Cryptography

Coding Technology

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Coding Technology

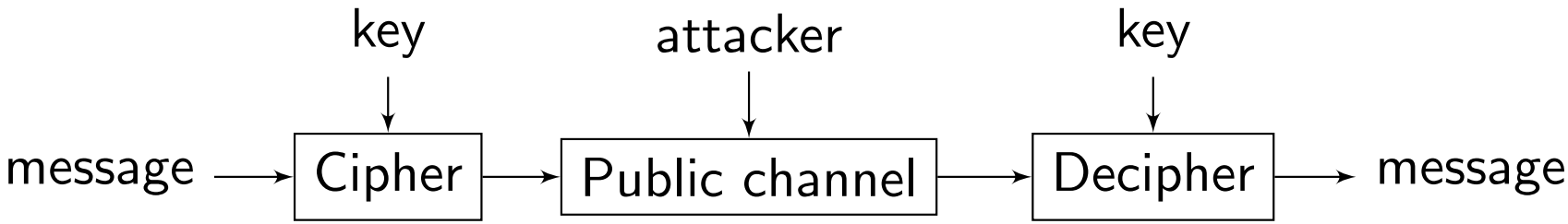
General setup.

During the Coding Technology course, our main focus is on various types of codes:

- Error correction codes adding redundancy to the messages that allow detection and/or correction of errors, allowing reliable communication over noisy channels. Also known as Error Control, or Channel Coding.
- Data compression codes identify patterns and eliminate redundancies in data. Also known as Source Coding.
- **Cryptography protocols** that turn readable information into unintelligible text, allowing secure private communication over public channels.

Cryptography

Main objective: secure communication over a public channel.



Construct algorithms that turn private information into unintelligible (obscure/nonsense) text (encryption) that can only be read by reversing the process (decryption).

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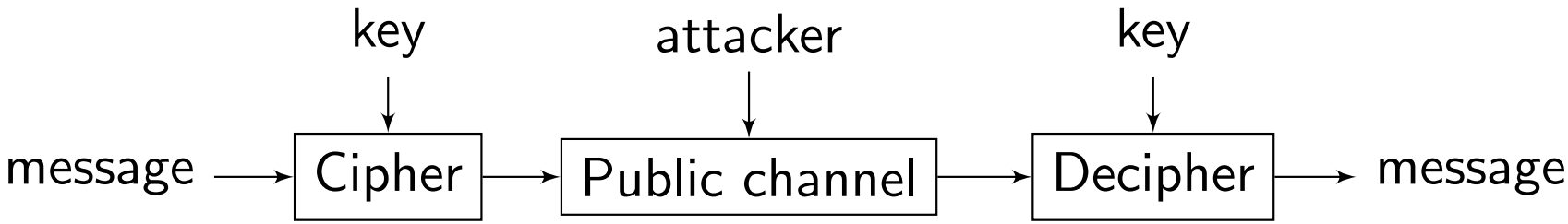

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(There are other applications, e.g. authentication, which are similar in spirit and use mostly similar techniques, but the setup is slightly different.)

Symmetric encryption

Symmetric encryption or *conventional* setup for secure communication:

- the key k is from a finite set of size K;
- $x = (x_1 \dots x_M)$: the original message or *plaintext*;
- $y = (y_1 \dots y_N)$: the encrypted message or *ciphertext*.

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For each k,

$$y = E_k(x)$$

where E_k is the encryption function, and

$$x = D_k(y)$$

where D_k is the decryption function, which is the inverse of E_k.

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We generally assume that the cryptography protocol used is fully known to all parties, except for the key, which is unknown to Eve.

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Passive attacks:

- ciphertext only attack: Eve has a series of $E_k(x_1), \ldots, E_k(x_L)$ ciphertexts encrypted with a common key k;

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Passive attacks usually exploit statistical patterns in either the plaintext or the ciphertext.

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- man-in-the-middle attack: getting between the communication of Alice and Bob and changing it in both directions;
- possibly other types of manipulation.

Additive cipher

Historical examples.

Additive cipher, also known as Caesar cipher. If the size of the alphabet is n (e.g. $n = 26$ for English texts),

$$E_k(x) = y = x + k \mod n,$$

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Decoding is

$$D_k(y) = x = y - k \mod n$$

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Just slightly better than additive ciphers, but still vulnerable to attacks.

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Attack it using a known plaintext attack.

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Not from a single equation, but if we put together several, then we get

$$Y = AX$$

where Y and X are matrices with columns $y_2 - y_1, y_3 - y_1, \ldots$ and $x_2 - x_1, x_3 - x_1, \ldots$ respectively.

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If X is $N \times N$ and invertible, then

$$A = YX^{-1}$$

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Two practical issues:

- some information is obtained about the key;
- the key needs to be changed frequently.

We will address both issues soon.

One-time pad

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One time pad (OTP): both the sender and the receiver have the same random bit sequence k; the encryption is bitwise addition of the message and the key.

Example:

$$\begin{array}{rcl} \mathrm{x} & = & 01001101 \\ +\mathrm{k} & = & 11010000 \\ \hline \mathrm{y} & = & 10011101 \end{array}$$

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Proof (sketch). The key must have entropy at least equal to entropy of the message; in other words, the key must be at least as long as the plaintext and at least as random as the plaintext.

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Computationally secure protocols are usually based on mathematical problems that are known to be computationally complex.

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NP-hard problems are typically good candidates, but even in that case, $\mathrm{P} \neq \mathrm{NP}$ is not actually proven, just a conjecture.

Computationally complex problems

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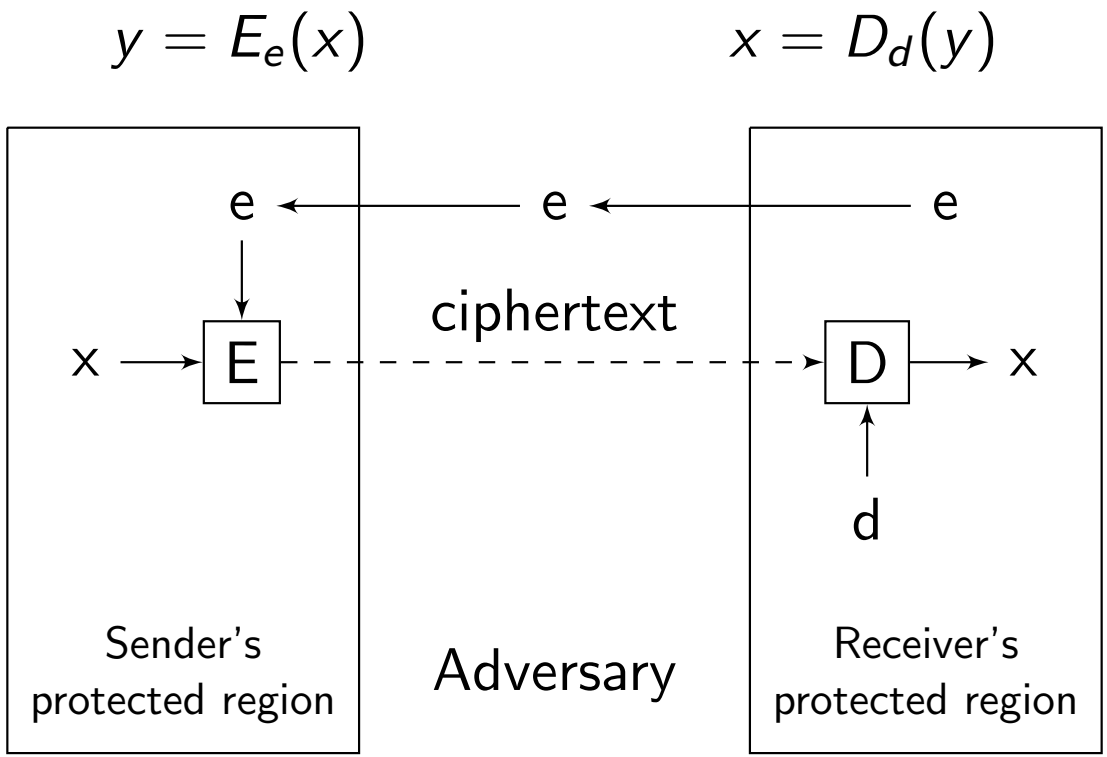
- Shortest vector problem. Given an n-dimensional lattice, find the shortest lattice vector. Used in post-quantum secure cryptosystems.

Public key cryptography

Instead of a common key k which is known by both the sender and the receiver, *public key cryptography* works the following way:

- the receiver has a $k = (d, e)$ pair of keys
- d is a private key known only by the receiver
- e is a public key known by everyone



RSA algorithm

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Key generation:

- select 2 large primes p and q; $n = pq$.
- $m = (p-1)(q-1)$.
- Select a coding exponent e so that $\gcd(e, m) = 1$ and $1 < e < m$.
- Solve $de = 1 \mod m$ to obtain the decoding key d.
- (n, e) is the public key;
- p, q, m and d are kept secret.

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Decryption:

- $x = y^d \mod n$.

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2. Encryption using the public key is computationally simple
3. Decryption using the private key is computationally simple
4. Encryption and decryption are indeed inverse operations
5. Attacking without the private key is computationally complex

RSA algorithm

1. Key generation is computationally simple:
   - Primality testing (checking whether a given number is a prime or not) is computationally simple.
   - There are many primes even among large numbers: the Prime Number Theorem states that among numbers of order N, on average 1 out of $\log(N)$ numbers is a prime.
   - So *finding* large prime numbers for p and q is easy: we can just start prime checking large numbers randomly, and we will soon find two large prime numbers.
   - gcd and $de = 1 \mod m$ can be solved fast using the Extended Euclidean Algorithm.

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Assume $a > b$; initialize $r_0 = a, r_1 = b$ and also $s_0 = 1, t_0 = 0, s_1 = 0, t_1 = 1$. In each step, we write

$$r_{k-1} = r_k \cdot q_{k+1} + r_{k+1} \qquad r_k = s_k \cdot a + t_k \cdot b,$$

where $0 \leq r_{k+1} < r_k$, and s_{k+1} and t_{k+1} are computed from

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The algorithm stops when $r_{k+1} = 0$; then $r_k = \gcd(a, b)$, and $\gcd(a, b) = s_k \cdot a + t_k \cdot b$.

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Assume $a > b$; initialize $r_0 = a, r_1 = b$ and also $s_0 = 1, t_0 = 0, s_1 = 0, t_1 = 1$. In each step, we write

$$r_{k-1} = r_k \cdot q_{k+1} + r_{k+1} \qquad r_k = s_k \cdot a + t_k \cdot b,$$

where $0 \leq r_{k+1} < r_k$, and s_{k+1} and t_{k+1} are computed from

$$s_{k+1} = s_{k-1} - q_k s_k, \qquad t_{k+1} = t_{k-1} - q_k t_k.$$

The algorithm stops when $r_{k+1} = 0$; then $r_k = \gcd(a, b)$, and $\gcd(a, b) = s_k \cdot a + t_k \cdot b$.

For $\gcd(n, e) = 1$, the algorithm gives $1 = \gcd(n, e) = s \cdot n + t \cdot e$, so $e^{-1} = t \bmod n$.

Extended Euclidean Algorithm

Compute the greatest common divisor (gcd) of $a = 8387$ and $b = 1243$, and also compute s and t so that

$$\gcd(8387, 1243) = s \cdot 8387 + t \cdot 1243.$$

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13 &= 2 \cdot 6 + 1
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2 &= 1 \cdot 2 + 0.
\end{aligned}$$

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Finally,

$$\gcd(8387, 1243) = 1 = -574 \cdot 8387 + 3873 \cdot 1243.$$

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Compute the multiplicative inverse of 1243 mod 8387.

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From the Extended Euclidean algorithm,

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this means

$$3873 \cdot 1243 = 1 \mod 8387,$$

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Theorem

The Extended Euclidean algorithm takes no more than $\log_{1.618}(\min(a, b))$ *steps.*

(No proof.)

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Any guess for what pairs of numbers it is slowest for, and where does the value 1.618 come from?

RSA algorithm

2-3. Encryption and decryption are computationally simple.

$x^e \bmod n$ (and $y^d \bmod n$) can be computed using *Fast modular exponentiation*:

- Write e in base 2 as

$$e = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_\ell}$$

  where $b_1 > b_2 > \cdots > b_\ell$ are integers.
- Compute $x^2 \bmod n, x^4 \bmod n, \ldots x^{2^{b_1}} \bmod n$ by taking the square of the previous term mod n in each step.
- Compute $x^e \bmod n$ as

$$x^e = x^{2^{b_1}} \cdot x^{2^{b_2}} \cdots x^{2^{b_\ell}} \mod n$$

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23^2 & = & 529 & = 5 \mod 131 \\
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23^{16} & = 101^2 = & 10201 & = 114 \mod 131
\end{array}
$$

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- $$23^{42} = 23^{32} \cdot 23^8 \cdot 23^2 = 27 \cdot 101 \cdot 5 = 11 \mod 131$$

RSA algorithm

4. Encryption and decryption are indeed inverse operations.

Let $\phi(n)$ be Euler's totient function:

$$\begin{aligned} n &= p_1^{k_1} \cdots p_r^{k_r} \\ \phi(n) &= p_1^{k_1-1}(p_1-1) \cdots p_r^{k_r-1}(p_r-1), \end{aligned}$$

specifically, if $n = pq$, then $\phi(n) = (p-1)(q-1)$.

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Theorem (Euler–Fermat)

If gcd(x, n)=1, then

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Theorem (Euler–Fermat)

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(No proof.)

For RSA, decryption and encryption are indeed inverse operations:

$$de = 1 \mod \phi(n) \quad \Longrightarrow \quad x^{de} = x \mod n.$$

RSA algorithm

5. Attacking without the private key is computationally complex.

Integer factorization is computationally difficult for large numbers. This is not a proven fact; it's just that no polynomial time algorithm is known. (Note that despite this, primality testing is computationally simple! That is, we can easily tell if a number is a prime or not, but if it is not a prime, we cannot factor it.)

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So even though n is public, p and q cannot be computed efficiently, and without p and q, $m = \phi(n)$ and d cannot be computed either. Overall, if p and q are sufficiently large, attacking RSA is computationally infeasible.

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(1991, RSA challenge to factorize 54 numbers; currently 23 has been factorized, the largest has 250 digits.)