# Uniform quantizer, vector quantization, Discrete cosine transform, Predictive coding

Coding Technology

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# Outline

1. Quantization and entropy
2. Uniform quantizer
3. Vector quantization
   - Lloyd-Max algorithm in higher dimension
   - Discrete cosine transform
   - Predictive coding

# Reminder: quantization

Irreversible data compression. Distortion vs compression ratio.

Quantization: rounding values from a large set to a small set. Quantizer, levels, domains.

Nearest neighbour rule, Center of gravity rule, Lloyd-Max property.

Optimal quantizer. Lloyd-Max algorithm.

# Quantization and entropy

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Basically, by quantization, we lose information.

Note that the theorem only applies for $X$ discrete.

# Uniform quantizer

Let $X$ be a continuous random variable with probability density function $f(x)$.

Assume $f(x)$ is continuous on $[0, A]$ and 0 outside it (so $X$ is concentrated on $[0, A]$).

The uniform $N$-level quantizer $Q_N$ for the interval $[0, A]$ is the following:

$$\text{domains: } B_i = \left[(i-1)\frac{A}{N}, i\frac{A}{N}\right], \quad i = 1, \ldots, N,$$

$$\text{levels: } x_i = (2i-1)\frac{A}{2N}, \quad i = 1, \ldots, N.$$

The uniform quantizer is a simple quantizer (but not optimal except if $X$ has uniform distribution on $[0, A]$).

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Proof (sketch). For large $N$, the conditional distribution of $X$ assuming $X \in B_i$ is close to the uniform distribution on $B_i$ since $f$ is continuous. The uniform distribution on an interval of length $L$ has variance $L^2/12$, so

$$D(Q_N) \approx \sum_{i=1}^{N} \int_{B_i} f(x) \frac{(A/N)^2}{12} \mathrm{d}x = \frac{A^2}{12N^2} \int_{-A}^{A} f(x) \mathrm{d}x = \frac{A^2}{12N^2}$$

# Differential entropy

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## Theorem

$$\lim_{N\to\infty}(H(Q_N(X)) + \log_2(A/N)) = H(f)$$

(No proof.)

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Accordingly, the above formula gives

$$\begin{aligned} H(f) &= \int_{-1/2}^{1/2} 1 \cdot 0 \mathrm{d}x = 0 \\ \log_2(A) &= \log_2(1) = 0 \\ \log_2(N) &= k \end{aligned}$$

and $H(Q_N(X)) = k$, which is indeed the entropy of $k$ bits.

# Lloyd-Max algorithm

Reminder: Lloyd-Max algorithm.

1. Initialize the levels.
2. Update the domains using the nearest neighbour property.
3. Compute $D(Q)$, if the decrease is below a given threshold, stop.
4. Update the levels using the center of gravity rule, repeat from step 2.

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What to do when there are multiple local optima?

- Start from a reasonably good initial condition.
- Start from multiple initial conditions, and select the best obtained quantizer.

# Vector quantization

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Quantizing height and weight jointly can do that. Quantizing them separately cannot do that.

# Vector quantization

Similar to the 1-dimensional case, a vector quantizer $Q : \mathbb{R}^d \to \mathbb{R}^d$ is described by the collection of output vectors $x^{(i)} \in \mathbb{R}^d$ and domains $B^{(i)} \subseteq \mathbb{R}^d$ such that

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We use the Euclidean distance between points $x, y \in \mathbb{R}^d$:

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2}$$

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For coordinates of different units (e.g. kg vs cm), we usually apply a linear scaling to bring all coordinates to the same order of magnitude first!

The distortion can now be defined as

$$D(Q) = \mathbb{E}(\|X - Q(X)\|^2) = \sum_{i=1}^{N} \int_{B^{(i)}} \|x - x^{(i)}\|^2 f(x) \mathrm{d}x,$$

where $f(x)$ is the $d$-dimensional density function of $X$.

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A quantizer satisfies the Lloyd-Max property if it satisfies both the nearest neighbour rule and the center of gravity rule.

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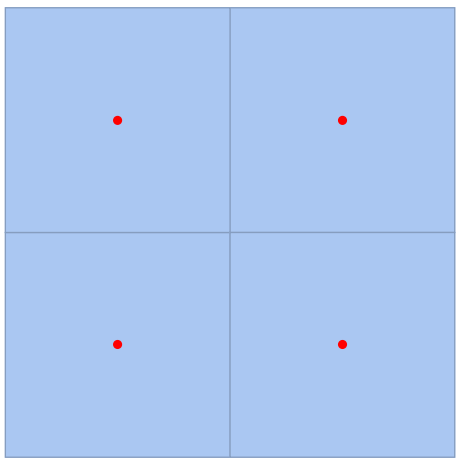
This is a good candidate:



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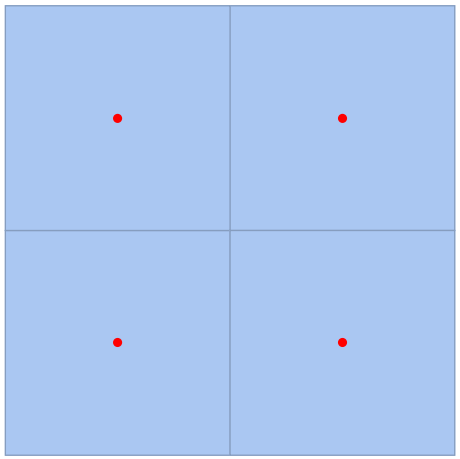


How could we find an optimal quantizer numerically?

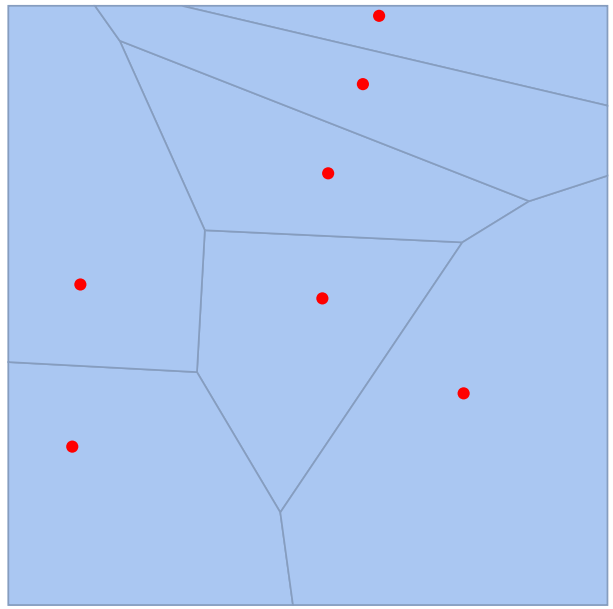
# Lloyd-Max algorithm in higher dimensions

The Lloyd-Max algorithm works the same for higher dimensions.

1. Initialize the levels.
2. Compute the domains for the output vectors using the nearest neighbour property.
3. Compute $D(Q)$, compare it to the old value of $D(Q)$ from the previous iteration, and if the difference is below a given threshold, stop.
4. Otherwise, from the domains, compute new values for the output vectors according to the center of gravity rule, then repeat from step 2.

# Lloyd-Max algorithm in higher dimensions

The nearest neighbor rule leads to the so-called Voronoi cells:



In $d = 2$ dimensions, the Voronoi cells are polygon-shaped.

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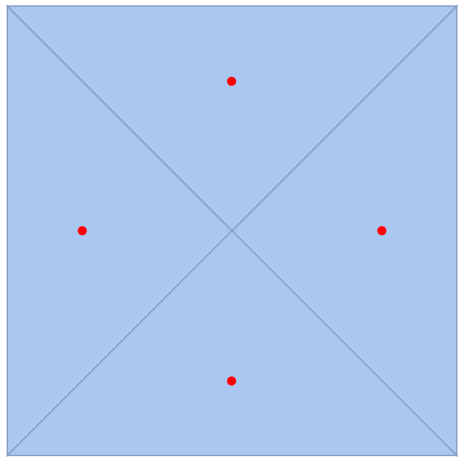

# Lloyd-Max algorithm in higher dimensions

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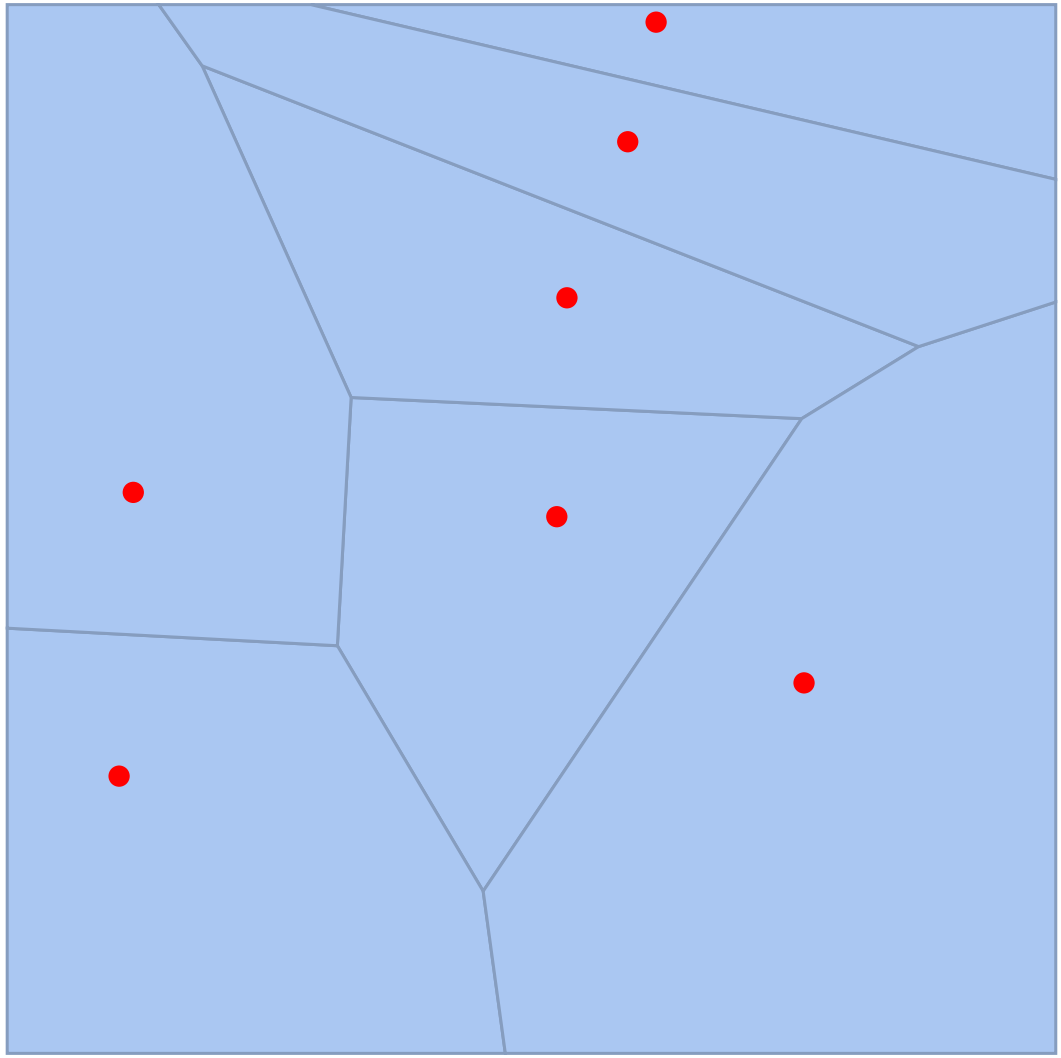
How could the optimal quantizer with $N = 7$ output vector look like for a uniform distribution over a square?

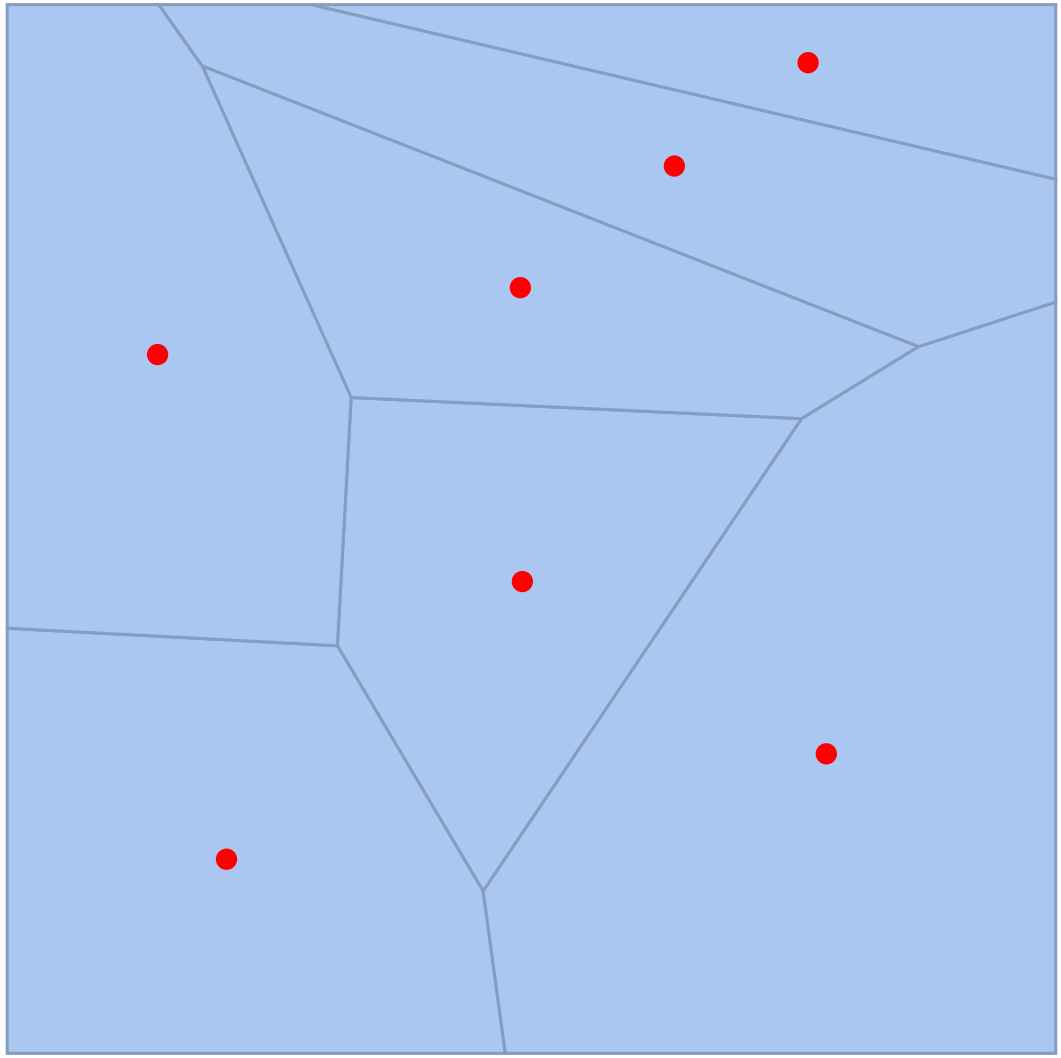
Let's see what the Lloyd-Max algorithm does! (We will test several different initializations to see if they converge to the same quantizer.)

# 2D Lloyd-Max algorithm



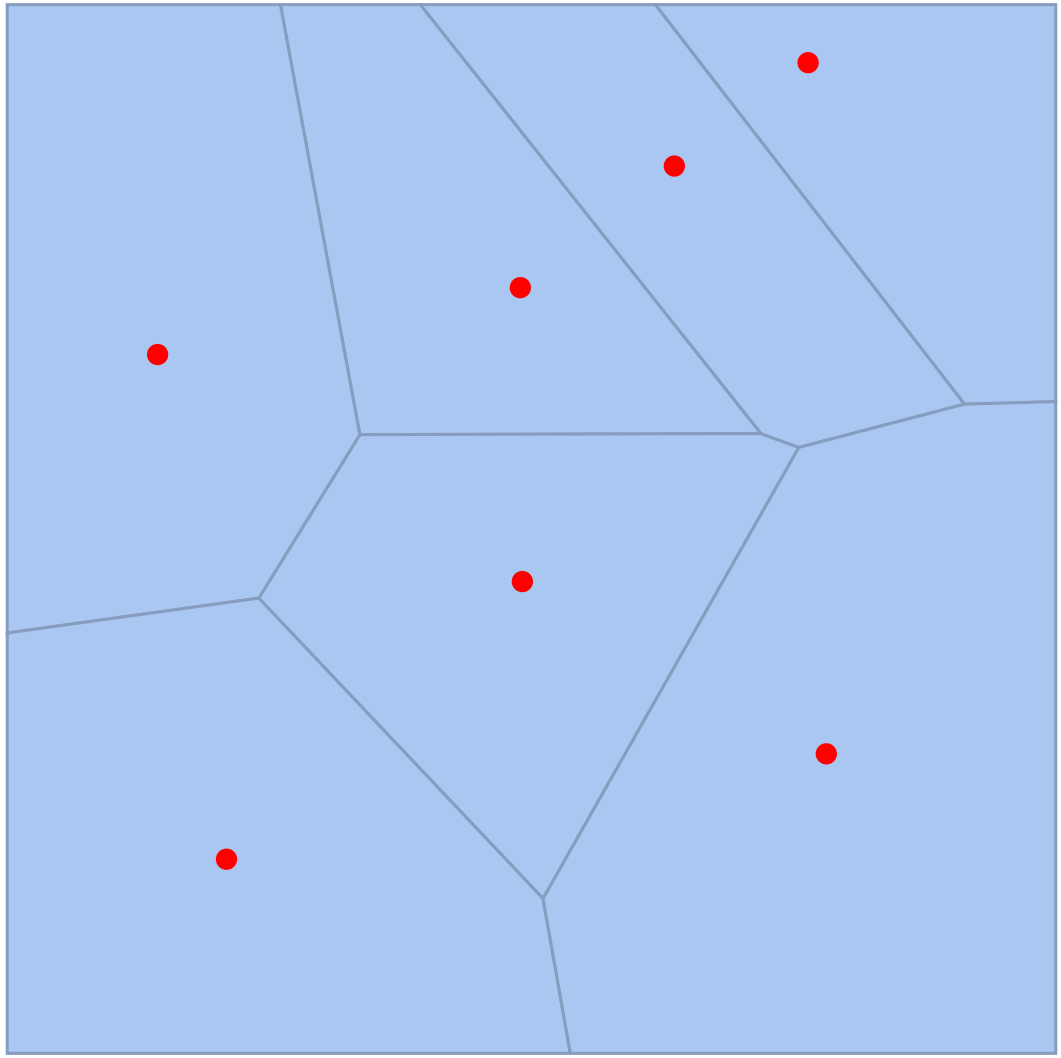
Initial points, domains from nearest neighbor rule

# 2D Lloyd-Max algorithm



Points updated 1×

# 2D Lloyd-Max algorithm



Points updated 1×, domains updated 1×

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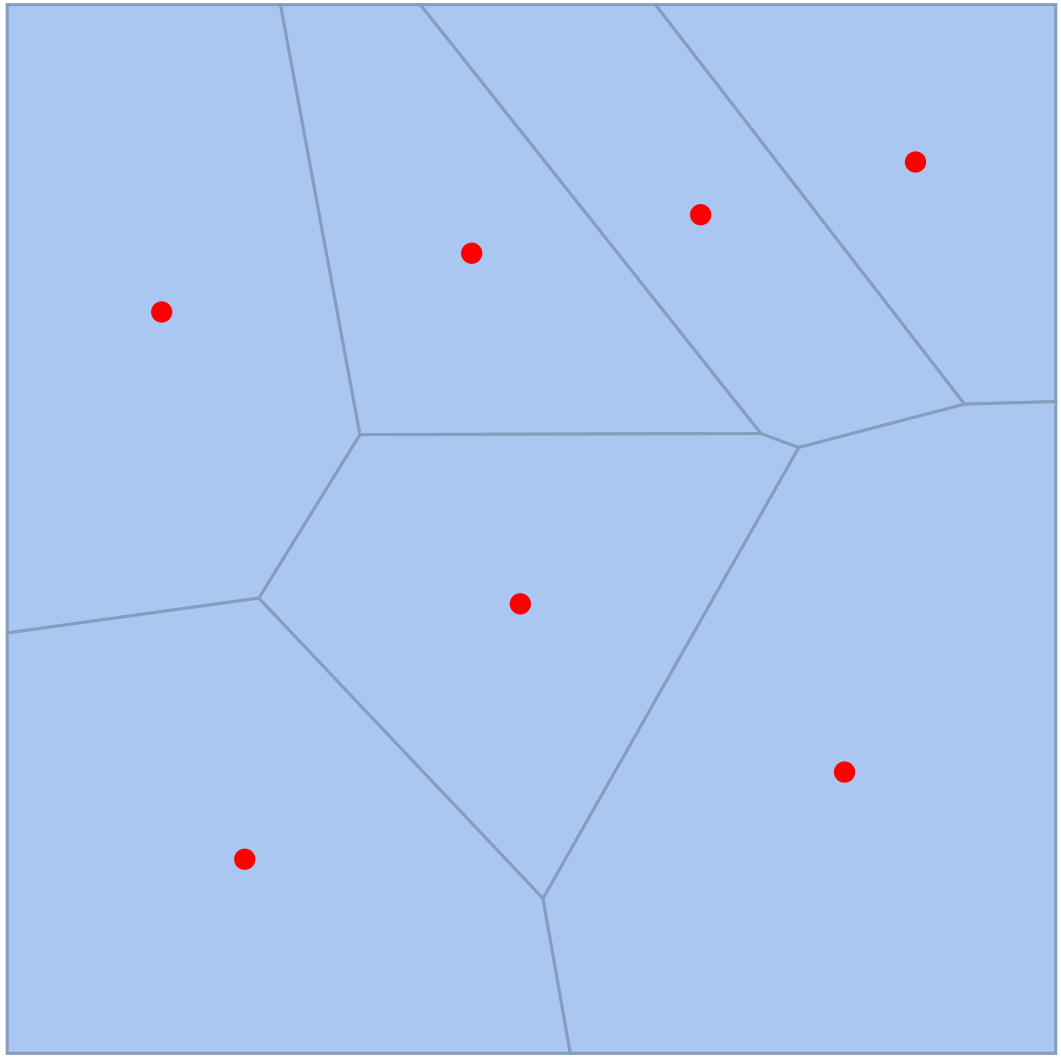
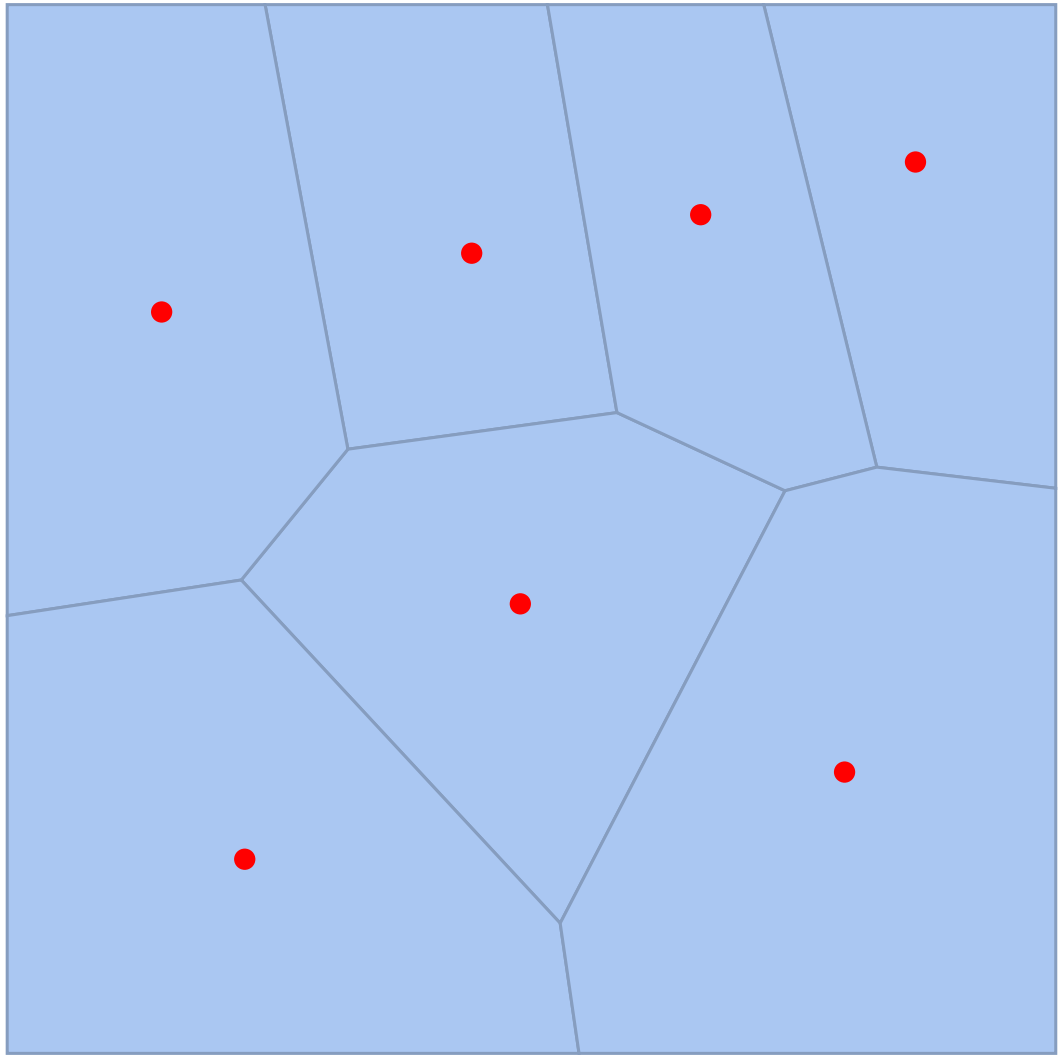


Points updated 2×, domains updated 1×

# 2D Lloyd-Max algorithm



Points updated 2×, domains updated 2×

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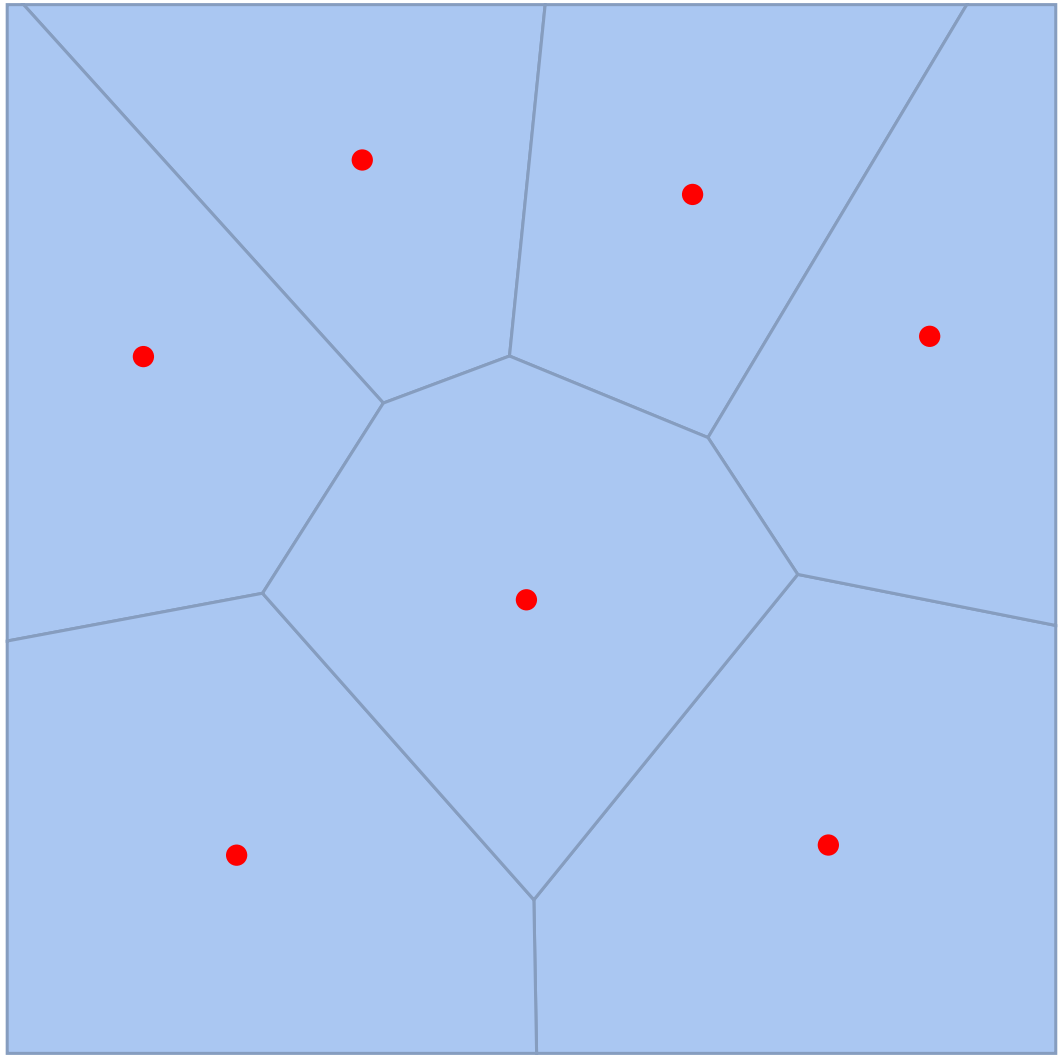

Points updated 10×, domains updated 10×

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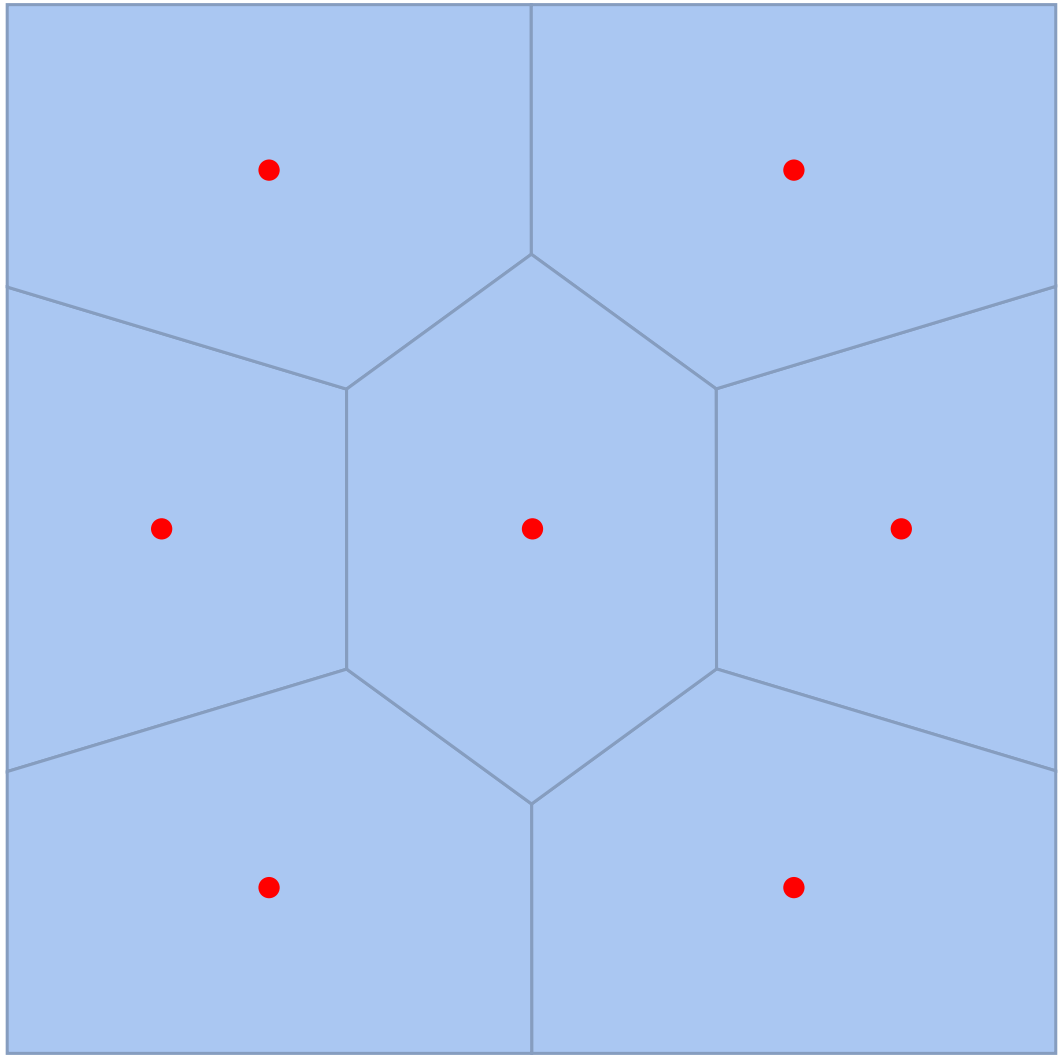
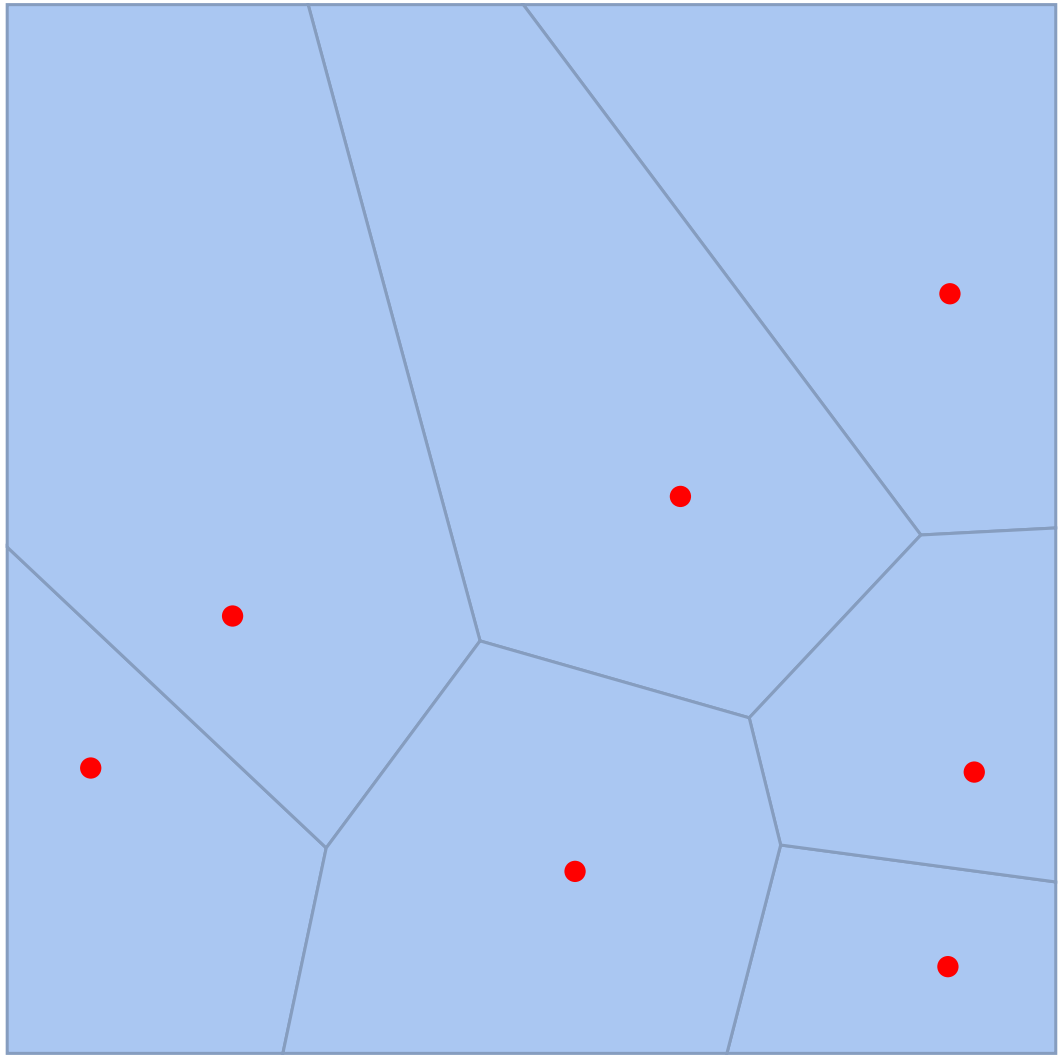


Points updated 50×, domains updated 50×

# 2D Lloyd-Max algorithm



Initial points, domains from nearest neighbor rule

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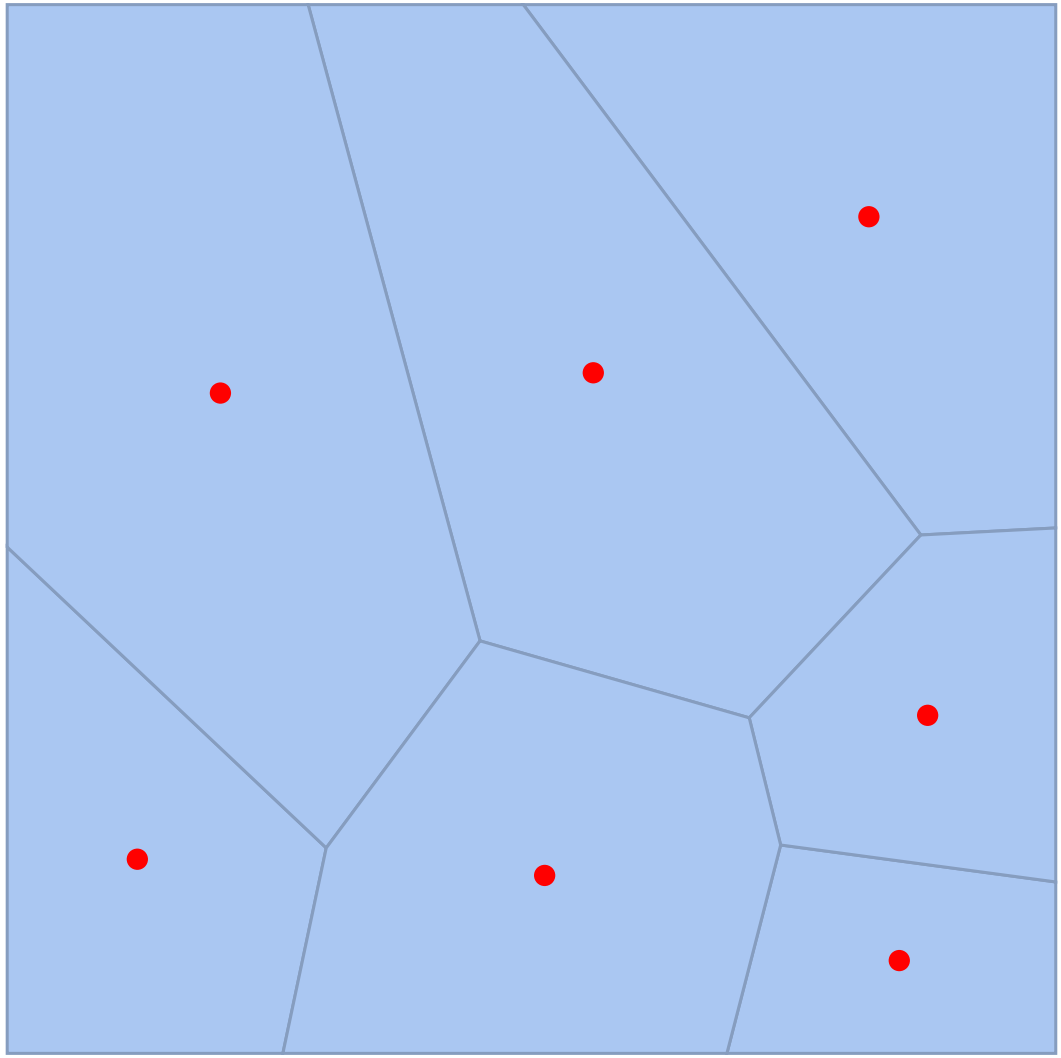

Points updated 1×

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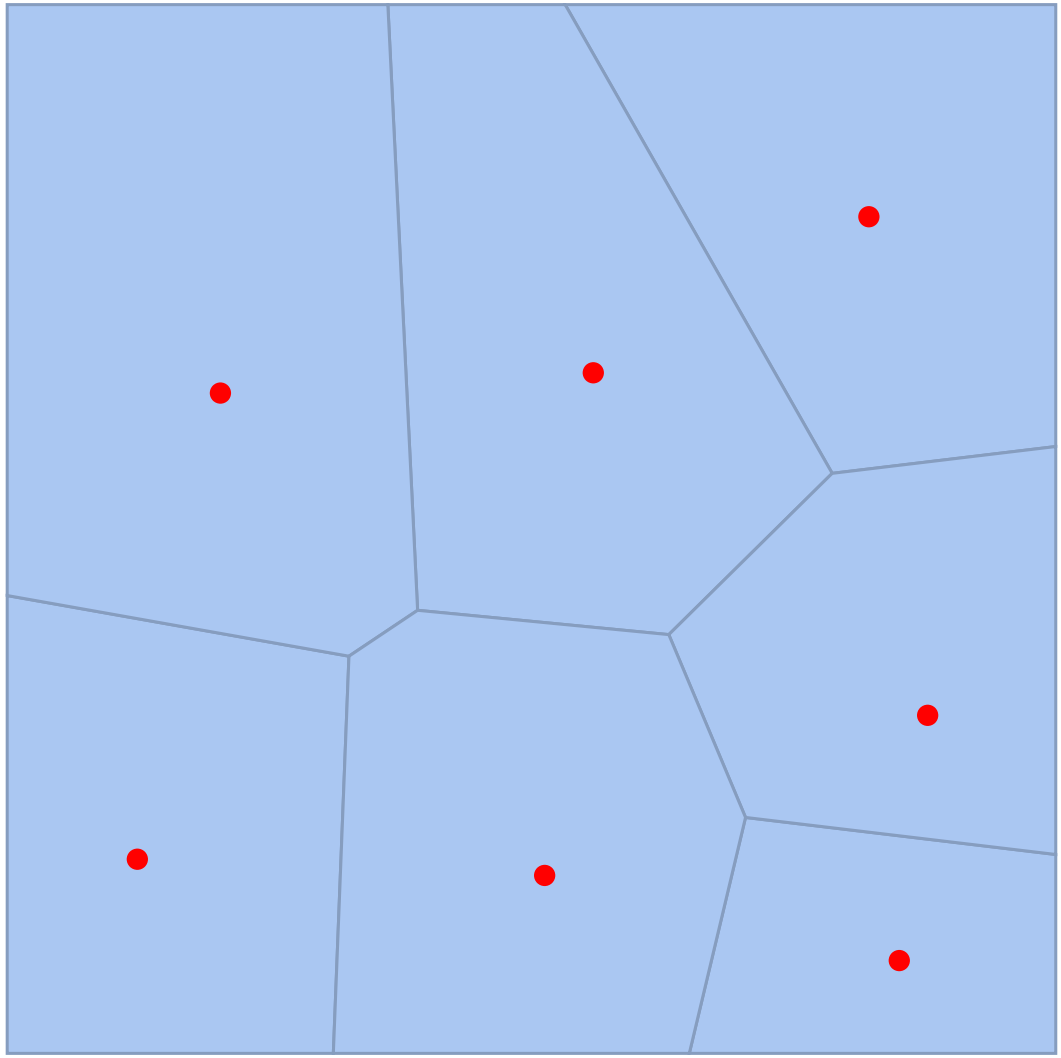

Points updated 1×, domains updated 1×

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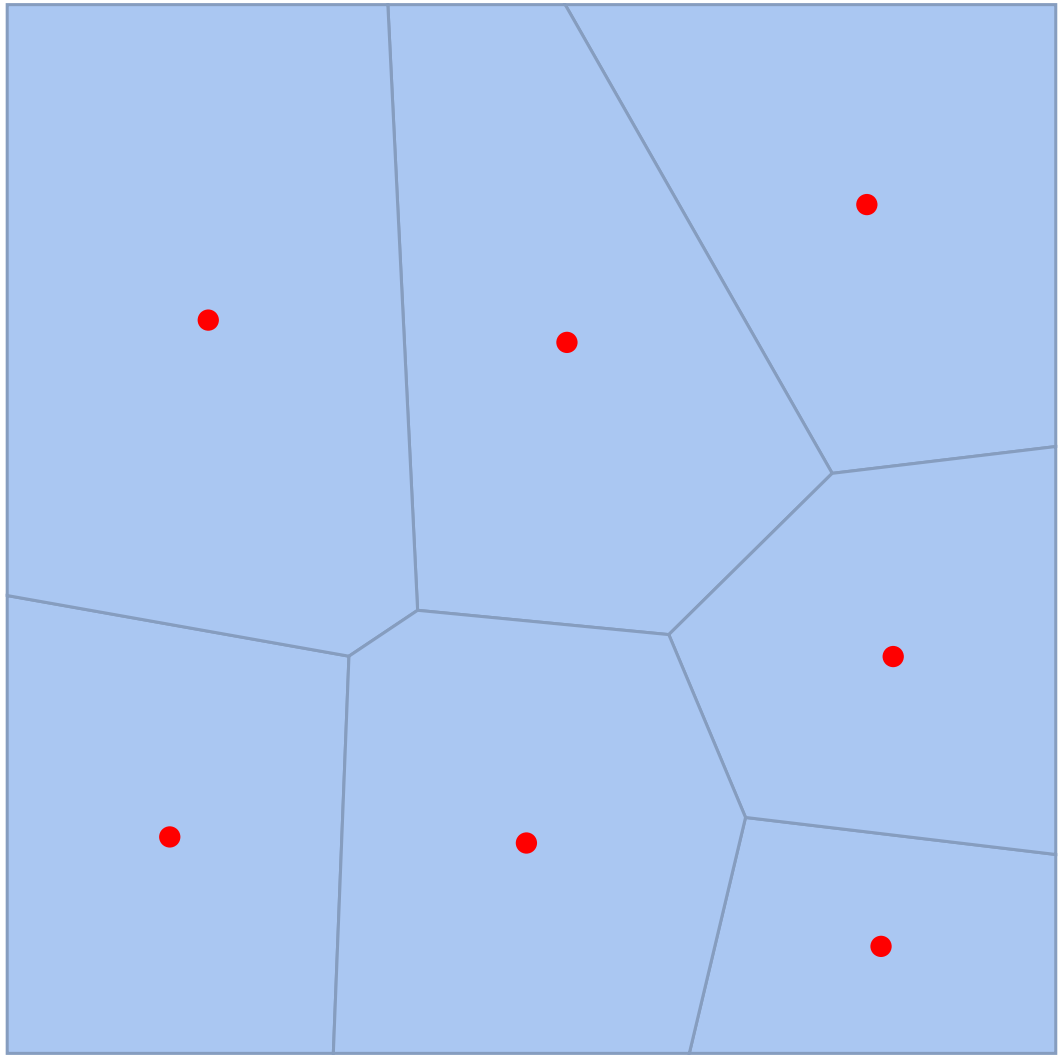

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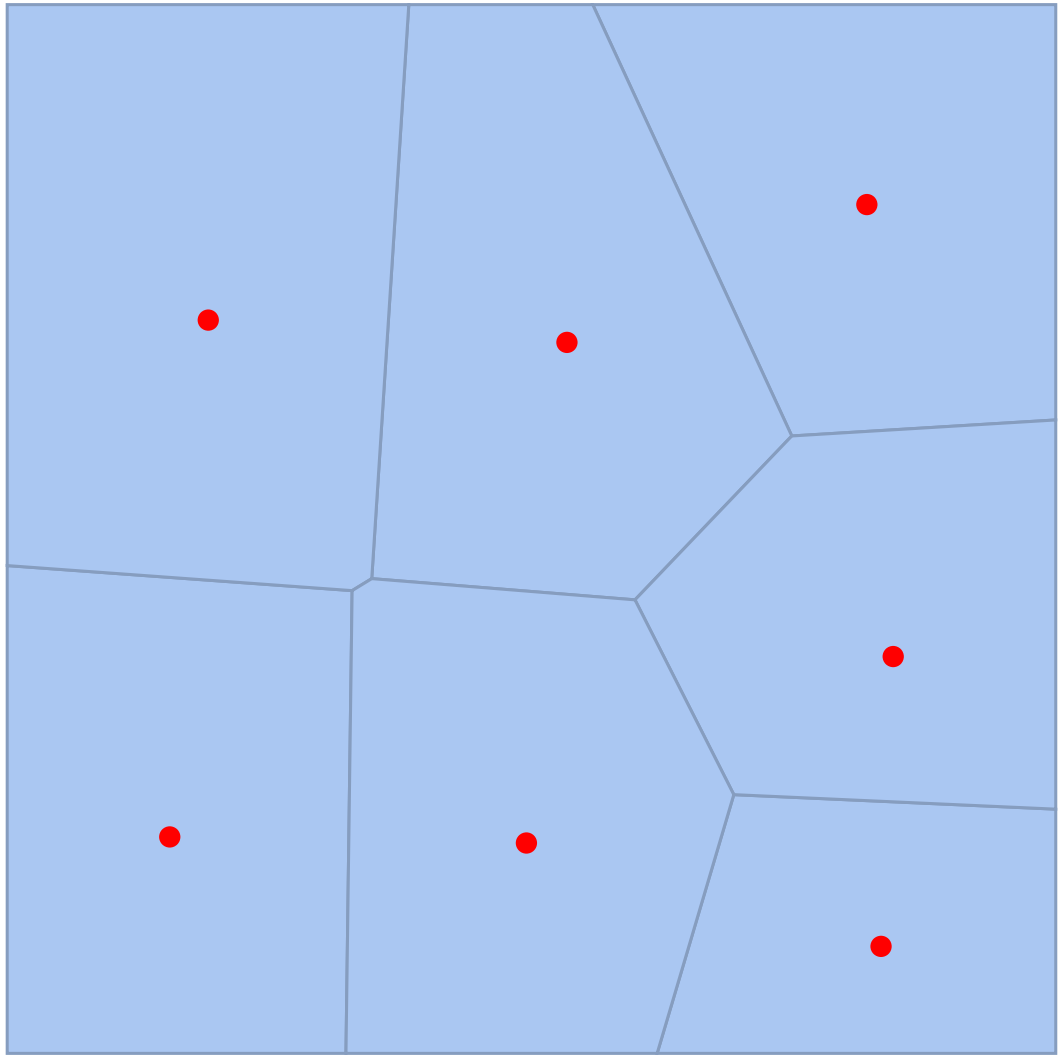

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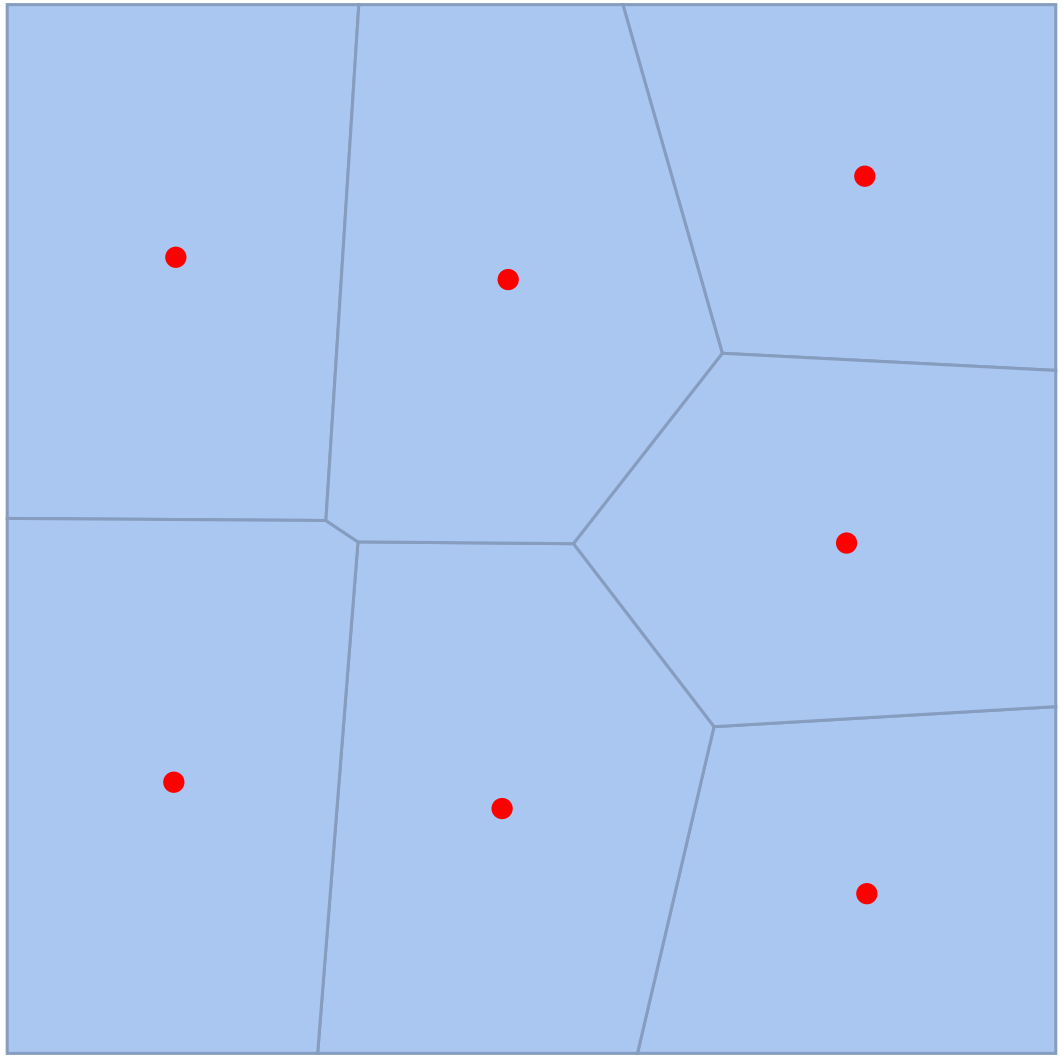

Points updated 10×, domains updated 10×

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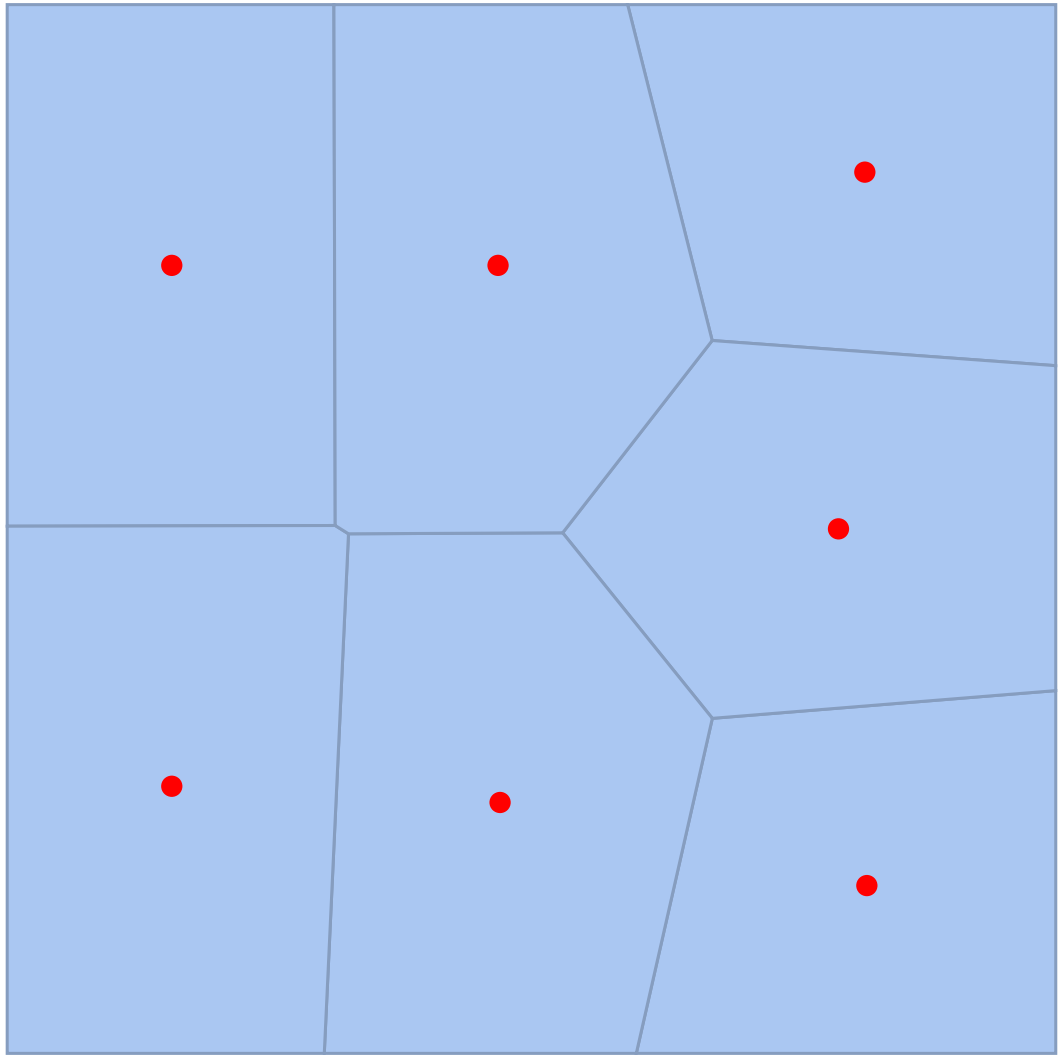

Points updated 50×, domains updated 50×

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Depending on the application, we might also settle for a locally optimal quantizer.

# Outlook: structured quantizers

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Tree-structured vector quantizers. First, only one coordinate is quantized, then for each domain, the next coordinate is quantized (possibly in a different manner), and so on.

Energy-form quantizers: in voice or music compression, usually there is a natural energy (volume) of the signal. The energy and the normalized signal (form) are quantized separately.

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We do not chase this direction.

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The general concept of transform coding is similar to text transforms: a transform is a reversible mapping of a signal such that the transformed signal is easier to compress than the original signal.

The most widely used such transform is the Discrete cosine transform (DCT), designed for image and video processing (accordingly, it is the basis of JPEG and MPEG), but also used in many other applications.

The main idea of DCT is to transform a block of $k$ values from the signal in a way that the first few values of the transform capture the main defining properties of the original signal block, and subsequent elements of the transform correspond to finer and finer details (which can then be compressed with higher distortion).

# DCT

Here we will discuss the version used for image processing.

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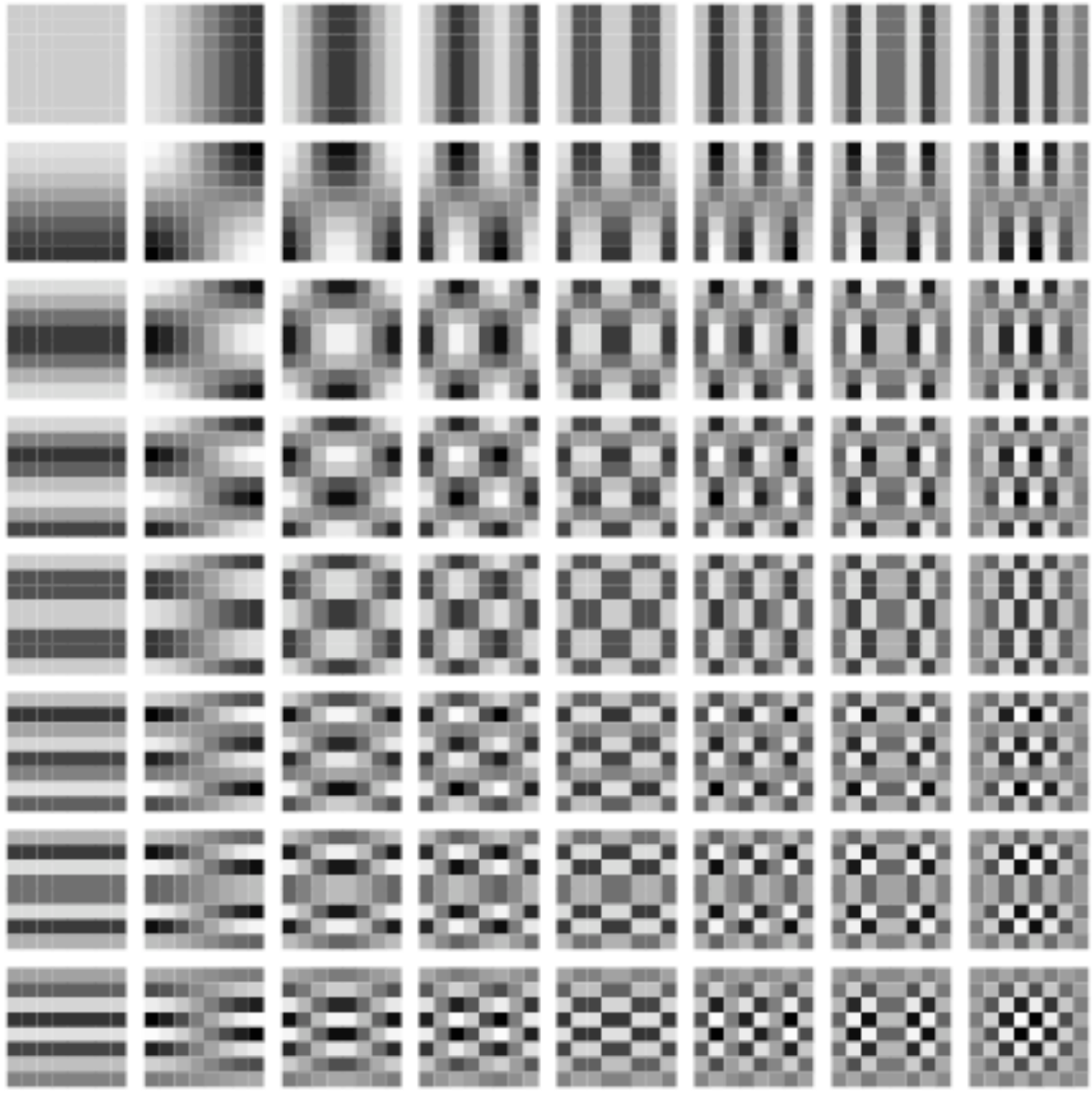
DCT then transforms the $k \times k$ values into another array of $k \times k$ values called *frequency coefficients*. The first of these coefficients (the *DC coefficient* or *main coefficient*) represents the average value of the original block, and subsequent coefficients represent deviations from the average in sinusoidal patterns with varying frequency.

# DCT



# DCT

Mathematically, the formula for the DCT is the following.

$X$ denotes the values of the original block, arranged in a $k \times k$ matrix. Then the DCT of $X$ is

$$Y = AXA^T,$$

where the matrix $A$ is the following:

- the first row of $A$ is $\begin{bmatrix} \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \cdots & \frac{1}{\sqrt{k}} \end{bmatrix}$
- for other rows,

$$a_{ij} = \sqrt{\frac{2}{k}} \cos\left(\frac{(2j-1)(i-1)\pi}{2k}\right)$$

# DCT

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Technically, the DCT does not compress; actual compression is done *after* applying DCT by quantizing the frequency coefficients.

# DCT

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Algorithmic complexity. Normally, computing the matrix product $AXA^T$ would take $O(k^4)$ operations, but this can be reduced to $O(k^2 \log(k))$ due to the special structure of $A$ using *Fast cosine transform* (FCT).

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FCT utilizes a factorization of $A$ as a product of simple matrices (similar to Fast Fourier transform (FFT)). We do not go into more details here.

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Example. Consider the following sequence of bytes:

$$147, 145, 141, 146, 149, 147, 143, 145$$

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Example. Consider the following sequence of bytes:

$$147, 145, 141, 146, 149, 147, 143, 145$$

Here, it makes sense to use each signal as a predictor for the next signal, and take the difference, so the transformed sequence is

$$147, -2, -4, 5, 3, -2, -4, 2$$

# Predictive coding

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More general versions of this idea may use predictors other than the previous signal; in this case, the difference between the predictor and the actual value are used for later signals.

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It is actually better to use

$$d_n = x_n - \hat{x}_{n-1}, \qquad d_n \to \hat{d}_n,$$

where $\hat{x}_{n-1}$ denotes the reconstructed version of $x_{n-1}$ based on $\hat{x}_0, \hat{d}_1, \ldots, \hat{d}_{n-1}$. In this order, the quantization error does not accumulate. (Essentially, the same information is used for compression and reconstruction.)