot \mathbb{1}\left(\max_{1 \leq l \leq T_{(0,0)}} X_l \geq N\right)\right] = \mathbb{E}\big[T_{(0,0)} \cdot \mathbb{1}\left(M^{T_{(0,0)}} \geq N\right)\big] \\
&= \mathbb{E}\big[T_{(0,0)} \bigm| M^{T_{(0,0)}} \geq N\big]\mathbb{P}\big(M^{T_{(0,0)}} \geq N\big).
\end{aligned}
$$

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$$= \mathbb{E}\big[T_{(0,0)} \mid M^{T_{(0,0)}} \geq N\big]\mathbb{P}\big(M^{T_{(0,0)}} \geq N\big).$$

Theorem: Upper and lower bounds for the approximation

$$0 \leq \mathbb{P}\big(X_\infty \geq x, Y_\infty \geq y\big) - \mathbb{P}\big(X_\infty^{(N)} \geq x, Y_\infty^{(N)} \geq y\big)$$
$$\leq \mathbb{E}\big[T_{(0,0)} \mid M^{T_{(0,0)}} \geq N\big] \frac{\mathbb{P}\big(M^{T_{(0,0)}} \geq N\big)}{\mathbb{E}\, T_{(0,0)}}.$$

Asymptotic upper bound

Main theorem

As $N \to \infty$,

$$\mathbb{P}\big(X_\infty \geq x, Y_\infty \geq y\big) - \mathbb{P}\big(X_\infty^{(N)} \geq x, Y_\infty^{(N)} \geq y\big) \lesssim KNe^{-\gamma N},$$

where

$$K = \left(\frac{1}{\mu_2 - \lambda \mathbb{E}B} \cdot \Big(\frac{(\breve{\mu}_1 - \mu_2)^+}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1} + \frac{(\mu_1 - \mu_2)^+}{\mu_1 - \lambda \mathbb{E}B}\Big) + \frac{1}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1} + \frac{1}{\mu_1 - \lambda \mathbb{E}B}\right) \times C_1 e^{\gamma}\left(1 - \frac{\lambda \mathbb{E}B}{\mu_1}\right),$$

and C_1 is a constant.

Proof

Step 1: Limit for the probability $\mathbb{P}\big(M^{T_{(0,0)}} \geq N\big)$

- $T_0 = \inf\{n \geq 1 : X_n = 0 \mid X_0 = 0\}$
- from extreme value theory:
$\max_{i=1,\dots,\frac{n}{\mathbb{E}T_{(0,0)}}} M_i^{T_{(0,0)}} \approx \max_{i=1,\dots,n} X_i \approx \max_{i=1,\dots,\frac{n}{\mathbb{E}T_0}} M_i^{T_0}$
- result:
$$\frac{\mathbb{P}\big(M^{T_{(0,0)}} \geq N\big)}{\mathbb{E}T_{(0,0)}} \sim \frac{\mathbb{P}\big(M^{T_0} \geq N\big)}{\mathbb{E}T_0}.$$

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Step 2: Limit for the probability $\mathbb{P}\left(M^{T_0} \geq N\right)$

- a *conspiracy* leads to a maximum value N
- an exponential change of measure gives $\breve{\lambda}$, $\breve{\mathbb{P}}(B = n)$, $\breve{\mu}_1$, and $\breve{\mu}_2$ (γ is the solution of the Lundberg equation).
- *Cramér-Lundberg approximation*: $e^{\gamma(N-1)}\mathbb{P}\left(M^{T_0} \geq N\right) \to C_1$

Proof (continued)

- ergodicity of X_n gives: $\mathbb{E}T_0 = 1/\mathbb{P}\big(X_\infty = 0\big)$
- *Little's formula*: $\mathbb{P}\big(X_\infty = 0\big) = 1 - \rho_1 = 1 - \lambda\mathbb{E}B/\mu_1$.

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Step 3: The conditional expectation $\mathbb{E}\left[T_{(0,0)} \,\middle|\, M^{T_{(0,0)}} \geq N \right]$

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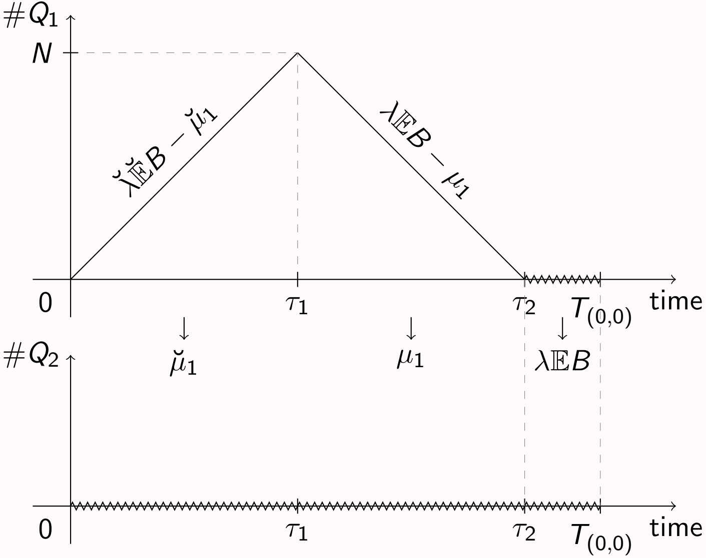
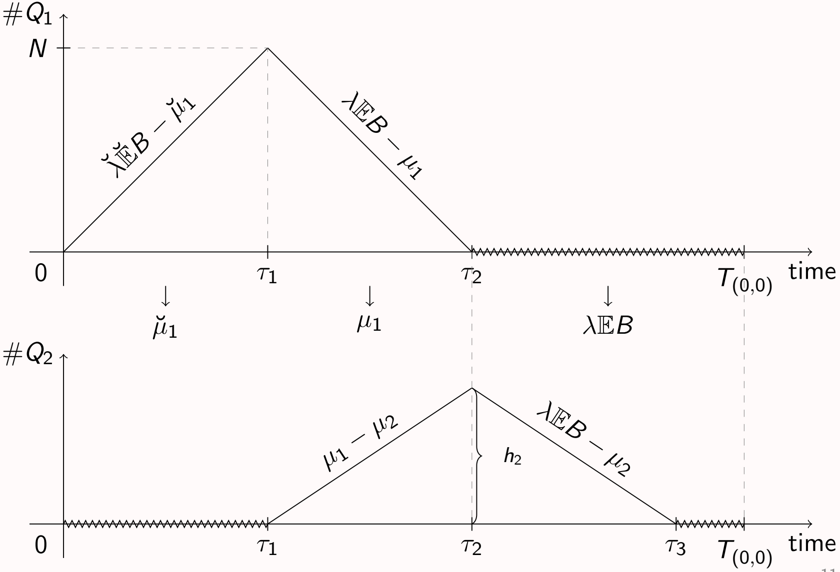


Proof (continued)



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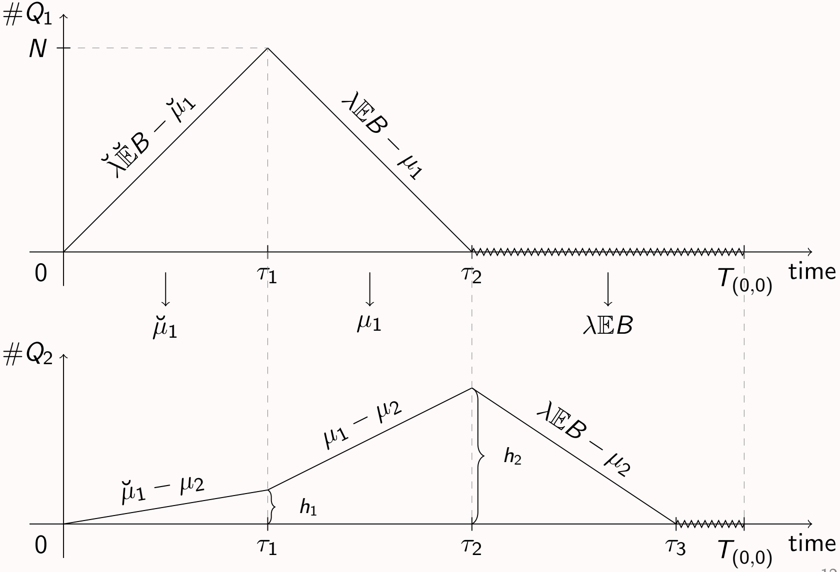


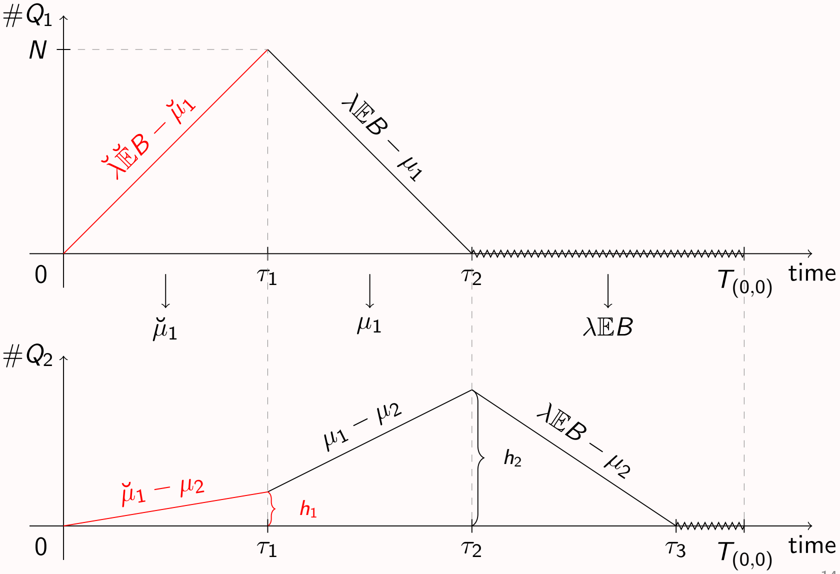
Proof (continued)

Distributions of the jumps/connection with random walks

$$Z_n = \begin{cases} 0, & \text{with probability } \mu_2, \\ -1, & \text{with probability } \mu_1, \\ m, & \text{with probability } \lambda p_m, m = 1, 2, \ldots, \end{cases}$$

$$W_n = \begin{cases} -1, & \text{if } Z_n = 0, \\ 1, & \text{if } Z_n = -1 \text{ and } X_{n-1} > 0, \\ 0, & \text{else.} \end{cases}$$

Behaviour in the time interval $[0, \tau_1]$



Behaviour in the time interval $[0, \tau_1]$

Proposition (for Q1)

As $N \to \infty$,

$$\mathbb{E}\left[\tau_1 \mid M^{T_{(0,0)}} \geq N\right] = \frac{1}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1}\big(N + o(N)\big).$$

Proof.

- Let z be s.t. $z > 1/\big(\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1\big)$. Then,

$$\mathbb{E}\left[\frac{\tau_1}{N}\middle|\tau_1 < T_{(0,0)}\right] = \int_0^z \mathbb{P}\Big(\tau_1 > yN \;\Big|\; \tau_1 < T_{(0,0)}\Big)dy + \int_z^\infty \mathbb{P}\Big(\tau_1 > yN \;\Big|\; \tau_1 < T_{(0,0)}\Big)dy.$$

- change of measure and use of $\lim\limits_{N\to\infty} \breve{\mathbb{E}}\left[\frac{\tau_1}{N}\right] = \dfrac{1}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1}$

□

Behaviour in the time interval $[0, \tau_1]$

Proposition (for Q2)

As $N \to \infty$,

$$\mathbb{E}\left[Y_{\tau_1} \mid M^{T_{(0,0)}} \geq N\right] \leq \frac{(\breve{\mu}_1 - \mu_2)^+}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1} N + o(N).$$

Proof.

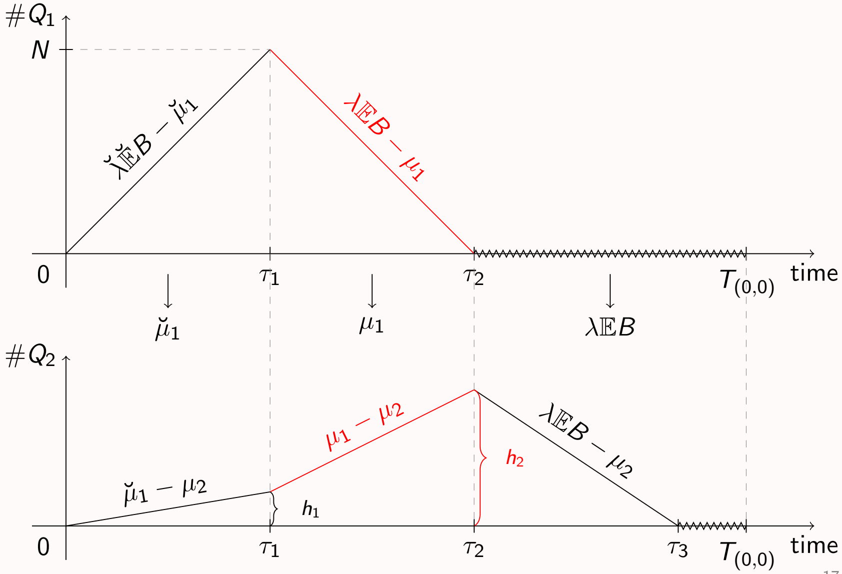
- kill dependence from X_n

$$W'_n = \begin{cases} -1, & \text{if } Z_n = 0, \\ K, & \text{if } Z_n = -1, \\ 0, & \text{else}, \end{cases}$$

- use properties of 2-dimensional random walks: $\frac{V'_{\tau(N)}}{N} \xrightarrow{\breve{\mathbb{P}}} \frac{\breve{\mathbb{E}}W'}{\breve{\mathbb{E}}Z}, \quad a.s. \quad N \to \infty$

□

Behaviour in the time interval $[\tau_1, \tau_2]$



Behaviour in the time interval $[\tau_1, \tau_2]$

Proposition (for Q1)

As $N \to \infty$,

$$\mathbb{E}\big[\tau_2 - \tau_1 \,\big|\, M^{T_{(0,0)}} \geq N\big] = \frac{1}{\mu_1 - \lambda \mathbb{E} B}\big(N + o(N)\big).$$

Proof.

definition of a recursive function and use of exponential change of measure □

Behaviour in the time interval $[\tau_1, \tau_2]$

Proposition (for Q2)

As $N \to \infty$,

$$\mathbb{E}\big[Y_{\tau_2} \mid M^{T_{(0,0)}} \geq N\big] = \left(\frac{(\breve{\mu}_1 - \mu_2)^+}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1} + \frac{(\breve{\mu}_1 - \mu_2)^+}{\mu_1 - \lambda \mathbb{E}B} \right) N + o(N).$$

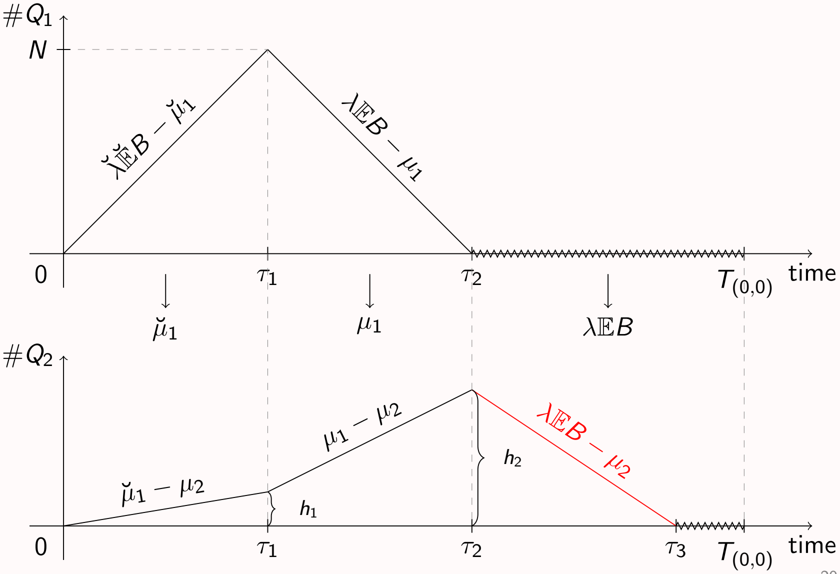
Proof.

- Q1 has always customers to feed Q2
- conditioning on $\{M^{T_{(0,0)}} \geq N\}$ and taking expectations

$$\mathbb{E}\big[Y_{\tau_2} \mid M^{T_{(0,0)}} \geq N\big] = \mathbb{E}\big[Y_{\tau_1} \mid M^{T_{(0,0)}} \geq N\big] + \underbrace{\mathbb{E}\left[\sum_{n=\tau_1+1}^{\tau_2} W_n' \,\middle|\, M^{T_{(0,0)}} \geq N \right]}_{Wald's\ \ equation}$$

□

Behaviour in the time interval $[\tau_2, \tau_3]$



Behaviour in the time interval $[\tau_2, \tau_3]$

Proposition (for Q2)

As $N \to \infty$,

$$\mathbb{E}\left[\tau_3 - \tau_2 \;\middle|\; M^{T_{(0,0)}} \geq N\right] = \frac{1}{\mu_2 - \lambda\mathbb{E}B} \cdot \left(\frac{(\breve{\mu}_1 - \mu_2)^+}{\breve{\lambda}\breve{\mathbb{E}}B - \breve{\mu}_1} + \frac{(\mu_1 - \mu_2)^+}{\mu_1 - \lambda\mathbb{E}B} \right) N + o(N).$$

Proof.

- $W_{n+1} = Y_{n+1} - Y_n$ conditionally independent given $(Z_i)_{i\geq 0}$
- $(X_n, S_n)_{n\geq 0}$, with $S_n = -\sum_{i=1}^n W_i$ and $X_0 = S_0 = 0$, is a Markov Additive Process (MAP)/ (*Markov Random Walk* (MRW)) and satisfies

$$\mathbb{P}\left(X_{n+1} \in A, S_{n+1} - S_n \in B \;\middle|\; X_n, W_n\right) = \mathbb{P}\left(X_n, A \times B\right).$$

- *Markov Renewal Theorem* for MAP

Behaviour in the time interval $[\tau_2, \tau_3]$

Proposition (for Q1)

$N \to \infty$,

$$\mathbb{E}\left[X_{\tau_3} \mid M^{T_{(0,0)}} \geq N\right] = \frac{\lambda \mathbb{E} B + \mu_2}{2(\mu_2 - \lambda \mathbb{E} B)} \big(1 + o(1)\big).$$

Proof.

- define the margingale
  $A_n = \sum_{i=1}^{n} Z_i \mathbb{1}\left(Z_i > 0\right) - \sum_{i=1}^{n} \mathbb{1}\left(Z_i = 0\right) - (\lambda \mathbb{E} B - \mu_2) n$
- use *Doob's optional sampling theorem*
- and *Wald's equation for Markov random walks*

□

Numerical example - Special case

- Geometric distribution for the batch sizes:
  $\mathbb{P}(B = n) = \beta(1-\beta)^{n-1}$, $n = 1, 2, \ldots$.
- $\gamma = -\ln\big((\lambda + \mu_1 - \beta\mu_1)/\mu_1\big)$

$$\begin{aligned} a.u.e.b. = N\Bigg(& \frac{\beta}{\beta\mu_2 - \lambda} \cdot \bigg(\lambda\Big(1 - \frac{\mu_2}{\lambda + \mu_1 - \beta\mu_1}\Big)^+ \\ & + \beta(\mu_1 - \mu_2)^+ \bigg) + \beta \\ & + \frac{\lambda}{(\lambda + \mu_1 - \beta\mu_1)} \Bigg) \Big(\frac{\lambda + \mu_1 - \beta\mu_1}{\mu_1} \Big)^{N-1} \rho_1(1 - \rho_1). \end{aligned}$$

Numerical example - Special case

Focus: on the marginal distribution of Q2

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Focus: on the marginal distribution of Q2

Parameter choise: $\{\beta = 0.5, \rho_1 = 0.7, \rho_2 = 0.8\}$

y	$N = 10$	$N = 20$	$N = 30$	$N = 40$	$N = 50$
5	0.128921	0.025536	0.005539	0.001551	0.000755
10	0.123171	0.029763	0.006556	0.001551	0.000517
15	0.086761	0.026535	0.006317	0.001419	0.000349
20	0.054454	0.020534	0.005432	0.001229	0.000237
25	0.032516	0.014616	0.004358	0.001069	0.000221
30	0.018948	0.009835	0.003276	0.000874	0.000195
a.u.e.b.	0.617191	0.243018	0.071766	0.018839	0.004636

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Conclusions

- The asymptotic error bound depends only on N and the model parameters; i.e. it is uniform in the values of x and y.
- The bound is conservative.
- The bound becomes more conservative as N increases.
- The undesirable behaviour of the bound is mostly attributed to N.
- Simply expression that converges to zero.

Thank you for your attention