Problems 1 - Binary Symmetric Channel, Block coding scheme, Binary linear codes

Coding Technology

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Problem 1

An input vector u is sent through a BSC with bit error probability $p_b = 0.01$.

(a) What is the error vector if the input vector is $u = (00100111)$ output vector is $v = (10100101)$?

(b) What is the conditional probability that the output vector is $v = (10100101)$, assuming that the input vector is $u = (00100111)$?

Solution.

(a) The error vector is input + output mod 2:

$$e = u \oplus v = (10000010)$$

(b)

$$P(v = (10100101) | u = (00100111)) =$$
$$p_b^2(1 - p_b)^6 = 0.01^2 \cdot 0.99^6 \approx 0.00009415.$$

Problem 2

A sequence of random bits has independent bits. The probability of 1 is $p_b = 0.03$, and the probability of 0 is $1 - p_b = 0.97$.

(a) What is the probability of the sequence 01000100?

(b) What is the probability that the number of 1's in an 8-bit sequence is exactly 2?

(c) What is the probability that the number of 1's in an 8-bit sequence is 2 or higher?

Solution.

(a)

$$\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0.97 \cdot & 0.03 \cdot & 0.97 \cdot & 0.03 \cdot & 0.97 \cdot & 0.03 \cdot & 0.97 \cdot & 0.03 \end{array} = 0.03^2 \cdot 0.97^6 \approx 0.00075.$$

Problem 2

Solution.

(b) The number of 8-bit sequences that contain exactly 2 ones is $\binom{8}{2}$, and each such sequence has probability equal to part (a), so

$$P(2 \text{ ones in an 8-bit sequence}) = \binom{8}{2} \cdot 0.03^2 \cdot 0.97^6 \approx 0.0210.$$

(c)

$$P(2 \text{ or more ones in an 8-bit sequence}) =$$
$$1 - 0.97^8 - \binom{8}{1} \cdot 0.03^1 \cdot 0.97^7 \approx 0.0223.$$

Problem 3

A code has codewords

$$10100, 01111, 11110, 00000.$$

(a) Calculate the n and k parameters of the code.
(b) What is the minimal Hamming distance between codewords?
(c) How many errors can the code detect? How many errors can the code correct?

Problem 3

Solution.

(a) The length of the codewords is $n = 5$, and the number of codewords is $2^k = 4$ (one for each message vector of length k), so $k = 2$. This is a $C(5, 2)$ code.

(b) Using pairwise comparison, the minimal Hamming distance is

$$d_{\min} = 2.$$

(c) A code with $d_{\min} = 2$ can detect

$$d_{\min} - 1 = 1$$

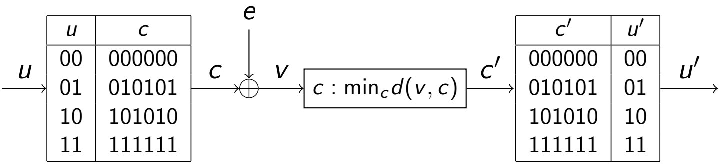
errors and correct

$$\left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = 0$$

errors.

Problem 4

We have the following coding scheme:



u	c
00	000000
01	010101
10	101010
11	111111

c'	u'
000000	00
010101	01
101010	10
111111	11

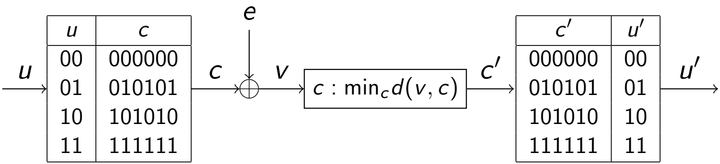
For $u = (11)$ and $e = (001000)$, determine the vectors c, v, c', u'.

Solution.

$$\begin{aligned} c &= (111111) \\ v &= c \oplus e = (111111) + (001000) = (110111), \\ c' &= \{c : \min_c d(v, c)\} = (111111) \\ u' &= (11) \end{aligned}$$

Problem 5

Use the same coding scheme:



u	c
00	000000
01	010101
10	101010
11	111111

$c : \min_c d(v, c)$

c'	u'
000000	00
010101	01
101010	10
111111	11

for $u = (01)$ and $e = (001011)$ to determine the vectors c, v, c', u'.

Solution.

$$\begin{aligned} c &= (010101) \\ v &= c \oplus e = (010101) + (001011) = (011110), \\ c' &= \{c : \min_c d(v, c)\} = (111111) \\ u' &= (11) \end{aligned}$$

Problem 6

For each of the following sets of codewords, give the appropriate (n, k, d) designation, where n is number of bits in each codeword, k is the number of message bits transmitted by each codeword and $d = d_{\text{min}}$ is the minimum Hamming distance between codewords. Also give the code rate.

(a) $\{111, 100, 010, 001\}$

(b) $\{00000, 01111, 10100, 11011\}$

(c) $\{00000\}$

Problem 6

Solution.

(a) $\{111, 100, 010, 001\}$

- $n = 3$ (the length of the codewords);
- $k = 2$ (the number of codewords is $4 = 2^k$);
- $d = d_{\min} = 2$ (from pairwise comparison).
- the code rate is $k/n = 2/3$.

(b) $\{00000, 01111, 10100, 11011\}$

$n = 5, k = 2, d = 2$, the code rate is $2/5$.

(c) $\{00000\}$

A bit of a trick question. $n = 5, k = 0$, d is undefined. The code rate is 0.

With only one codeword, no useful information can be transmitted since the receiver knows in advance what the codeword is going to be.

Problem 7

The Registrar has asked for an encoding of classes according to year ("Freshman", "Sophomore", "Junior", "Senior") that allows single error correction. Give an appropriate 5-bit binary encoding for each of the four years.

Solution. We need a $C(5,2)$ block code with $d_{\text{min}} = 3$.

We have seen one like that during lecture, that code is suitable for this problem too: $\{(00000), (00111), (11100), (11011)\}$.

Problem 8

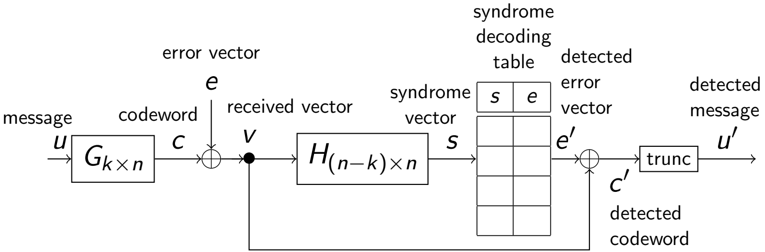
A perfect code with $n = 15$ corrects $t = 1$ error. What is the value of k?

Solution. For a perfect code, there is equality in the Hamming bound:

$$\begin{aligned} \sum_{i=0}^{t} \binom{n}{i} &= 2^{n-k} \\ n+1 &= 2^{n-k} \\ 16 &= 2^{15-k}, \end{aligned}$$

so $k = 11$.

The binary linear coding scheme



message u

$G_{k\times n}$

codeword c

error vector e

received vector v

$H_{(n-k)\times n}$

syndrome vector s

syndrome decoding table

s e

detected error vector e'

detected codeword c'

trunc

detected message u'

Problem 9

A systematic binary linear code has generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

(a) What are the n and k parameters of the code?

(b) What are the codewords of the code?

(c) Compute the parity check matrix H.

(d) What are the error detecting and correcting capabilities of the code?

(e) Compute the syndrome and error group of $e = (0100)$.

Problem 9

Solution.

(a) $n = 4, k = 2$.

(b) The codewords are

$$c_1 = \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

$$c_3 = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

$$c_4 = \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem 9

(c)

$$G = \left[\begin{array}{cc|cc} \underset{}{1} & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \multicolumn{2}{c}{I} & \multicolumn{2}{c}{B} \end{array}\right] \rightarrow H = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \multicolumn{2}{c}{B^T} & \multicolumn{2}{c}{I} \end{array}\right]$$

(d)

$$d_{\min} = \min_{c \neq (00...0)} w(c) = 2,$$

so the code can

- detect $d_{\min} - 1 = 1$ error, and
- correct $\lfloor (d_{\min} - 1)/2 \rfloor = 0$ errors.

Problem 9

(e) $s = e \cdot H^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

$E_{(10)} =$
$\{(0100), (0100) + (0110), (0100) + (1001), (0100) + (1111)\} =$
$\{(0100), (0010), (1101), (1011)\}$

(How can we tell from the error group that the code cannot correct 1 error?)

Problem 10

A systematic binary linear code is given by its codewords:

$$c^{(0)} = (000000), \quad c^{(1)} = (010101),$$
$$c^{(2)} = (101010), \quad c^{(3)} = (111111).$$

(a) What is the type of the code?

(b) Compute the error detection and error correction capabilities of this code.

(c) Compute the generator and parity check matrices.

(d) Compute the syndrome and the error group of $e = (010100)$. What is the group leader?

(e) Execute the coding and decoding for $u = (11)$ and $e = (010100)$.

Problem 10

$$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$$

(a) The type of the code is C(6,2) as the length of the codewords is $n = 6$ and the number of codewords is $4 = 2^k$, so $k = 2$.

(b) For linear codes,

$$d_{\min} = \min_{c \neq (00\ldots0)} w(c),$$

so the error detection capability of this code is $3 - 1 = 2$, and the code correction capability is

$$\left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = 1,$$

that is, the code can correct 1 error.

Problem 10

$$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$$

(c) Row 1 of the generator matrix G is $c^{(2)}$, and row 2 is $c^{(1)}$:

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The code is systematic, since the leftmost 2 × 2 block of G is the identity matrix.

$$G = \left[\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}\right].$$

Problem 10

$$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$$

(c)

$$G = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \underbrace{\boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}}}_{B}\right]$$

For systematic codes, the parity check matrix H can be written as

$$H = (B^T, I_{n-k}) = \left[\underbrace{\boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{array}}}_{B^T} \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Problem 10

The syndrome is

$$eH^T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned} E_{(0001)} = &\{(010100), (010100) + (010101), \\ &(010100) + (101010), (010100) + (111111)\} = \\ &\{(010100), \color{red}{(000001)}, (111110), (101011)\}. \end{aligned}$$

Problem 10

(e)

$$\begin{aligned}
u &= (11)\\
c &= (111111)\\
v &= c \oplus e = (111111) + (010100) = (101011),\\
s &= vH^T = (101011) \cdot \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\\
e' &= (000001)\\
c' &= v - e' = (101010)\\
u' &= (10)
\end{aligned}$$

The decoding was incorrect (due to $e = (010100)$ not being a group leader).

Problem 11

For a binary linear code, some of the message-codeword pairs are the following:

u	c
(001)	(001111)
(011)	(011100)
(110)	(110101)

(a) What is the type of the code (n and k parameters)?

(b) Determine the generator matrix G and parity check matrix H.

Problem 11

Solution.

(a) What is the type of the code (n and k parameters)?

$n = 6$ (length of codewords) and $k = 3$ (length of messages).

(b) Determine G and H.

We calculate the codewords corresponding to the messages (010) and (100). Since the code is linear,

$$(010)\cdot G = (011)\cdot G + (001)\cdot G = (011100) + (001111) = (010011),$$
$$(100)\cdot G = (110)\cdot G + (010)\cdot G = (110101) + (010011) = (100110).$$

Finally,

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 12

A binary linear code has parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(a) What are the n and k parameters of the code?

(b) Is the code systematic?

(c) Compute the generator matrix G.

(d) What are the codewords of the code?

(e) What are the error detecting and correcting capabilities of the code?

(f) Is this code perfect? Is it MDS?

Problem 12

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(a) What are the n and k parameters of the code?

The parity check matrix is $(n-k) \times n$, so $n = 7, n - k = 3$, and $k = 4$.

(b) Is the code systematic?

Yes, because the rightmost 3×3 block of H is the identity matrix.

(c) Compute the generator matrix G.

$$H = \left[\underbrace{\begin{matrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{matrix}}_{B} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \right] \rightarrow G = \left[\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}}_{B^T} \right]$$

Problem 12

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(d) What are the codewords of the code?

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \cdot G &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot G &= \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \cdot G &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \cdot G &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \cdot G &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \cdot G &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \\
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\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot G &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Problem 12

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(e) What are the error detecting and correcting capabilities of the code?

$d_{\text{min}} = \min_{c \neq (00...0)} w(c) = 3$, so the code can detect $3 - 1 = 2$ errors, and correct $\left\lfloor \frac{d_{\text{min}}-1}{2} \right\rfloor = 1$ error.

(f) Is this code perfect? Is it MDS?

$1 + \binom{n}{1} = 1 + 7 = 8 = 2^{7-4}$, so the code is perfect.

$d_{\text{min}} = 3 \neq n - k + 1 = 7 - 4 + 1 = 4$, so the code is not MDS.

(This code is known as the C(7,4) Hamming code.)