# An algorithmic approach to branching processes with countably infinitely many types


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# Material of the talk

The material of this talk is taken from

S. Hautphenne, G. Latouche and G. Nguyen. Extinction probabilities of branching processes with countably infinitely many types. *Advances in Applied Probability*, 45(4) : 1068-1082, 2013.

and

P. Braunsteins, G. Decrouez, and S. Hautphenne. A pathwise iterative approach to the extinction of branching processes with countably many types. *arXiv preprint arXiv :1605.03069*, 2016.

# Multi-type Galton-Watson process

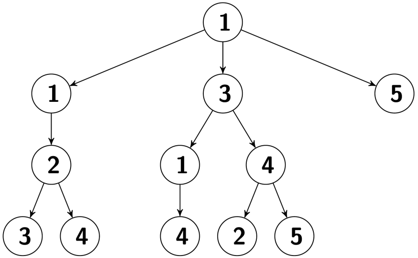
- Each individual has a type $i \in \mathcal{S} \equiv \mathbb{N}$
- The process initially contains a single individual of type $\varphi_0$
- Each individual lives for a single generation
- At death individuals of type $i$ have children according to the progeny distribution : $p_i(\mathbf{r}) : \mathbf{r} = (r_1, r_2, \ldots)$, where

  $p_i(\mathbf{r}) =$ probability that a type $i$ gives birth to $r_1$ children of type 1, $r_2$ children of type 2, etc.
- All individuals are independent

# Multi-type Galton-Watson process

Population size : $\mathbf{Z}_n = (Z_{n1}, Z_{n2}, \ldots), n \in \mathbb{N}$, where

$Z_{ni}$ : # of individuals of type $i$ at the $n$th generation




In this example $\mathbf{Z}_3 = (0, 1, 1, 2, 1, 0, 0, \ldots)$.

$\{\mathbf{Z}_n\}$ : $\infty$-dim Markov process with state space $(\mathbb{N}_0)^\infty$ and an absorbing state $\mathbf{0} = (0, 0, \ldots)$.

# Multi-type Galton-Watson process

Progeny generating vector $\mathbf{G}(\mathbf{s}) = (G_1(\mathbf{s}), G_2(\mathbf{s}), G_3(\mathbf{s}), \ldots)$, where $G_i(\mathbf{s})$ is the progeny generating function of an individual of type $i$

$$G_i(\mathbf{s}) = \sum_{\mathbf{r} \in (\mathbb{N}_0)^\infty} p_i(\mathbf{r})\, \mathbf{s}^{\mathbf{r}} = \sum_{\mathbf{r} \in (\mathbb{N}_0)^\infty} p_i(\mathbf{r}) \prod_{k=1}^{\infty} s_k^{r_k}, \qquad \mathbf{s} \in [0,1]^\infty$$

Mean progeny matrix $M$ with elements

$$M_{ij} = \left.\frac{\partial G_i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=\mathbf{1}}$$
$$= \text{expected number of direct offspring of type } j \text{ born to a parent of type } i$$

There is a path from type $i$ to $j$ $\Leftrightarrow$ there exists $\ell$ such that $(M^\ell)_{ij} > 0$.

# Global extinction probability

Global extinction probability vector $\mathbf{q} = (q_1, q_2, q_3, \ldots)$, with entries

$$q_i = \mathbb{P}\left[\lim_{n\to\infty} \mathbf{Z}_n = \mathbf{0} \,\middle|\, \varphi_0 = i\right]$$

The vector $\mathbf{q}$ is the (componentwise) minimal nonnegative solution of

$$\mathbf{s} = \mathbf{G}(\mathbf{s}), \qquad \mathbf{s} \in [0,1]^\infty$$

# Partial extinction probability

Partial extinction probability vector $\widetilde{\mathbf{q}} = (\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3, \ldots)$, with

$$\widetilde{q}_i = \mathbb{P}\left[\forall \ell : \lim_{n\to\infty} Z_{n\ell} = 0 \,\middle|\, \varphi_0 = i\right]$$

We have

$$\mathbf{0} \leq \mathbf{q} \leq \widetilde{\mathbf{q}} \leq \mathbf{1}.$$

The vector $\widetilde{\mathbf{q}}$ also satisfies the fixed point equation

$$\mathbf{s} = \mathbf{G}(\mathbf{s}), \qquad \mathbf{s} \in [0,1]^\infty$$

# Example 1

Suppose

$$p_1(\mathbf{r}) = \begin{cases} 1/6, & \mathbf{r} = 3\mathbf{e}_1 \\ 1/6, & \mathbf{r} = 3\mathbf{e}_2 \\ 2/3, & \mathbf{r} = \mathbf{0} \end{cases}$$

and

$$p_i(\mathbf{r}) = \begin{cases} 1/75, & \mathbf{r} = 3\mathbf{e}_{i-1} \\ 1/6, & \mathbf{r} = 3\mathbf{e}_i \\ 1/6, & \mathbf{r} = 3\mathbf{e}_{i+1} \\ 49/75, & \mathbf{r} = \mathbf{0} \end{cases}$$

for $i \geq 2$.

# Example 1

The mean progeny matrix has entries

$$M_{11} = M_{12} = 1/2$$

and

$$M_{i,i-1} = 1/25, \quad M_{i,i} = M_{i,i+1} = 1/2$$

for $i \geq 2$.

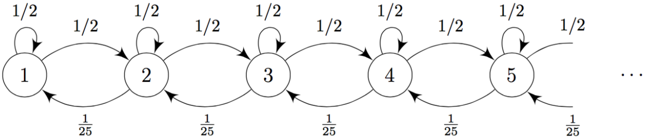
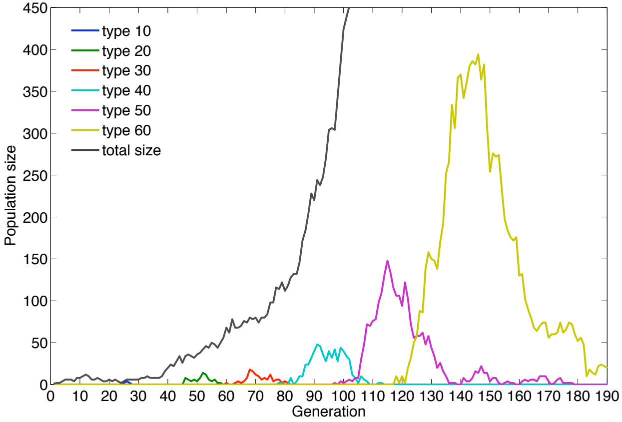



FIGURE : A graphical representation of the mean progeny matrix.

# Example 1




Question : How to compute $\mathbf{q}$ and $\tilde{\mathbf{q}}$ ?

# Example 1

The progeny generating vector, $\mathbf{G}(\mathbf{s})$, has the form

$$\begin{aligned}
G_1(\mathbf{s}) &= \frac{s_1^3}{6} + \frac{s_2^3}{6} + \frac{2}{3} \\
G_2(\mathbf{s}) &= \frac{s_1^3}{75} + \frac{s_2^3}{6} + \frac{s_3^3}{6} + \frac{49}{75} \\
&\vdots \\
G_i(\mathbf{s}) &= \frac{s_{i-1}^3}{75} + \frac{s_i^3}{6} + \frac{s_{i+1}^3}{6} + \frac{49}{75} \\
&\vdots
\end{aligned}$$

# Example 1

The fixed point equation, $\mathbf{s} = \mathbf{G}(\mathbf{s})$, is

$$
\begin{aligned}
s_1 &= \frac{s_1^3}{6} + \frac{s_2^3}{6} + \frac{2}{3} \\
s_2 &= \frac{s_1^3}{75} + \frac{s_2^3}{6} + \frac{s_3^3}{6} + \frac{49}{75} \\
&\vdots \\
s_i &= \frac{s_{i-1}^3}{75} + \frac{s_i^3}{6} + \frac{s_{i+1}^3}{6} + \frac{49}{75} \\
&\vdots
\end{aligned}
$$

# Example 1

Take the first $k$ elements of $\mathbf{G}(\mathbf{s})$

$$s_1 = \frac{s_1^3}{6} + \frac{s_2^3}{6} + \frac{2}{3}$$

$$s_2 = \frac{s_1^3}{75} + \frac{s_2^3}{6} + \frac{s_3^3}{6} + \frac{49}{75}$$

$$\vdots$$

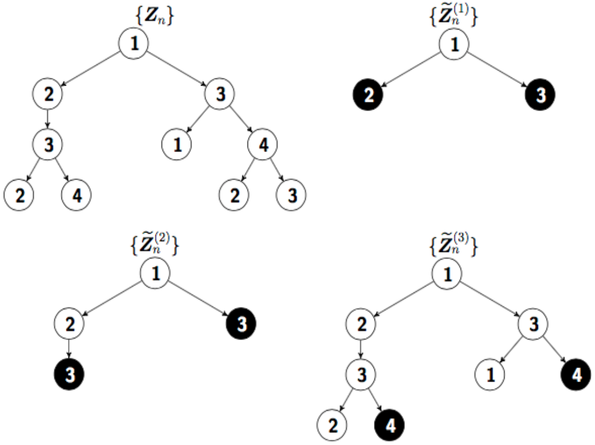
$$s_i = \frac{s_{i-1}^3}{75} + \frac{s_i^3}{6} + \frac{s_{i+1}^3}{6} + \frac{49}{75}$$

$$\vdots$$

$$s_k = \frac{s_{k-1}^3}{75} + \frac{s_k^3}{6} + \frac{s_{k+1}^3}{6} + \frac{49}{75}$$

# Computing $\widetilde{\mathbf{q}}$

Define $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$ by modifying $\{\mathbf{Z}_n\}$ such that all types $> k$ are sterile



# Computing $\widetilde{\mathbf{q}}$

- Denote $\widetilde{\mathbf{q}}^{(k)}$ : the (global) extinction probability of $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$

$$\widetilde{\mathbf{q}}^{(k)} \searrow \widetilde{\mathbf{q}} \quad \text{as } k \to \infty$$

- The proof is an application of the monotone convergence theorem

- For each $k$, $\widetilde{\mathbf{q}}^{(k)}$ can be computed, for instance using functional iteration

# Computing $\widetilde{\mathbf{q}}$

In Example 1 the progeny generating vector, $\tilde{\mathbf{G}}^{(k)}(\mathbf{s})$, is

$$\tilde{G}_1^{(k)}(\mathbf{s}) = \frac{s_1^3}{6} + \frac{s_2^3}{6} + \frac{2}{3}$$

$$\tilde{G}_2^{(k)}(\mathbf{s}) = \frac{s_1^3}{75} + \frac{s_2^3}{6} + \frac{s_3^3}{6} + \frac{49}{75}$$

$$\vdots$$

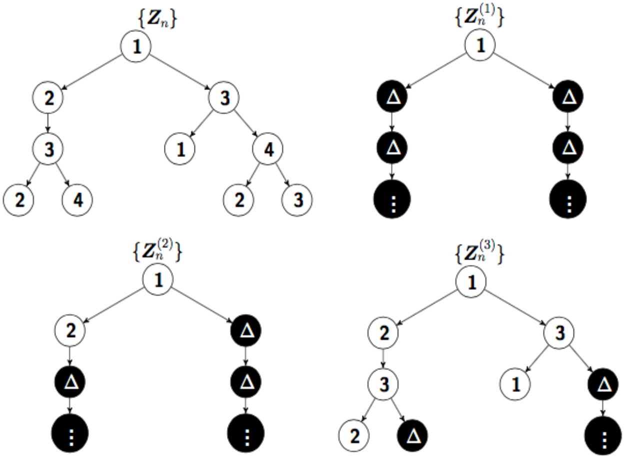
$$\tilde{G}_i^{(k)}(\mathbf{s}) = \frac{s_{i-1}^3}{75} + \frac{s_i^3}{6} + \frac{s_{i+1}^3}{6} + \frac{49}{75}$$

$$\vdots$$

$$\tilde{G}_k^{(k)}(\mathbf{s}) = \frac{s_{k-1}^3}{75} + \frac{s_k^3}{6} + \frac{1}{6} + \frac{49}{75}$$

## Computing $\mathbf{q}$

Define $\{\mathbf{Z}_n^{(k)}\}$ by modifying $\{\mathbf{Z}_n\}$ such that all types $> k$ are replaced by an immortal type $\Delta$



# Computing $\mathbf{q}$

- Denote $\mathbf{q}^{(k)}$ : the (global) extinction probability of $\{\mathbf{Z}_n^{(k)}\}$

$$\mathbf{q}^{(k)} \nearrow \mathbf{q} \quad \text{as } k \to \infty$$

- The proof is again an application of the monotone convergence theorem

- For each $k$, $\mathbf{q}^{(k)}$ can be computed, for instance using functional iteration

# Computing $\mathbf{q}$

In Example 1 the progeny generating vector, $\mathbf{G}^{(k)}(\mathbf{s})$, is

$$G_1^{(k)}(\mathbf{s}) = \frac{s_1^3}{6} + \frac{s_2^3}{6} + \frac{2}{3}$$

$$G_2^{(k)}(\mathbf{s}) = \frac{s_1^3}{75} + \frac{s_2^3}{6} + \frac{s_3^3}{6} + \frac{49}{75}$$

$$\vdots$$

$$G_i^{(k)}(\mathbf{s}) = \frac{s_{i-1}^3}{75} + \frac{s_i^3}{6} + \frac{s_{i+1}^3}{6} + \frac{49}{75}$$

$$\vdots$$

$$G_k^{(k)}(\mathbf{s}) = \frac{s_{k-1}^3}{75} + \frac{s_k^3}{6} + 0 + \frac{49}{75}$$

# Random replacement

Define $\{\bar{\mathbf{Z}}_n^{(k)}\}$ by modifying $\{\mathbf{Z}_n\}$ such that

- All types $> k$ are replaced by a type in $\{1, 2 \ldots, k\}$
- The types of the replaced individuals are selected independently using the probability distribution $\boldsymbol{\alpha}^{(k)} = \left(\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_3^{(k)}\right)$

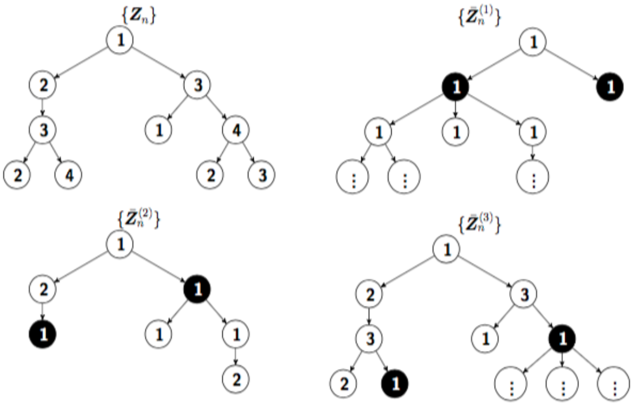
For example

- $\boldsymbol{\alpha}^{(k)} = \mathbf{e}_1$ : replacement by type 1
- $\boldsymbol{\alpha}^{(k)} = \mathbf{1}/k$ : replacement by a type uniformly distributed on $\{1, \ldots, k\}$
- $\boldsymbol{\alpha}^{(k)} = \mathbf{e}_k$ : replacement by type $k$

Denote $\bar{\mathbf{q}}^{(k)}$ : the (global) extinction probability of $\{\bar{\mathbf{Z}}_n^{(k)}\}$

# Random replacement

An illustration when $\alpha^{(k)} = \mathbf{e}_1$



# Random replacement

In Example 1 the progeny generating vector, $\bar{\mathbf{G}}^{(k)}(\mathbf{s})$, is

$$\bar{G}_1^{(k)}(\mathbf{s}) = \frac{s_1^3}{6} + \frac{s_2^3}{6} + \frac{2}{3}$$

$$\bar{G}_2^{(k)}(\mathbf{s}) = \frac{s_1^3}{75} + \frac{s_2^3}{6} + \frac{s_3^3}{6} + \frac{49}{75}$$

$$\vdots$$

$$\bar{G}_i^{(k)}(\mathbf{s}) = \frac{s_{i-1}^3}{75} + \frac{s_i^3}{6} + \frac{s_{i+1}^3}{6} + \frac{49}{75}$$

$$\vdots$$

$$\bar{G}_k^{(k)}(\mathbf{s}) = \frac{s_{k-1}^3}{75} + \frac{s_k^3}{6} + \frac{\left(\sum_{\ell=1}^{k} \alpha_\ell^{(k)} s_\ell\right)^3}{6} + \frac{49}{75}$$

# Random replacement

What conditions on $\{\mathbf{Z}_n\}$ and $\{\alpha^{(k)}\}$ are required for

$$\bar{\mathbf{q}}^{(k)} \to \mathbf{q}$$

as $k \to \infty$?

# Assumptions

**Assumption (1)**

$$\inf_{i \in \mathcal{S}} q_i > 0$$

**Assumption (2)**

*There exists constants $N_1, N_2 \geq 1$ and $a > 0$, all independent of $k$, such that*

$$\sum_{i=1}^{\min\{N_1, k\}} \alpha_i^{(k)} \geq a$$

*for all $k \geq N_2$.*

# Main result

**Theorem**

*Suppose Assumptions 1 and 2 hold. In addition, assume that there exists $N_1$ such that either*

- $\tilde{q}_j < 1$ *for all* $j \in \{1, \ldots, N_1\}$*, or*
- $\tilde{q}_j = 1$ *for all* $j \in \{1, \ldots, N_1\}$*, and there is a path from any* $j \in \{1, \ldots, N_1\}$ *to the initial type* $i$*.*

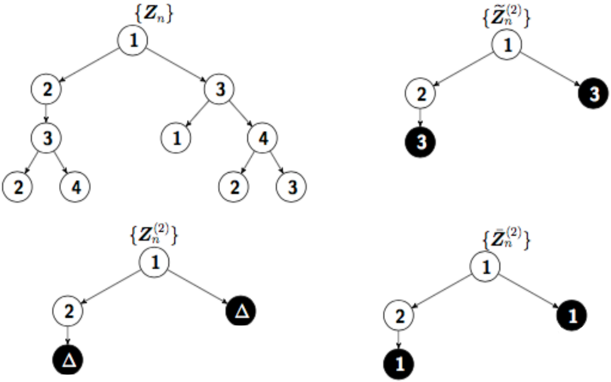
*Then*

$$\lim_{k \to \infty} \bar{q}_i^{(k)} \to q_i$$

*for any initial type* $i$*.*

# Coupling of the branching processes

We place $\{\mathbf{Z}_n\}$, $\{\mathbf{Z}_n^{(k)}\}$, $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$, and $\{\bar{\mathbf{Z}}_n^{(k)}\}$ on the same probability space, for all $k \geq 1$.



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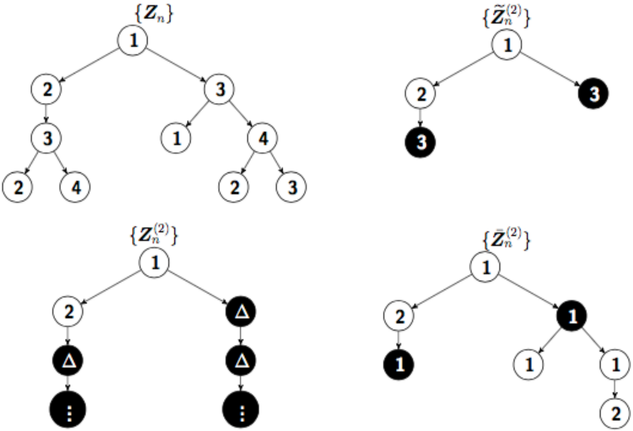
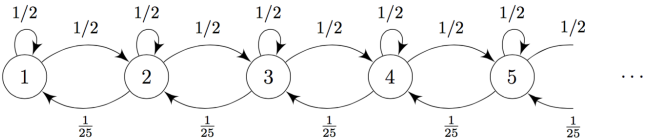
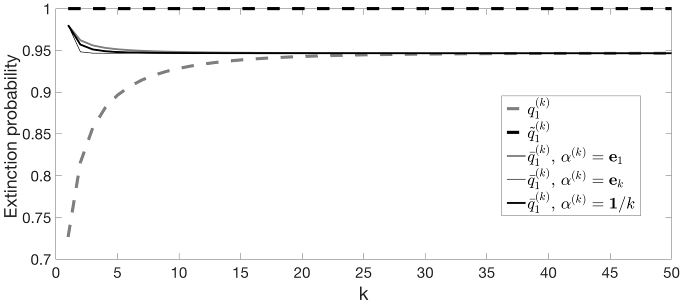


# Example 1


1/2
1
2
3
4
5
1/25
...


Extinction probability
k
1
0.95
0.9
0.85
0.8
0.75
0.7
0
5
10
15
20
25
30
35
40
45
50
$q_1^{(k)}$
$\tilde{q}_1^{(k)}$
$\bar{q}_1^{(k)}$, $\alpha^{(k)} = \mathbf{e}_1$
$\bar{q}_1^{(k)}$, $\alpha^{(k)} = \mathbf{e}_k$
$\bar{q}_1^{(k)}$, $\alpha^{(k)} = \mathbf{1}/k$

# Example 2

Consider the branching process with progeny generating function $\mathbf{G}(\mathbf{s})$ such that $a, c > 0$, $d > 1$ and

$$G_1(\mathbf{s}) = \frac{cd}{t}\, s_2^t + 1 - \frac{cd}{t},$$

and for $i \geq 2$,

$$G_i(\mathbf{s}) = \begin{cases} \dfrac{cd}{u}\, s_{i+1}^u + \dfrac{ad}{u}\, s_{i-1}^u + 1 - \dfrac{d(a+c)}{u} & \text{when } i \text{ is odd,} \\ \dfrac{c}{dv}\, s_{i+1}^v + \dfrac{a}{dv}\, s_{i-1}^v + 1 - \dfrac{(a+c)}{dv} & \text{when } i \text{ is even,} \end{cases}$$

where $t = \lceil dc \rceil + 1$, $u = \lceil d(c+a) \rceil + 1$ and $v = \lceil (c+a)/d \rceil + 1$.

# Example 2

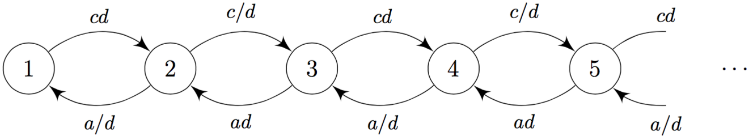
When $i \geq 2$ the mean progeny matrix $M$ has entries,

$$M_{i,i-1} = ad \quad \text{and} \quad M_{i,i+1} = cd$$

for $i$ odd and

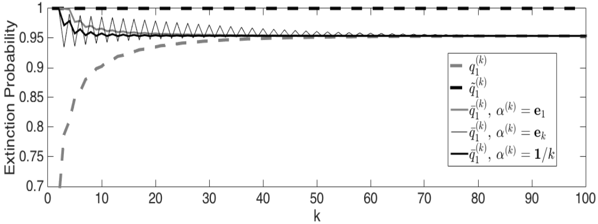
$$M_{i,i-1} = a/d \quad \text{and} \quad M_{i,i+1} = c/d$$

for $i$ even.

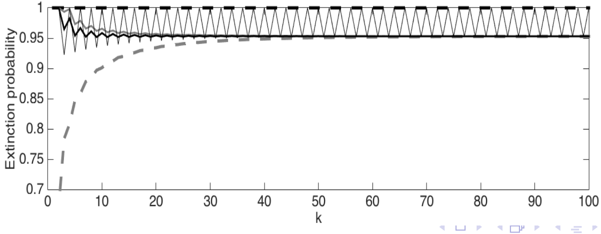


# Example 2

$a = 1/6,\ c = 7/8$ and $d^{-1} = 0.95$




$a = 1/6,\ c = 7/8$ and $d^{-1} = 0.93$



# Concluding remarks

- Example 2 demonstrates that when $\boldsymbol{\alpha}^{(k)} = \mathbf{e}_k$, $\lim_{k\to\infty} \bar{\mathbf{q}}^{(k)}$ does not necessarily exist
- For this example we can prove that when $\boldsymbol{\alpha}^{(k)} = \mathbf{e}_k$ for any $a, c > 0$ and $d > 1$,

$$\liminf_{k\to\infty} \bar{\mathbf{q}}^{(k)} = \mathbf{q}.$$

  Under Assumption 1 we believe this to be true in general.
- When $\boldsymbol{\alpha}^{(k)} = \mathbf{1}/k$, we can construct an example where $\mathbf{q} < \lim_{k\to\infty} \bar{\mathbf{q}}^{(k)} = \widetilde{\mathbf{q}}$.

Questions ?