Uncertainties on the Mean Up Time : What are the Consequences on the Unavailability ?

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1 Introduction

Estimating a mean up time (MUT) for a new component generally relies on an extrapolation of informations obtained on one or several matured components of the same family. In charge of such a task, who did not hesitate thinking to himself : « why 5 000 hours rather than 4 000 hours? or 6 000 hours? ».

This estimation of the MUT will be used to estimate the steady state availability of this new component and from that elaborate the estimation of the availability of the global system in which this subsystem is included. Moreover, for tailored complex highly dependable systems (think for example to air traffic control systems), there is a high sensitivity between the steady state availability of this new component and the amount of spares to produce when introducing the new complex system on the field. An over-estimation of the MUT and the customer will not obtain the required availability; an under-estimation of the MUT and very costly spares might be produced for no need.

In order to take this risk into consideration, we propose to consider the steady state mean up time (MUT) as a random variable. This implies that the steady state availability becomes itself a random variable.

In this paper we first consider the MUT as uniformly distributed on a time interval $[a_0, b_0]$; when no additional information is available, this is meaningful to use the uniform distribution (this is the distribution that maximizes the entropy function of the information). Then we consider the so called « triangular » distribution. For such a distribution, the density function is continuous and non nul over the interval $[x_{min}, x_{max}]$, linearly increasing on $[x_{min}, x_{mod}]$ and linearly decreasing on $[x_{mod}, x_{max}]$.

Because we deal with highly available systems, it is more meaningful to study the relative variation of the unavailability as a function the different parameters. Here, the mean down time (easier to evaluate) is supposed to be constant and the steady state unavailability can be written as :

$$\overline{A} = \frac{d}{MUT + d}$$

where d is the constant mean down time.

We determine the expectation and the probability distribution of this steady state unavailability as a function of the probability distribution of the steady state mean up time for the two different cases, respectively in sections 2 and 3. Sensitivities of the different parameters are pointed out with graphical illustrations of numeral cases. Section 4 gives some insights on the differences induced by the two distinct distributions.

2 The MUT follows a uniform distribution

In this section, the random variable MUT follows the uniform distribution $U[a_0, b_0]$. Letting $a = (a_0 + d)$ and $b = (b_0 + d)$ the random variable unavailability $\overline{A}$ takes its values on the interval $\overline{A} \in [\overline{A}_{\min}, \overline{A}_{\max}]$ where $\overline{A}_{\min} = \dfrac{d}{b}, \overline{A}_{\max} = \dfrac{d}{a}$. It is interesting to represent the obtained results as a function of the ratio b/a and therefore we let $\beta = b/a$, $\beta > 1$. On this interval, the density function of $\overline{A}$ is equal to

$$f_{\overline{A}}(x) = \frac{d}{a(\beta - 1)x^2}, \qquad x \in [\overline{A}_{\min}, \overline{A}_{\max}]$$

From this result we get the expectation

$$\mathbb{E}[\overline{A}] = \frac{d}{a(\beta - 1)} \ln(\beta)$$

Expressing x as a function of a parameter α, $\alpha \in [0, 1]$ in the following way

$$x(\alpha) = \overline{A}_{\min} + \alpha(\overline{A}_{\max} - \overline{A}_{\min}) = d\frac{1 + \alpha(\beta - 1)}{\beta a},$$

it is possible to show that

$$\mathbb{P}(\overline{A} > x(\alpha)) = \frac{(1 - \alpha)}{(1 - \alpha) + \alpha\beta} \qquad (1)$$

and also that

$$\mathbb{P}(\overline{A} > x(\alpha)) < (1 - \alpha)$$

Figure 1 gives the variation of the probability $\mathbb{P}(\overline{A} \leq x(\alpha))$ as a function of α, $\alpha \in [0, 1]$, for several values of β. The line segment denoted $\beta = 1$ corresponds to the limit curve when β tends to one, *i.e.*, when b_0 tends towards a_0. It is possible to check that the probability $\mathbb{P}(\overline{A} > x(\alpha))$ decreases significantly when parameter β increases. However we have to remember that interval $[\overline{A}_{\min}, \overline{A}_{\max}]$ increases with β and therefore that, globally the uncertainty stays penalizing.

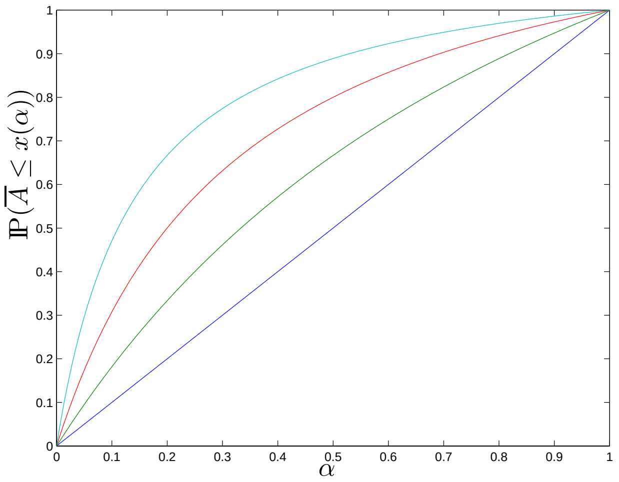

Figure 1: Variation of $\mathbb{P}(\overline{A} \leq x(\alpha))$ as a function of α, $\alpha \in [0, 1]$. Values of β : $1; 2; 4; 8$ (bottom-up). MUT following a uniform distribution.

Given a small value of γ, we can determine the value of α such that $\mathbb{P}(\overline{A} > x(\alpha)) = \gamma$. We get the function :

$$\alpha = \frac{(1-\gamma)}{(1-\gamma)+\gamma\beta}$$

Note that this function is an involution ($f(f(x)) = x$). Figure 2 gives this variation of α as a function of γ, $\gamma \in [0\ ,\ 0,2]$. for several values of β. We observe that α decreases significantly when parameter β increases.

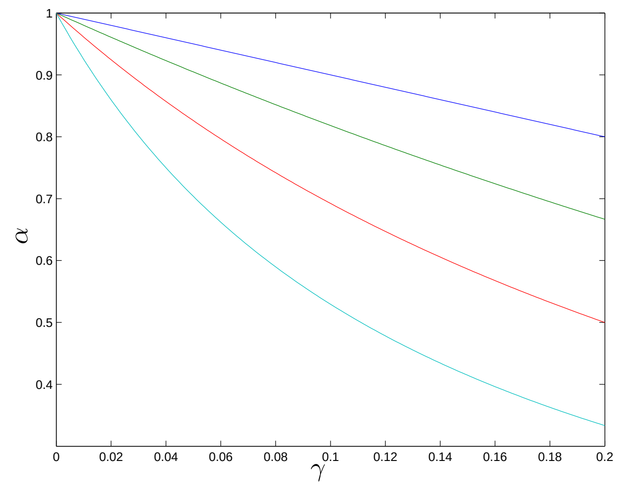

Figure 2: Variation of α as a function of γ, $\gamma \in [0\ ,\ 0,2]$. Values of β : $1; 2; 4; 8$ (top-down). MUT following a uniform distribution.

Note that if we were considering a deterministic value of MUT equal to its expectation, then the steady state unavailability would be deterministic and equal to

$$\widehat{A} = \frac{2d}{a(\beta+1)}$$

It can be shown that increasing the uncertainty on the mean up time (without changing its expectation) decreases the expectation of the steady state availability.

3 The MUT follows a triangular distribution

We assume now that the random variable MUT follows a triangular distribution (see a representation on figure 3).

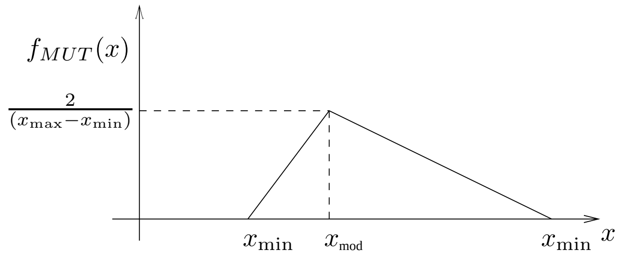

Figure 3: Illustration of the density function of the « triangular » distribution.

This distribution is entirely characterized by the triplet $(x_{\min}, x_{\max}, x_{\text{mod}})$.

The mean down time is still supposed to be constant (MDT $= d$) and, in order to simplify the writting, we let $a = x_{\min} + d$, $b = x_{\max} + d$, $\Delta = x_{\max} - x_{\min}$ and $\rho = \dfrac{(x_{\text{mod}} - x_{\min})}{(x_{\max} - x_{\min})}$.

Considering the random variable $Y = \dfrac{\overline{A}}{d}$, it can be shown that this random variable has the following density function :

$$f_Y(y) = \begin{cases} 0 & if \quad y < 1/b \\ \dfrac{2}{(1-\rho)\Delta^2}\dfrac{1}{y^2}(b-\dfrac{1}{y}) & if \quad 1/b \leq y \leq \dfrac{1}{a+\rho\Delta} \\ \dfrac{2}{\rho\Delta^2}\dfrac{1}{y^2}(\dfrac{1}{y}-a) & if \quad \dfrac{1}{a+\rho\Delta} \leq y \leq 1/a \\ 0 & if \quad y > 1/a \end{cases}$$

This density function is represented on figure 4 for three values of ρ ($1/4; 1/2$ and $3/4$). The cumulative probability function is obtained by integration :

$$F_Y(y) = \begin{cases} 0 & if \quad y < 1/b \\ \dfrac{1}{(1-\rho)\Delta^2}\left(b - \dfrac{1}{y}\right)^2 & if \quad 1/b \leq y \leq \dfrac{1}{a+\rho\Delta} \\ 1 - \dfrac{1}{\rho\Delta^2}\left(\dfrac{1}{y} - a\right)^2 & if \quad \dfrac{1}{a+\rho\Delta} \leq y \leq 1/a \\ 1 & if \quad y > 1/a \end{cases}$$

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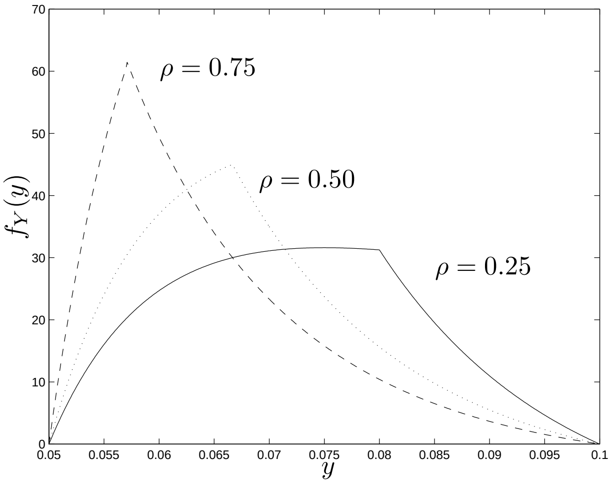

Figure 4: Density function of Y for three values of ρ ; with $a = 10$ and $b = 20$. MUT follows a « triangular » distribution.

With these previous notations, we still have $\overline{A} \in [\overline{A}_{\min}, \overline{A}_{\max}]$ where $\overline{A}_{\min} = \dfrac{d}{b}, \overline{A}_{\max} = \dfrac{d}{a}$, as in the previous section. Using again the notations $\beta = b/a$, $\beta > 1$ and $x(\alpha) = \overline{A}_{\min} + \alpha(\overline{A}_{\max} - \overline{A}_{\min})$, we get again

$$x(\alpha) = d\frac{1+\alpha(\beta-1)}{\beta a},$$

We can show that the expectation of $\overline{A}$ has the following expression when ρ satisfies the condition $0 < \rho < 1$:

$$\begin{aligned} \mathbb{E}[\overline{A}] &= \frac{2d}{a(1-\rho)(\beta-1)^2} \times \\ &\left(\beta\ln(\beta) - \frac{(1-\rho)+\rho\beta}{\rho}\ln((1-\rho)+\rho\beta)\right) \end{aligned}$$

Let α^* be the value of α such that $x(\alpha^*) = \dfrac{d}{a+\rho\Delta}$, *i.e.*, the value of α corresponding to the unavailability that we obtained when MUT $= x_{\text{mod}}$. This value is :

$$\alpha^* = \frac{(1-\rho)}{(1-\rho)+\rho\beta}$$

When α takes its value between α^* and 1, we can show that :

$$\mathbb{P}(\overline{A} \leq x(\alpha)) = 1 - \frac{1}{\rho}\left(\frac{(1-\alpha)}{(1-\alpha)+\alpha\beta}\right)^2. \qquad (2)$$

If $\alpha = \alpha^*$, we have :

$$\mathbb{P}(\overline{A} \leq x(\alpha^*)) = (1-\rho), \qquad \forall\beta,$$

and when α takes its value between 0 and α^*, we can show that :

$$\mathbb{P}(\overline{A} \leq x(\alpha)) = \frac{1}{(1-\rho)}\left(\frac{\alpha\beta}{(1+\alpha(\beta-1))}\right)^2. \qquad (3)$$

Figure 5 gives the variation of the probability $\mathbb{P}(\overline{A} \leq x(\alpha))$ as a function of α and ρ ; $\rho = 0.5$, $\alpha \in [0.5\ ,1]$, for several values of β. Again, the curve denoted $\beta = 1$ corresponds to the limit curve when β tends to one, *i.e.*, when b_0 tends towards a_0. We can see that the probability $\mathbb{P}(\overline{A} > x(\alpha))$ decreases significantly when parameter β increases. But again we have to remember that interval $[\overline{A}_{\min}, \overline{A}_{\max}]$ increases with β and therefore that, globally the uncertainty stays penalizing.

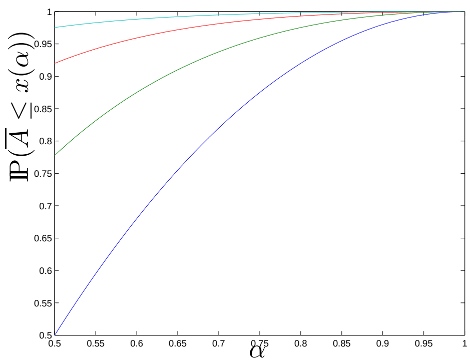

Figure 5: Variation of $\mathbb{P}(\overline{A} \leq x(\alpha))$ as a function of α, $\alpha \in [0.5\ ,1]$. $\rho = 0.5$. Values of β : $1; 2; 4; 8$ (bottom-up), $\alpha \in [0.5\ ,1]$. MUT follows a « triangular » distribution.

Given a small value of γ (satisfying $\gamma < \rho$), we can determine the value of α such that $\mathbb{P}(\overline{A} > x(\alpha)) = \gamma$. We get the function :

$$\alpha = \frac{(1-\sqrt{\rho\gamma})}{(1-\sqrt{\rho\gamma})+\sqrt{\rho\gamma}\beta}$$

Figure 6 gives this variation of α as a function of γ, $\gamma \in [0\ ,\ 0,2]$. for several values of β. For example, if $\beta = 2$, $\rho = 0.5$ and $\gamma = 0.1$, then α equals 0.63. We observe that α decreases significantly when parameter β increases. This figure can be compared to figure 2 relative to the variation of α in the case of the uniforme distribution.

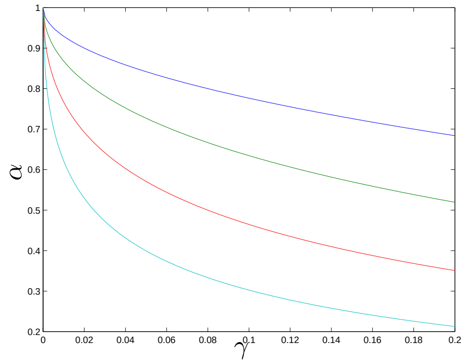

Figure 6: Variation of α as a function of γ, $\gamma \in [0\ ,\ 0.2]$. $\rho = 0.5$. Values of β : $1; 2; 4; 8$ (top-down). MUT follows a « triangular » distribution.

On the other side, given a small value of δ (satisfying $\delta < (1-\rho)$), we can determine the value of α such that $\mathbb{P}(\overline{A} \leq x(\alpha)) = \delta$. We get the function :

$$\alpha = \frac{\sqrt{(1-\rho)\delta}}{\beta - (\beta - 1)\sqrt{(1-\rho)\delta}}$$

4 Comparison

In order to make comparisons between the two distributions, let $\mathbb{P}_U(\overline{A} > x(\alpha))$ and $\mathbb{P}_T(\overline{A} > x(\alpha))$ denote the respective probabilities corresponding to the uniform and to the triangular distributions of the random variable MUT. From previous equations 1 and 2 , we get, for a given value of α, $\alpha > \alpha^*$:

$$\mathbb{P}_T(\overline{A} > x(\alpha)) = \frac{1}{\rho}\left(\mathbb{P}_U(\overline{A} > x(\alpha))\right)^2. \tag{4}$$

For $\alpha = \alpha^*$, we have :

$$\mathbb{P}_T(\overline{A} > x(\alpha)) = \mathbb{P}_U(\overline{A} > x(\alpha)) = \rho. \tag{5}$$

While for α, $\alpha < \alpha^*$:

$$\mathbb{P}_T(\overline{A} \leq x(\alpha)) = \frac{1}{(1-\rho)}\left(\mathbb{P}_U(\overline{A} \leq x(\alpha))\right)^2. \tag{6}$$

Figure 7 shows, for $\beta = 2$, the advantage of the « triangular » distribution on the uniform distribution. Increasing the value of parameter β would increase this advantage.

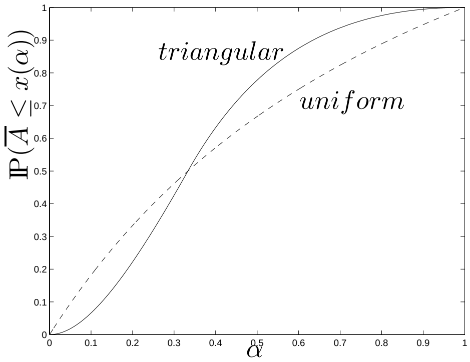

Figure 7: Variation of $\mathbb{P}(\overline{A} \leq x(\alpha))$ as a function of α, $\alpha \in [0.0\ ,1]$. Comparison of the uniform distribution with the « triangular » distribution. With $\rho = 0.5$; $\beta = 2$.

5 Conclusions

In this study, we have pointed out a way to determine the probability $\mathbb{P}(\overline{A} > x)$ when, due to a lake of knowledge, the mean up time is considered as a random variable. Simple expressions have been obtained for two cases of probability distributions : the uniform distribution and the « triangular » distribution. Expressions of x satisfying the equality $\mathbb{P}(\overline{A} > x) = \gamma$, for a given γ have also been exhibited.

We conducted this research with the aim to help the engineer to understand the consequences of such uncertainties on the mean up time.

Finally, note that, starting from these initial guesses on probability distributions, we may then improve the evaluation by using the Bayesian approach, making uses of the data returning from the field experiences.

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