Waiting Times in BMAP/BMAP/1 Queues

MAM-9, Budapest

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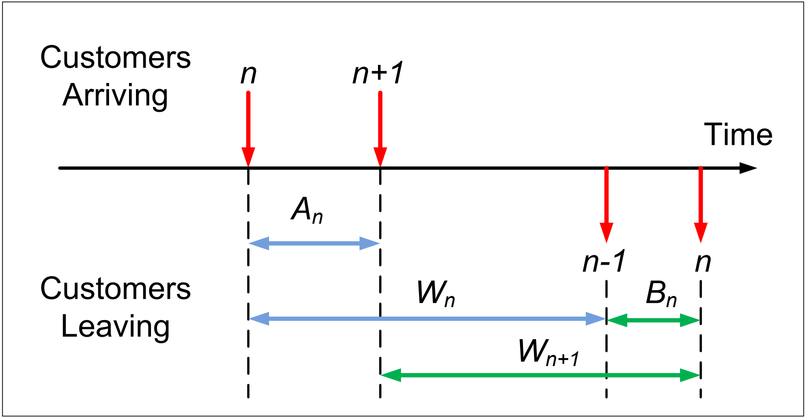
Continuous-Valued Lindley Process



Lindley Equation for Waiting Times

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- $\rho = E[B]/E[A] < 1$

ME Distribution

- The non-negative random variable $X \sim \mathrm{ME}(v, T, h, d)$ has a PDF $f_X(x)$

$$f_X(x) = ve^{Tx}h + d\,\delta(x) \tag{1}$$

  where
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- The MGF $g_X(s) = E[e^{-sX}]$ is rational

$$g_X(s) = \int_{0^-}^{\infty} e^{-sx} f_X(x)dx = v(s\mathbf{I} - T)^{-1}h + d \tag{2}$$

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- $E[X^i] = (-1)^{i+1}\, i!\, vT^{-(i+1)}h$

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- $\Delta_k = T_{k+1} - T_k$ (interarrival times: modulated process)

$$
\begin{aligned}
P\{X_{k+1} = j, T_{k+1} - T_k \leq t \,|\, X_0, \cdots, X_k = i; T_0, \ldots, T_k\} \\
&= P\{X_{k+1} = j, T_{k+1} - T_k \leq t \,|\, X_k = i\} \\
&= F_{ij}(t)
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- Semi-Markov Kernel

$$F(t) = \{F_{ij}(t)\}$$

Generation of the MRP

- We are in state i for the current arrival

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- With probability $F_{ij}(\infty)$, the discrete-time background process moves to state j associated with the next arrival
- The interarrival-time CDF is $F_{ij}(t)/F_{ij}(\infty)$.

MRP-ME Process

- An MRP with a kernel in ME form is an MRP-ME process

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- n gives us the size of the kernel or the order of the MRP-ME
- m gives us the the number of modes, i.e., mode count

MRP-ME Process Cont'd

- Kernel density $G(t)$

$$
\begin{aligned}
G(t) = \frac{d}{dt}F(t) &= Ve^{Tt}TS + (F + VS)\delta(t), t \geq 0 \\
&= Ve^{Tt}H + D\delta(t), t \geq 0
\end{aligned}
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- Laplace transform

$$G^*(s) = \int_{0^-}^{\infty} e^{-ts}G(t)dt = V(sI - T)^{-1}H + D$$

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- D parameter is crucial in modeling batch arrivals

MRP-ME Examples

- Phase-type renewal process X

$$\begin{bmatrix} T & t \\ 0 & 0 \end{bmatrix}, \text{initial vector}(v, \alpha), X \sim \text{MRP-ME}(v, T, t, \alpha)$$

  - $n = 1$ (order)
  - m (mode count)

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- Renewal process with ME-type inter-arrival times

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  - $n = 1$ (order)
  - m (mode count)
- Renewal process with ME-type inter-arrival times
  - same as PH-type with not necessarily probabilistic interpretation
  - $n = 1$, m arbitrary

MRP-ME Examples Cont'd

- Poisson process X with intensity λ

$$X \sim \text{MRP-ME}(1, -\lambda, \lambda, 0)$$

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- Poisson process X with intensity λ and geometric batch arrivals with parameter p

$$X \sim \text{MRP-ME}(p, -\lambda, \lambda, 1 - p)$$

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- Poisson process X with intensity λ

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- Poisson process X with intensity λ and geometric batch arrivals with parameter p

$$X \sim \text{MRP-ME}(p, -\lambda, \lambda, 1-p)$$

- Poisson process X with intensity λ and batch size of 2

$$X \sim \text{MRP-ME}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda, \begin{bmatrix} 0 & \lambda \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}).$$

MRP-ME Examples Cont'd

Modified Hyper-exponential Distribution

$$G(t) = p_1 \mu_1 e^{-\mu_1 t} + p_2 \mu_2 e^{-\mu_2 t} + (1 - p_1 - p_2)\delta(t)$$

$$G(t) = \underbrace{\begin{bmatrix} p_1 & p_2 \end{bmatrix}}_{V} \exp\left(\underbrace{\begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix}}_{T} t \right) \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}}_{H}$$

$$+ \underbrace{(1 - p_1 - p_2)}_{D} \delta(t)$$

MRP-ME Examples Cont'd

- MAP characterized with (D_0, D_1)

$$F(t) = (e^{D_0 t} - I)D_0^{-1} D_1, X \sim \text{MRP-ME}(I, D_0, D_1, 0) \quad (3)$$

- n gives both order and mode count

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- n gives both order and mode count
- RAP (Rational Arrival Process) generalizes MAP in the same way as ME distributions generalize PH-type distributions
- RAP still characterized with kernel of the form (3)

BMAP as an MRP-ME

- BMAP X with characterizing matrices $D_k, 0 \leq k \leq K$ of size m

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- BMAP X with characterizing matrices $D_k, 0 \le k \le K$ of size m
- $X \sim$ MRP-ME(V, T, H, D)
- $$V = \begin{bmatrix} I_{m\times m} \\ 0_{m(K-1)\times m(K-1)} \end{bmatrix}, H = \begin{bmatrix} D_1 & D_2 & \cdots & D_K \end{bmatrix},$$
$$T = D_0, D = \begin{bmatrix} 0_{m\times m(K-1)} & 0_{m\times m} \\ I_{m(K-1)\times m(K-1)} & 0_{m(K-1)\times m} \end{bmatrix}.$$

BMAP as an MRP-ME

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- order $= n = mK$, mode count$=m$

BMAP as an MRP-ME Cont'd

- In a BMAP, an event corresponding to the parameter matrix D_k is bound to a batch arrival with size k.
- In a GBMAP, we allow an event corresponding to $D_k, 1 \leq k \leq K$ to be a batch arrival corresponding to class-k traffic with discrete PH-type distribution with matrix pair (α_k, S) where the sub-stochastic matrix S of size $l \times l$ is shared by all classes.
- $X \sim \text{MRP-ME}(V, T, H, D)$

$$V = \begin{bmatrix} I_{m\times m} \\ 0_{ml\times m} \end{bmatrix}, H = \begin{bmatrix} \sum_{k=1}^K D_k \gamma_k & \sum_{k=1}^K \beta_k \otimes D_k, \end{bmatrix}$$

$$T = D_0, D = \begin{bmatrix} 0_{m\times m} & 0_{ml\times m} \\ s \otimes I_m & S \otimes I_m \end{bmatrix},$$

$$s = (I - S)1_{l\times 1}, \gamma_k = \alpha_k s, \beta_k = \alpha_k S.$$

Algorithm for Steady-state Waiting Time

Algorithm based on N. Akar, K. Sohraby, System-theoretical Algorithmic Solution to Waiting Times in Semi-Markov Queues, Performance Evaluation, vol. 66, no. 11, pp. 587-606, November 2009.

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- $A \sim$ MRP-ME(V_A, T_A, H_A, D_A), order n_A, mode count m_A
- $B \sim$ MRP-ME(V_B, T_B, H_B, D_B), order n_B, mode count m_B

Algorithm

1. Find the stationary vector of the underlying arrival process $\pi_A = \pi_A(-V_A T_A^{-1} H_A + D_A),\ \pi_A e = 1.$

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Algorithm

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2. Find the stationary vector of the underlying service process $\pi_B = \pi_B(-V_B T_B^{-1} H_B + D_B), \; \pi_B e = 1.$
3. Obtain $\tilde{\pi} = \pi_A \otimes \pi_B$

Algorithm for Steady-state Waiting Time Cont'd

4.

$$\begin{aligned} \tilde{V}_A &= V_A \otimes I_{n_B}, \\ \tilde{T}_A &= T_A \otimes I_{n_B}, \\ \tilde{H}_A &= H_A \otimes I_{n_B}, \\ \tilde{D}_A &= D_A \otimes I_{n_B}, \end{aligned}$$

Algorithm for Steady-state Waiting Time Cont'd

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\end{aligned}$$

5. $\tilde{D}_{AB} = (I - \tilde{D}_A \tilde{D}_B)^{-1}$ $\tilde{D}_{BA} = (I - \tilde{D}_B \tilde{D}_A)^{-1}$

Steady-state Solution for W_∞ Cont'd

6. Obtain the coupling matrix T_F:

$$T_F = \begin{bmatrix} -\tilde{T}_A - \tilde{H}_A\tilde{D}_B\tilde{D}_{AB}\tilde{V}_A & -\tilde{H}_A\tilde{D}_{BA}\tilde{V}_B \\ \tilde{H}_B\tilde{D}_{AB}\tilde{V}_A & \tilde{T}_B + \tilde{H}_B\tilde{D}_A\tilde{D}_{BA}\tilde{V}_B \end{bmatrix}$$

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7. Obtain

$$V_F = \begin{bmatrix} \tilde{D}_B \tilde{D}_{AB} \tilde{V}_A & \tilde{D}_{BA} \tilde{V}_B \end{bmatrix}$$

8.

$$H_F = \begin{bmatrix} -\tilde{H}_A \tilde{D}_{BA} \\ \tilde{H}_B \tilde{D}_A \tilde{D}_{BA} \end{bmatrix}.$$

Steady-state Solution for W_∞ Cont'd

9. Obtain the Schur decomposition of T_F:

$$Q_F^T T_F Q_F = \begin{bmatrix} T_{F,++} & T_{F,+-} \\ 0 & T_{F,--} \end{bmatrix},$$

$\sigma(T_{F,--}) \in$ Open Left Half Plane, $T_{F,--}$ is square of size $n_A m_B$.

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10. Partition Q_F as

$$Q_F = \begin{bmatrix} Q_{F,++} & Q_{F,+-} \\ Q_{F,-+} & Q_{F,--} \end{bmatrix}$$

11. Solve for x_0 and d_0 from $\begin{bmatrix} 0 & \tilde{\pi} \end{bmatrix} =$

$$\begin{bmatrix} x_0 & d_0 \end{bmatrix} \begin{bmatrix} Q_{F,+-} & Q_{F,+-} \, T_{F,--}^{-1} \, Q_{F,-}^T \, H_F \\ V_F Q_{F,-} & \left(-V_F \, Q_{F,-} \, T_{F,--}^{-1} \, Q_{F,-}^T \, H_F + \tilde{D}_{BA}\right) \end{bmatrix}$$

Steady-state Solution for W_∞ Cont'd

12. Obtain the waiting time ME-type density $g_W(t)$

$$f_W(t) = ve^{t\ T}h + d$$

where

$$\begin{aligned} v &:= x_0\, Q_{F,+-} + d_0\, V_F\, Q_{F,-} \\ T &:= T_{F,--} \\ h &:= Q_{F,-}^T\, H_F e \\ d &:= d_0\, \tilde{D}_{BA} e \end{aligned}$$

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13. $\mathcal{O}((n_A m_B + m_B n_A)^3)$

Conclusions

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Conclusions

- A numerically stable and efficient algorithm is already available for steady-state waiting times in MRP-ME/MRP-ME/1 queues
- Waiting time distribution can be obtained directly without any need to construct the embedded chain or obtain the G matrix
- The same algorithm can be used for (G)BMAP/(G)BMAP/1 queues
- The instrument to use is the so-called D parameter of MRP-MEs in modeling batches of arrivals (or services)

Future Work

- Recall the Lindley equation

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