Exact Stationary Tail Asymptotics for a Markov Modulated Two-Demand Model — In Terms of a Kernel Method

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at MAM9, June 28–30, 2016

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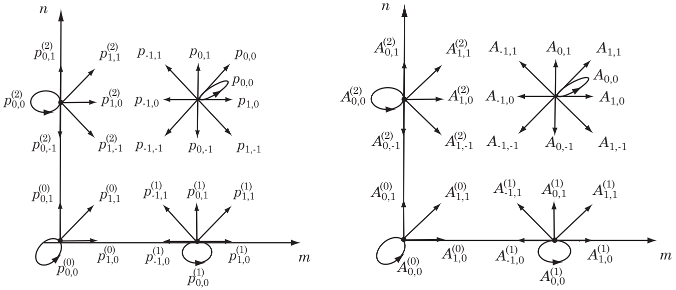
(Based on joint work with Y. Liu and P. Wang)

Outline

1. Model: from scalar to block
2. Kernel Method
3. Methods for tail
4. RW-Block Case
5. Exasmple

Transition diagrams for (scalar) RW and MMRW in QP

Transition diagrams of a (usual) random walk in the quarter plane, and its generalization (two-dimensional QBD process)



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As two-dimensional QBD

If m as level and n as background or phase, then the transition matrix P is given by:

$$P = \begin{pmatrix} B_0 & B_1 & & & \\ A_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$B_i = \begin{pmatrix} A_{i,0}^{(0)} & A_{i,1}^{(0)} & & & \\ A_{i,-1}^{(2)} & A_{i,0}^{(2)} & A_{i,1}^{(2)} & & \\ & A_{i,-1}^{(2)} & A_{i,0}^{(2)} & A_{i,1}^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$A_i = \begin{pmatrix} A_{i,0}^{(1)} & A_{i,1}^{(1)} & & & \\ A_{i,-1} & A_{i,0} & A_{i,1} & & \\ & A_{i,-1} & A_{i,0} & A_{i,1} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Exact tail asymptotics

- $\pi_{m,n;k}$ ($m, n = 0, 1, \ldots$, and $k = 1, 2, \ldots M$): Stationary distribution under a stability condition
- Exact tail asymptotic along m-direction: for fixed n and k, looking for a function $f(m)$ such that $\pi_{m,n;k}$ and $f(m)$ have the same exact tail asymptotic property, or

$$\lim_{m\to\infty} \pi_{m,n;k}/f(m) = 1, \quad \text{denoted by} \quad \pi_{m,n;k} \sim f(m)$$

- Exact tail asymptotic along n-direction: for fixed m and k, looking for a function $g(n)$ such that $\pi_{m,n;k}$ and $g(n)$ have the same exact tail asymptotic property, or

$$\lim_{n\to\infty} \pi_{m,n;k}/g(n) = 1, \quad \text{denoted by} \quad \pi_{m,n;k} \sim g(n)$$

KM: A bit of history:

- In combinatorics, first introduced by Knuth (1969) and later developed as the kernel method by Banderier *et al.* (2002)
- Fundamental form:

$$K(x,y)F(x,y) = A(x,y)G(x) + B(x,y)$$

  where $F(x,y)$ and $G(x)$ are unknown functions.
- Key idea in the kernel method: to find a branch $y = y_0(x)$, such that $K(x, y_0(x)) = 0$. When analytically substituting this branch into RHS, we then have $G(x) = -B(x, y_0(x))/A(x, y_0(x))$, and hence,

$$F(x,y) = \frac{-A(x,y)B(x,y_0(x))/A(x,y_0(x)) + B(x,y)}{K(x,y)}$$

KM: for RW (scalar)

- Unknown GFs:

$$\pi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \pi_{m,n} x^{m-1} y^{n-1},$$

$$\pi_1(x) = \sum_{m=1}^{\infty} \pi_{m,0} x^{m-1}, \quad \pi_2(y) = \sum_{n=1}^{\infty} \pi_{0,n} y^{n-1}.$$

- Fundamental form:

$$-h(x,y)\pi(x,y) = h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}$$

  Instead of one, we have two unknown functions $\pi_1(x)$ and $\pi_2(y)$ on RHS.

- When we consider a branch $Y = Y_0(x)$, such that $h(x, Y_0(x)) = 0$, analytically substituting this branch into RHS only leads to a relationship between the two unknown functions.

Determination of unknown functions

- Brute force method (e.g., Jackson networks)
- Boundary value problems (e.g., 2 by 2 switches; symmetric JSQ)
- Uniformization method (e.g., 2 by 2 swithches; 2-demand model; JSQ)
- Algebraic approach (e.g., 2-demand model)

In general, the determination of the unknown function is expressed in terms of a singular integral, based on which tail asymptotic properties in probabilities could be studied.

Tail asymptotics

Advantage: Without a determination of the unknown function. Instead, we only need: (1) location and (2) its detailed property of the dominant singularity.

- Kernel equation: $h = 0$, leading to branch point x_3, a candidate of the dominant singularity (decay rate $1/x_3$), and branches $Y_0(x)$ and $Y_1(x)$)
- Interlace of two unknown functions $\pi_1(x)$ and $\pi_2(y)$, leading to analytic continuation of unknown functions (dominant singularity and its asymptotic property
- Tauberian-like theorem (relationship between asymptotic property of a function and asymptotic property of its coefficients, or probabilities)

Four types of tail asymptotics

For non-singular genus one RW, if it is not X-shaped, then one of the following holds:

- Exact geometric:

$$\pi_{n,j} \sim c\theta^n$$

- Geometric with subgeometric factor $n^{-3/2}$:

$$\pi_{n,j} \sim cn^{-3/2}\theta^n$$

- Geometric with subgeometric factor $n^{-3/2}$:

$$\pi_{n,j} \sim cn^{-1/2}\theta^n$$

- Geometric with subgeometric factor n:

$$\pi_{n,j} \sim cn\theta^n$$

Methods for tail asymptotics

- Analytic and algebraic: Generating function methods: Malyshev 1972, 1973; Flatto and McKean 1977; Fayolle and Iasnogorodski 1979; Fayolle, King and Mitrani 1982; Cohen and Boxma 1983; Flatto and Hahn 1984; Flatto 1985; Fayolle, Iasnogorodski and Malyshev 1991; Wright 1992; Kurkova and Suhov 2003; Leeuwaarden 2005; Morrison: 2007; Guillemin and Leeuwaarden 2009; Miyazawa and Rolski; Li and Zhao 2010
- Large deviations (LD): Borovkov and Mogul'skii (2001)
- Markov additive processes (MAP) and LD: McDonald 1999; Foley and McDonald 2001, 2005; Khachi 2008, 2009; Adan, Foley and McDonald (2009)
- Matrix analytic methods (MAP and mtraix): Takahashi, Fujimoto and Makimoto 2001; Haque 2003; Miyazawa 2004; Miyazawa and Zhao 2004; Kroese, Scheinhardt and Taylor 2004; Haque, Liu and Zhao 2005; Motyer and Taylor 2006; Li, Miyazawa and Zhao 2007; He, Li and Zhao 2008
- Non-linear optimization (N-LP) (MAP and N-LP): Miyazawa 2007, 2008, 2009; Kobayashi and Miyazawa 2010
- Kernel methods (analytic combinatorics and asymptotic analysis): Bousquet-Melou 2005; Mishna 2006; Hou and Mansour 2008; Flajolet and Sedgewick 2009

KM: for RW (block)

- Fundamental form:

$$-\Pi(x,y)H(x,y) = \Pi_1(x)H_1(x,y)+\Pi_2(y)H_2(x,y)+\Pi_0 H_0(x,y)$$

- All H, H_1, H_2 and H_0 are given matrices, for example, $H(x,y) = xy\left(I - \sum_{i=-1}^{1}\sum_{j=-1}^{1} x^i y^j A_{ij}\right)$
- $\Pi(x,y)$, $\Pi_1(x)$ and $\Pi_2(y)$ are unknown vector functions, for example, $\Pi_1(x) = \left(\sum_{i=1}^{\infty} \pi_{i,0;1}x^{i-1}, \sum_{i=1}^{\infty} \pi_{i,0;2}x^{i-1}, \ldots, \sum_{i=1}^{\infty} \pi_{i,0;M}x^{i-1}\right)_{1\times M}$

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Challenges from scalar from block

1. **Kernel equation:** $\Pi(x,y)H(x,y) = 0$
   - For scalar case,

     $$-h(x,y)\pi(x,y) = h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}$$

     There exit enough (x,y) such that $h(x,y) = 0$
   - For block case,

     $$-\Pi(x,y)H(x,y) = \Pi_1(x)H_1(x,y) + \Pi_2(y)H_2(x,y) + \Pi_0 H_0(x,y)$$

     We need to show that there exist enough (x,y) such that $\Pi(x,y)H(x,y) = 0$.
   - This is not immediate. For specific simple examples (incl MM 2-demand model), a direct method may prevail, but for a general case, we need a different treatment (for example, based on analytic continuation to construct analytic functions that satisfy the FF, and then use the uniqueness theorem)

2. Factorization of $\det H(x, y) = 0$

- $\det H(x, y) = 0$ for (x, y) such that $\Pi(x, y) \neq 0$.
- Factorization:

$$\begin{aligned}\det H(x, y) =& [a(x)y^2 + b(x)y + c(x)]q(x, y) \\ =& [\tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y)]q(x, y) = 0,\end{aligned}$$

- Proof based on properties of:
  (1) Perron-Frobenius eigenvalue of

$$C(x, y) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} x^i y^j A_{i,j}$$

  (2) Convex property of $\bar{\Gamma} = \{(s_1, s_2) \in \mathbb{R}^2 : \chi(e^{s_1}, e^{s_2}) \leq 1\}$;
  (3) Polynomial $\det H(x, y) = 0$.

3. Analytic continuation of $\Pi_1(x)$

- Based on

$$\Pi_1(x)H_1(x, Y_0(x)) = -[\Pi_2(Y_0(x))H_2(x, Y_0(x)) + \Pi_0 H_0(x, Y_0(x))]$$

  the dominant singularity of $\Pi_1(x)$ is either the branch point x_3, or a zero of $\det H_1(x, Y_0(x)) = 0$ or the dominant singularity of $\Pi_2(Y_0(x))$.

- Interlace between $\Pi_1(x)$ and $\Pi_2(y)$ leads to that the dominant singularity of $\Pi_1(x)$ is either the branch point x_3, or a zero of $\det H_1(x, Y_0(x)) = 0$, or $\tilde{x}_1$ such that $Y_0(\tilde{x}_1)$ is a zero of $\det H_2(X_0(y), y) = 0$.

4. Asymptotic properties of $\Pi_1(x)$

- $\det H_1(x, y) = 0$ can be factored as

$$\begin{aligned}\det H_1(x, y) =& [b_1(x)y + c_1(x)]q_1(x, y)\\ =& [\tilde{a}_1(y)x^2 + \tilde{b}_1(y)x + \tilde{c}_1(y)]q_1(x, y),\end{aligned}$$

or $h_1(x, y) = b_1(x)y + c_1(x) = \tilde{a}_1(y)x^2 + \tilde{b}_1(y)x + \tilde{c}_1(y)$ is a polynomial of degree one in y and degree two in x.

- Similarly,

$$\begin{aligned}\det H_2(x, y) =& [a_2(x)y^2 + b_2(x)y + c_2(x)]q_2(x, y)\\ =& [\tilde{b}_2(y)x + \tilde{c}_2(y)]q_2(x, y),\end{aligned}$$

or $h_2(x, y) = a_2(x)y^2 + b_2(x)y + c_2(x) = \tilde{b}_2(y)x + \tilde{c}_2(y)$ is a polynomial of degree one in x and degree two in y.

Convert to scalar case

- Consider $h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0} = 0$. We want to claim that $\pi_1(x)$ has the same asymptotic property as that of a component of $\Pi_1(x)$, and $\pi_2(y)$ has the same asymptotic property as that of a component of $\Pi_2(y)$.
- We finally claim that the tail asymptotic problem for the block fundamental form:

$$-\Pi(x,y)H(x,y) = \Pi_1(x)H_1(x,y) + \Pi_2(y)H_2(x,y) + \Pi_0 H_0(x,y)$$

can be solved through asymptotic problem of the scalar fundamental form:

$$-h(x,y)\pi(x,y) = h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}$$

MM two-demand model

- Arrival rate is λ_k when the modulating MC is in state k. For example, for two-state MC (state 0 and state 1), its transition matrix is given by

$$J = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} p & \bar{p} \\ \bar{q} & q \end{bmatrix} \end{array},$$

where $\bar{a} = 1 - a$, and $0 < p, q < 1$ to avoid triviality.

Factorization

$$H(x,y) = \begin{bmatrix} xy(1-\lambda_1) - pg_0(x,y) & -\bar{p}g_0(x,y) \\ -\bar{q}g_1(x,y) & xy(1-\lambda_0) - qg_1(x,y) \end{bmatrix},$$

where

$$g_k(x,y) = x^2y^2\lambda_k + x\mu_2 + y\mu_1$$

For simplicity, assume $p = q = 1/2$, which leads to

$$\det H(x,y) = -\frac{x^2y^2}{2}h(x,y),$$

where

$$\begin{aligned} h(x,y) =& [\lambda_0(1-\lambda_0) + \lambda_1(1-\lambda_1)]x^2y^2 - 2(1-\lambda_0)(1-\lambda_1)xy \\ &+ [(1-\lambda_0) + (1-\lambda_1)](\mu_2 x + \mu_1 y). \end{aligned}$$

$$\det H_1(x,y) = \left(-\frac{x}{2}\right) h_1(x,y),$$

where

$$\begin{aligned} h_1(x,y) =& \big[(\lambda_0+\mu_1)\lambda_1 + (\lambda_1+\mu_1)\lambda_0\big] yx^2 - 2(\lambda_0+\mu_1)(\lambda_1+\mu_1)x \\ &+ \big[(\lambda_0+\mu_1) + (\lambda_1+\mu_1)\big]\mu_1. \end{aligned}$$

$$\det H_2(x,y) = \left(-\frac{y}{2}\right) h_2(x,y),$$

where

$$\begin{aligned} h_2(x,y) =& \big[(\lambda_0+\mu_2)\lambda_1 + (\lambda_1+\mu_2)\lambda_0\big] xy^2 - 2(\lambda_0+\mu_2)(\lambda_1+\mu_2)y \\ &+ \big[(\lambda_0+\mu_2) + (\lambda_1+\mu_2)\big]\mu_2. \end{aligned}$$

Dominant singularity

Recall

$$a(x) = [\lambda_0(1-\lambda_0) + \lambda_1(1-\lambda_1)]\, x^2, \tag{1}$$
$$b(x) = \mu_1(2-\lambda_0-\lambda_1) - 2(1-\lambda_0)(1-\lambda_1)x, \tag{2}$$
$$c(x) = \mu_2(2-\lambda_0-\lambda_1)x, \tag{3}$$

and the discriminant $D_1(x) = b^2(x) - 4a(x)c(x)$, which is a cubic polynomial. We can first show that $D_1(x)$ has three branch points: $0 < x_1 < x^* < x_2 < 1 < x_3 < +\infty$, where

$$x^* = \frac{\mu_1(2-\lambda_0-\lambda_1)}{2(1-\lambda_0)(1-\lambda_1)}$$

is the unique solution to $b(x) = 0$.

We are then to show:

1. $h_1(x, Y_0(x))$ has a unique zero x^* that is greater than one;
2. $h_2(X_0(y), y)$ does not have any zero y such that $y = X_0(\tilde{x}_1)$ for some $\tilde{x}_1 > 1$.

Tail asymptotic properties

Finally, based on which one is the dominant singularity, there are three types of tail asymptotic properties for $\pi_{m,0}$:

Type one: If $x^* < x_3$, then

$$\pi_{m,0} \sim c(1/x^*)^m;$$

Type two: If $x_3 < x^*$, then

$$\pi_{m,0} \sim cm^{-3/2}(1/x_3)^m;$$

Type three; If $x^* = x_3$, then

$$\pi_{m,0} \sim cm^{-1/2}(1/x^*)^m = cm^{-1/2}(1/x_3)^m.$$

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Thanks You!