Monotonicity, Convexity and Comparability of Some Functions Associated with Block-Monotone Markov Chains and Their Applications to Queueing Systems

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Outline

1. **Introduction**
2. **Block-Monotonicity of Probability Vectors and Stochastic Matrices**
3. **Block-Monotone Markov Chains**
4. **Application to the *GI/Geo/1* Queue**
5. **Numerical Examples**
6. **Conclusion**

1. Introduction

- **Queueing Systems & Univariate Markov chains**
  - Research in this area started in 1950's
  - Classical queues: *M/M/1*, *M/G/*1, *GI/M/1*, *GI/Geo/*1,…
- **Imbeeded Markov chain:** (e.g., Kendall 1951,1953), Hunter (1983), Tian and Zhang (2002))
- **Stochastic comparison methods:** (e.g.,Stoyan (1983))
- **Monotonicity and convexity of functions** *w.r.t.* **univariate Markov chains**(Yu, He and Zhang (2006))
- **Question**: Whether above results may be extended to **bivariate Markov chains?**

1. Introduction (continued)

- **Bivariate Markov chains with block-structured transition matrices (e.g., QBD)**
  - Research in this area started in 1980's.
  - Continuous-time queues: *GI/PH/1*, *PH/G/1*,*MAP/PH/1*,…
  - Discrete-time queues: *GI/Geo/1*, *GI/G/1*,…
  - Methods: Matrix-analytic method (MAM) vs. block-monotone Markov chain
- **MAM:** (Neuts (1981,1989), Gibson and Seneta (1987), Zhao, Li and Alfa (2000);
  Tweedie(1998), Liu (2010), He(2014), Alfa (2016), etc.)

1. Introduction (continued)

- **Block-monotone Markov chain approach**
  - First introduced the stochastic vectors based on F-orderings (Li and Shaked (1994))
  - Defined the block-increasing order, and proved the stationary distributions of its truncations converge to that of the original Markov chain with monotone transition matrix (Li and Zhao (2000))
  - Provided error bounds for augmented truncations of discrete-time block-monotone Markov chains under geometric drift conditions (Masuyama (2015)).
  - Continuous-time block-monotone Markov chain(Masuyama (2016)).

2. Block-Monotonicity of Probability Vectors and Stochastic Matrices

- **Definition:Block Stochastic orders**
  - $\boldsymbol{a}=(\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots)$, $\boldsymbol{b}=(\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots)$, where $\boldsymbol{a}_n=(a_n(1), a_n(2), \ldots, a_n(m))$, $\boldsymbol{b}_n=(b_n(1), b_n(2), \ldots, b_n(m))$.

    $\boldsymbol{a} \le_{m-s-st} \boldsymbol{b}$, if $\boldsymbol{a}\,\mathrm{E}_m^s \le_{el} \boldsymbol{b}\,\mathrm{E}_m^s$, for $s = 1, 2$

  where

$$E_m = \begin{pmatrix} I_m & O & O & O & \cdots \\ I_m & I_m & O & O & \cdots \\ I_m & I_m & I_m & O & \cdots \\ I_m & I_m & I_m & I_m & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad E_m^2 = \begin{pmatrix} I_m & O & O & O & \cdots \\ 2I_m & I_m & O & O & \cdots \\ 3I_m & 2I_m & I_m & O & \cdots \\ 4I_m & 3I_m & 2I_m & I_m & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{1}$$

Note: For $s = 1$, Li and Zhao (2000) defined the order $\le_{m-1-st}$

2. Block-Monotonicity of Probability Vectors and Stochastic Matrices(continued)

Example 1.

1) $\boldsymbol{a}_0$ =(0.3,0.2), $\boldsymbol{a}_1$ =(0.3,0.2),

$\boldsymbol{b}_0$ =(0.2,0.2), $\boldsymbol{b}_1$ =(0.4,0.2), then $\boldsymbol{a} \leq_{m-1-st} \boldsymbol{b}$.

2) $\boldsymbol{a}=(\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots)$, $\boldsymbol{b}=(\boldsymbol{a}_0+\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots)$, then $\boldsymbol{a} \leq_{m-1-st} \boldsymbol{b}$

Example 2.

$\boldsymbol{a}_0$ =(0.25, 0.25), $\boldsymbol{a}_1$ =(0.15, 0.15), $\boldsymbol{a}_2$ =(0.1, 0.1),

$\boldsymbol{b}_0$ =(0.21, 0.21), $\boldsymbol{b}_1$ =(0.17, 0.17), $\boldsymbol{b}_3$ =(0.12, 0.12),

then $\boldsymbol{a} \leq_{m-2-st} \boldsymbol{b}$.

2. Block-Monotonicity of Probability Vectors and Stochastic Matrices(continued)

- **Definition: Block-Monotone Stochastic Matrices**

Let $\boldsymbol{P}= (A_{k,l})$, $A_{k,l} = (A_{k,l}(i,j))$, $i,j=1,2, \ldots,m$ $k,l=0,1,2, \ldots$

$$\mathrm{P} \in M_{m-s-st} \text{, if } \mathrm{E}_m^{-1}\mathrm{P}\mathrm{E}_m^s \geq_{el} \mathrm{O} \text{ for } s = 1, 2.$$

- For n = 1, Definition of $\mathrm{P} \in M_{m-1-st}$ (Masuyama (2015))
- For n = 1, $\mathrm{P} \in M_{m-1-st} \Leftrightarrow \boldsymbol{a} \leq_{m-1-st} \boldsymbol{b}$ implies $\boldsymbol{a}\boldsymbol{P} \leq_{m-1-st} \boldsymbol{b}\boldsymbol{P}$ (Li and Zhao (2000))

2. Block-Monotonicity of Probability Vectors and Stochastic Matrices(continued)

- **Example 3 :QBD**

$$P = \begin{pmatrix} B & C & O & O & \cdots \\ F & A_1 & A_0 & O & \cdots \\ O & A_2 & A_1 & A_0 & \cdots \\ O & O & A_2 & A_1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

i)If $F \le_{el} B$,$F \le_{el} A_2$, and $B + C \le_{el} F + A_1 + A_0$,

then $\mathrm{P} \in M_{m-1-st}$

ii)If $F \le_{el} B + A_0$, $F \le_{el} 2A_2 + A_1 + A_0$,and

$B + 2C \le_{el} F + 2A_1 + 3A_0$,then $\mathrm{P} \in M_{m-2-st}$

- **Example 4. QBD** for *GI/Geo/1* queue (Alfa 2016)

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_m \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$F = A_2 = \mu B$$

$$A_1 = \overline{\mu} B + \mu C$$

$$A_0 = \overline{\mu} C$$

Then $\mathrm{P} \in M_{m-1-st}$, $\mathrm{P} \in M_{m-2-st}$.

2. Block-Monotonicity of Probability Vectors and Stochastic Matrices(continued)

- **Property 1:** Suppose that $P \in M_{m-s-st}$, s=1,2, then

i. $a \leq_{m-s-st} b \Rightarrow aP^n \leq_{m-s-st} bP^n$ for n=1,2, …..

ii. $a \leq_{m-s-st} aP \Rightarrow aP^n \leq_{m-s-st} aP^{n+1}$ for n=1,2, ….

$aP \leq_{m-s-st} a \Rightarrow aP^{n+1} \leq_{m-s-st} aP^n$ for n=1,2, ….

3. Block-Monotone Markov Chains (continued)

- **Theorem 1:** For a DTMC $\boldsymbol{Z}=\{Z_n, n \in \mathbb{N}_0\}$ with transition matrix *P* having block size *m* and initial distribution v.
Assume that $\mathsf{E}_v[f(Z_n)] = vP^n f^T < \infty$
  1) If ***Condition I$_{m\text{-}s\text{-}st}$*** **,** ***II$_{m\text{-}s\text{-}st}$*** and ***III$_{m\text{-}s\text{-}st}$*** in Eq.(2) hold, then $\mathsf{E}_v[f(Z_n)]$ is increasing concave in *n* for *n=0,1,2,…* and *s=1*,2.
  2) If ***Condition I$_{m\text{-}s\text{-}st}$*** ***,II'$_{m\text{-}s\text{-}st}$*** and ***III$_{m\text{-}s\text{-}st}$*** in Eq.(2) hold , then $\mathsf{E}_v[f(Z_n)]$ is decreasing convex in *n* for *n=0,1,2,…* and *s=1*,2.

3. Block-Monotone Markov Chains(continued)

For $s = 1, 2,$

***Condition I$_{m\text{-}s\text{-}st}$*:** $\quad \mathrm{P} \in M_{m-s-st}$

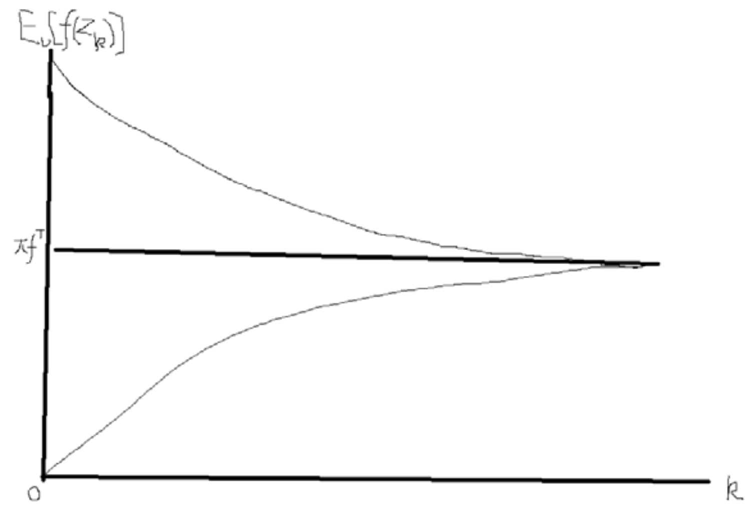
***Condition II$_{m\text{-}s\text{-}st}$*:** $\quad v \leq_{m-s-st} v\mathrm{P}$

***Condition II'$_{m\text{-}s\text{-}st}$*:** $\quad v\mathrm{P} \leq_{m-s-st} v$ (2)

***Condition III$_{m\text{-}1\text{-}st}$*:** $\mathrm{h}_{\mathrm{f}}^{T}$ is decreasing *w.r.t.* orders $\leq_{m-s-st}$

where

$$\mathrm{h}_{\mathrm{f}}^{T} = \mathrm{Pf}^{T} - \mathrm{f}^{T}$$



3. Block-Monotone Markov Chains (continued)

- **Theorem 2:** For two DTMCs $Z=\{Z_n, n \in \mathbb{N}_0\}$ and $\tilde{Z}=\{\tilde{Z}_n, n \in \mathbb{N}_0\}$ with the same state space and initial distribution ν, If their transition matrices P and $\tilde{P}$ *satisfy:* $P \leq_{m-s-st} \tilde{P}$, and either $P \in M_{m-s-st}$ or $\tilde{P} \in M_{m-s-st}$, then $E_\nu[f(Z_n)] \leq E_\nu[f(\tilde{Z}_n)]$ for $n=0,1,2,\ldots$, and $s=1,2$.

- **Proof based on Property 2:**
- **Property 2:** Suppose $P \leq_{m-s-st} \tilde{P}$ (i.e.,$PE_m^s \leq_{m-s-st} \tilde{P}E_m^s$) and either $P \in M_{m-s-st}$ or $P \in M_{m-s-st}$, then $P^n \leq_{m-s-st} \tilde{P}^n$

  Note: Li and Zhao (2000) gave result in property 2 for *s*=1

4. Application to the *GI/Geo/1* Queue

- There is a single server. Service times follow the geometric distribution with parameter μ , $0 < \mu < 1$
- Service discipline: first-come-first-served (FCFS).
- Inter-arrival times are follow a general distributions, $\mathbf{g}=(g_1, g_2, \ldots, g_m)$, $m<\infty$, and with DPH representation (β, B) of order m, where $\beta=(g_1, g_2, \ldots, g_m)$, B is given in Eq. (4).

4. Application to the *GI/Geo/1* Queue (continued)

- DTMC **Z**=$\{(I_n, J_n), n=0,1,2,\ldots\}$
- I_n : the number of customers in the system at time n
- J_n: the remaining inter-arrival time at time n
- The transition matrix P given by

$$P = \begin{pmatrix} B & C & O & O & \cdots \\ A_2 & A_1 & A_0 & O & \cdots \\ O & A_2 & A_1 & A_0 & \cdots \\ O & O & A_2 & A_1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \qquad (3)$$

4. Application to the *GI/Geo/1* Queue (continued)

where

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad (4)$$

$$C = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_m \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (5)$$

$$A_2 = \mu B, \qquad A_1 = \bar{\mu} B + \mu C, \qquad A_0 = \bar{\mu} C \quad (6)$$

4. Application to the *GI/Geo/1* Queue (continued)

For the *GI/Geo/1* Queue

- **Corollary 1:**

  1)If ***Condition I**$_{m\text{-}1\text{-}st}$* in Eq.(7), ***II**$_{m\text{-}1\text{-}st}$* in Eq.(8) and ***II**$_{m\text{-}1\text{-}st}$* in Eq.(9) hold, then $\mathsf{E}_v[\mathrm{f}(Z_n)]$ is increasing concave in *n* .

  2)If Conditions ***I**$_{m\text{-}1\text{-}st}$* in Eq.(7), ***II'**$_{m\text{-}1\text{-}st}$* in Eq.(8') and ***II**$_{m\text{-}1\text{-}st}$* in Eq.(9) hold, then $\mathsf{E}_v[\mathrm{f}(Z_n)]$ is decreasing convex in *n*.

4. Application to the *GI/Geo/1* Queue (continued)

***Condition I$_{m\text{-}1\text{-}st}$*:** $0 < \mu < 1$ (7)

***Condition II$_{m\text{-}1\text{-}st}$*:** For j=1,2,…, m-1,

$$\bar{g}_j \sum_{i=0}^{k} v_i(j) + \bar{\mu} g_j v_k(j) - \left(\sum_{i=0}^{k} v_i(j+1) + \mu v_{k+1}(j+1) \right) \geq 0$$

$$v_0(j) - v_0(j+1) - \mu v_0(j+1) \geq 0 \quad (8)$$

$$\sum_{i=0}^{k} v_i(m) - g_m \left(\sum_{i=0}^{k-1} v_i(m) + \mu v_k(m) \right) \geq 0$$

***Condition II'$_{m\text{-}1\text{-}st}$*:** For j=1,2,…, m-1,

$$\bar{g}_j \sum_{i=0}^{k} v_i(j) + \bar{\mu} g_j v_k(j) - \left(\sum_{i=0}^{k} v_i(j+1) + \mu v_{k+1}(j+1) \right) \leq 0$$

$$v_0(j) - v_0(j+1) - \mu v_0(j+1) \leq 0 \quad (8')$$

$$\sum_{i=0}^{k} v_i(m) - g_m \left(\sum_{i=0}^{k-1} v_i(m) + \mu v_k(m) \right) \leq 0$$

4. Application to the *GI/Geo/1* Queue (continued)

***Condition III**$_{m\text{-}1\text{-}st}$*:

$$\Delta f_{k+1}(1) \geq \mu \sum_{j=1}^{m} g_j [\Delta f_{k+1}(j) - \Delta f_{k+2}(j)] + \sum_{j=1}^{m} g_j \Delta f_{k+2}(j) \quad \text{for } k=1,2,\ldots$$

$$\Delta f_{k+1}(j+1) - \Delta f_{k+1}(j) \geq \mu [\Delta f_k(j) - \Delta f_{k+1}(j)] \qquad (9)$$

$$\Delta f_1(1) \geq \overline{\mu} \sum_{j=1}^{m} g_j \Delta f_1(j)$$

for k=1,2,…,j=1,2,…, m-1.

$$\Delta f_1(j+1) \geq \overline{\mu} \Delta f_1(j)$$

for j=1,2,…m-1.

- For example, taking , $f_n(j)=n$ for n=0,1,2,…, j=1,2,…, m. then the function $\boldsymbol{f}=(\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_2, \ldots)$ satisfies *Condition III*$_{m\text{-}1\text{-}st}$

4. Application to the *GI/Geo/1* Queue (continued)

- **Corollary 2:** For two *GI(n)/Geo(n)/1* queues

  -- service rates μ_n and $\tilde{\mu}_n$

  -- inter-arrival times $g_n = (g_n(1), g_n(m), ..., g_n(m))$

  $\tilde{g}_n = (\tilde{g}_n(1), \tilde{g}_n(m), ..., \tilde{g}_n(m))$

  Suppose that $\mu_{n+1} \le g_n(j) / g_{n+1}(j)$, $\mu_n \ge \tilde{\mu}_n$

  and $g_n(j) \le \tilde{g}_n(j))$ for *n*=0,1,2,…, *j*=1,2,…, *m*,

  then $\mathsf{E}_v[\mathrm{f}(Z_n)] \le \mathsf{E}_v[\mathrm{f}(\tilde{Z}_n)]$ for all *n*=0,1,2,…,

5. Numerical Examples

- **Consider the *GI/Geo/1* Queue system with** $\boldsymbol{g} = (0.5, 0.5)$, $\lambda = 2/3$, $\boldsymbol{v} = (v_0, 0, 0, \ldots)$, $v_0 = (3/4, 1/4)$, $f_n(j) = n$, $\boldsymbol{\mu = 0.8}$,

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0.8 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.4 & 0.4 \\ 0.2 & 0 \end{pmatrix}, A_0 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}.$$

- **Consider another *GI/Geo/1* Queue system with** $\boldsymbol{g} = (0.5, 0.5)$, $\lambda = 2/3$, $\boldsymbol{v} = (v_0, 0, 0, \ldots)$, $v_0 = (3/4, 1/4)$, $f_n(j) = n$, $\boldsymbol{\mu' = 0.75}$.

n	0	1	2	3	4	5	6
$E_v[f(Z_n)]$	0	0.7500	0.7750	0.7950	0.8150	0.8350	0.8550
$E_v[f(Z_n^{'})]$	0	0.7500	0.8125	0.8325	0.8525	0.8725	0.8925

6. Conclusion

- Sufficient conditions to assure the monotonicity and convexity of function for block-monotone Markov chain can be obtained.
- Our approach can be used to analyze those complex queueing systems, e.g.,
  - *GI/G/1 queue*, (Alfa 2016 Section 5.12)
  - *GI(n)/G(n)/1 queue*
  - *GI/Geo/1 queue with server vacations*

Thank you very much!