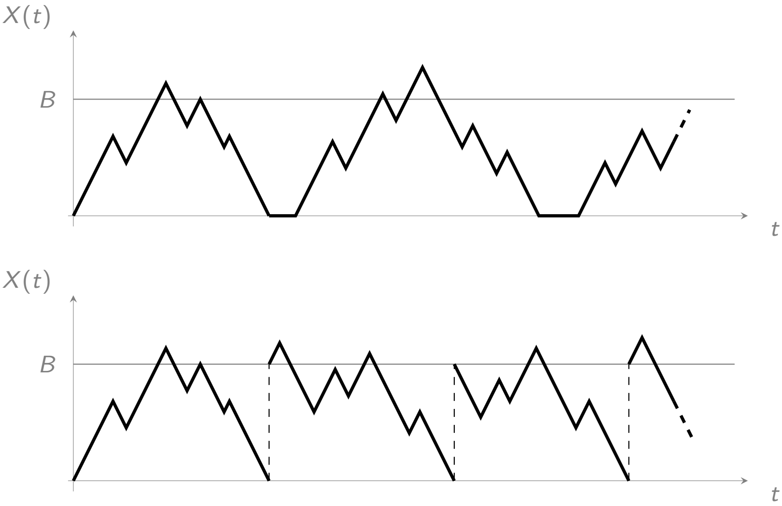
Fluid flows with jumps at the boundary

Eleonora Deiana
Guy Latouche Marie-Ange Remiche

Université de Namur

The Ninth International Conference on Matrix-Analytic Methods in Stochastic Models
Budapest, June 28 - 30, 2016

Classic fluid flow VS Fluid flow with jumps



Outline

Definition and notations: FLUID FLOW

Two-dimensional process:

$$\{X(t), \phi(t)\}_{t \geq 0}$$

- $X(t) \in \mathbb{R}^+$: level;
- $\phi(t) \in S = S^+ \cup S^-$: phase process.

Evolution of the level $X(t)$:

- $X(t) > 0$, $\phi(t) = i \in S$:

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = c_i$$

- $X(t) = 0$: instantaneous jump to a fixed level B.

Matrices

Transition and rate matrices:

$$T = \begin{bmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{bmatrix}, \text{ and } C = \begin{bmatrix} C_{+} & 0 \\ 0 & C_{-} \end{bmatrix}.$$

Matrix of the change of phases in the jump:

$$W = \begin{bmatrix} W_{-+} & W_{--} \end{bmatrix}.$$

OBJECTIVE

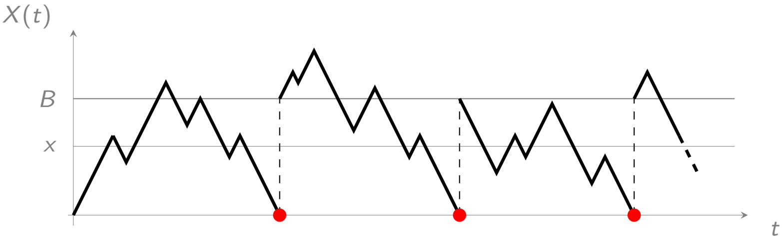
- Calculation of the stationary distribution:

$$\Pi_j(x) = \lim_{t\to\infty} P\left[X(t) < x, \phi(t) = j\right].$$

How?

REGENERATIVE APPROACH

Regenerative approach



Sequence of regeneration points $\{h_n\}_{n\geq 0}$ defined as:

$$h_0 = 0,$$
$$h_{n+1} = \inf\{t > h_n | X(t) = 0\}.$$

Stationary distribution:

$$\mathbf{\Pi}(x) = (\boldsymbol{\nu}\boldsymbol{m})^{-1}\boldsymbol{\nu}M(x).$$

- $\boldsymbol{\nu}$: stationary distribution of phases in the regeneration points:

$$\boldsymbol{\nu}H = \boldsymbol{\nu}, \quad \text{where}$$
$$H_{ij} = P[\phi(h_{n+1}) = j|\phi(h_n) = i], \quad i,j \in S^{-}.$$

- $M(x)$: mean sojourn time in $[0, x]$ between two regeneration points;
- $\boldsymbol{m} = M(B)\mathbf{1}$: mean sojourn time between two regenerative points given the phase of departure.

Stationary distribution:

$$\boldsymbol{\Pi}(x) = (\boldsymbol{\nu}\boldsymbol{m})^{-1}\boldsymbol{\nu}M(x).$$

- $\boldsymbol{\nu}$: stationary distribution of phases in the regeneration points:

$$\boldsymbol{\nu}H = \boldsymbol{\nu}, \quad \text{where}$$

$$H_{ij} = P[\phi(h_{n+1}) = j | \phi(h_n) = i], \quad i, j \in S^{-}.$$

- $M(x)$: mean sojourn time in $[0, x]$ between two regeneration points;
- $\boldsymbol{m} = M(B)\mathbf{1}$: mean sojourn time between two regenerative points given the phase of departure.

- $\boldsymbol{\nu}$: stationary distribution of phases in the regeneration points:

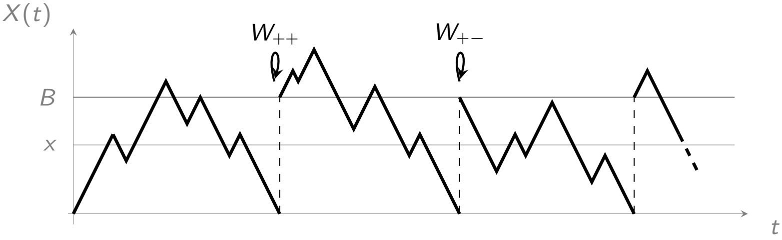
Transition matrix of phases between two regeneration points:

$$H = \begin{bmatrix} W_{-+} & W_{--} \end{bmatrix} \begin{bmatrix} \Psi \\ I \end{bmatrix} e^{Ub}.$$

Where:

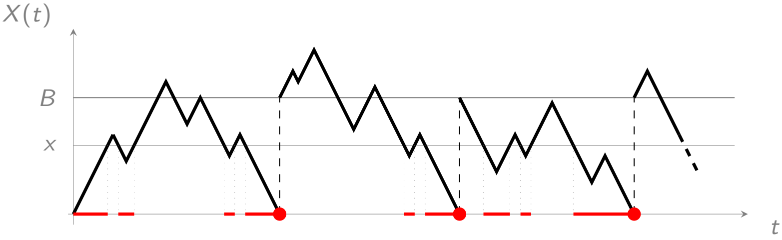
- Ψ: probability of the first return to the initial level,
- e^{Ux}: probability, starting from a fixed level x, to reach level 0 in a finite time.

$$H = \left[\begin{array}{cc} W_{-+} & W_{--} \end{array} \right] \left[\begin{array}{c} \Psi \\ I \end{array} \right] e^{Ub}.$$



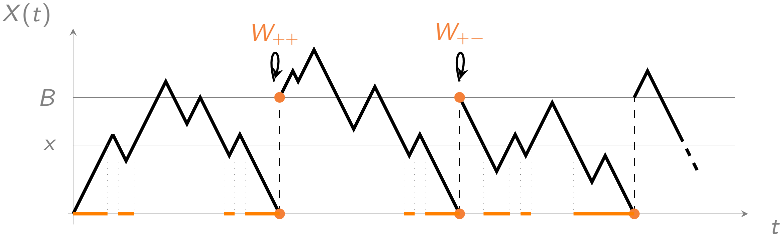
Mean sojourn time $M(x)$

$$M(x) = \begin{bmatrix} W_{-+} & W_{--} \end{bmatrix} \begin{bmatrix} \widetilde{M}_{+}(x) \\ \widetilde{M}_{-}(x) \end{bmatrix}.$$

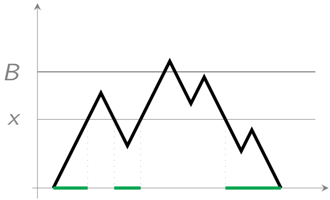


Mean sojourn time $M(x)$

$$M(x) = \begin{bmatrix} W_{-+} & W_{--} \end{bmatrix} \begin{bmatrix} \widetilde{M}_{+}(x) \\ \widetilde{M}_{-}(x) \end{bmatrix}.$$



$\Gamma(x)$: mean sojourn time in $[0, x]$ before the first return to the initial level, starting in level 0;

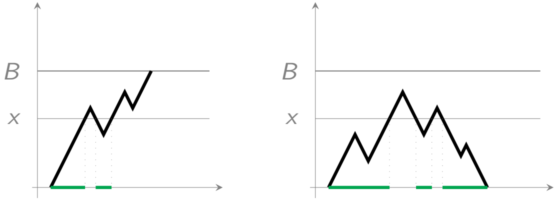


defined as:

$$\Gamma(x) = \int_0^x e^{Ku} du \left[\begin{array}{cc} C_+^{-1} & \Psi |C_-^{-1}| \end{array} \right],$$

with e^{Kx}: expected number of crossing of level x, starting from level 0 before the first return to the initial level.

$H_+^b(x)$: mean sojourn time in $[0, x]$ before reaching either level 0 or level B, starting in level 0;



Similarly...

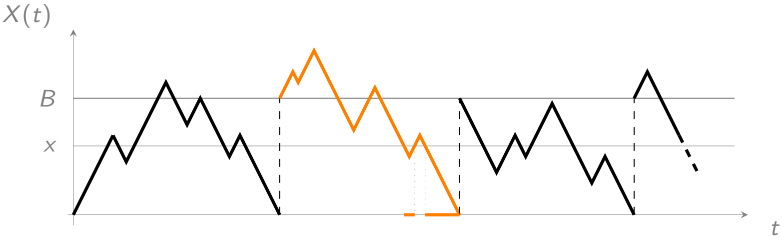
- $\widehat{\Gamma}(x)$: mean sojourn time in $[0, x]$ before the first return to the initial level, starting in level B;
- $H_{-}^{b}(x)$: mean sojourn time in $[0, x]$ before reaching either level 0 or level B, starting in level B.

These quantities can be putted together in the system:

$$\begin{bmatrix} \Gamma(x) \\ \widehat{\Gamma}(x) \end{bmatrix} = \begin{bmatrix} I & e^{Kb}\Psi \\ e^{\widehat{K}b}\widehat{\Psi} & I \end{bmatrix} \begin{bmatrix} H_{+}^{b}(x) \\ H_{-}^{b}(x) \end{bmatrix},$$

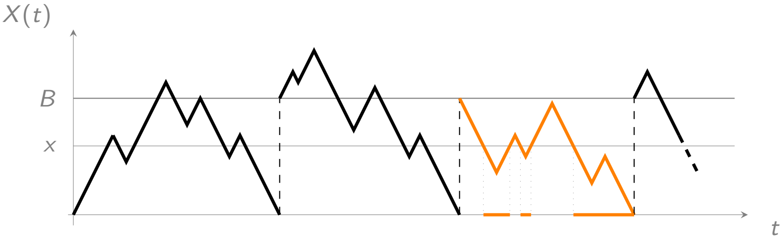
$0 < x < b$

$$\begin{cases} \widetilde{M}_{+}(x) = \Psi \widetilde{M}_{-}(x) \\ \widetilde{M}_{-}(x) = H_{-}^{b}(x) + \widehat{\Psi}^{b} \widetilde{M}_{+}(x) \end{cases}$$



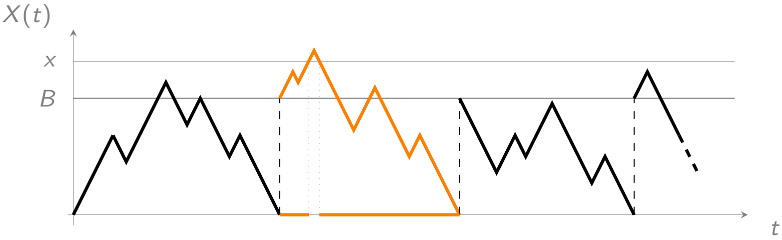
$0 < x < b$

$$\begin{cases} \widetilde{M}_+(x) = \Psi \widetilde{M}_-(x) \\ \widetilde{M}_-(x) = H_-^b(x) + \widehat{\Psi}^b \widetilde{M}_+(x) \end{cases}$$



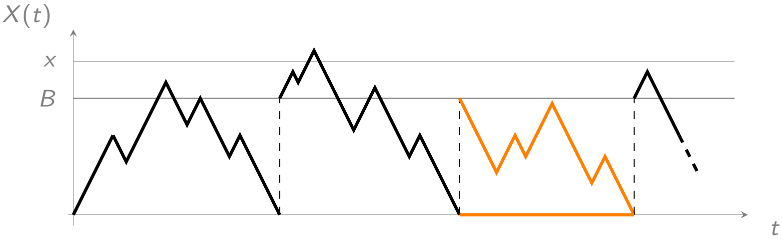
$x \geq b$

$$\begin{cases} \widetilde{M}_+(x) = \Gamma(x-b) + \Psi \widetilde{M}_-(x) \\ \widetilde{M}_-(x) = H_-^b(b) + \widehat{\Psi}^b \widetilde{M}_+(x). \end{cases}$$



$x \geq b$

$$\begin{cases} \widetilde{M}_+(x) = \Gamma(x-b) + \Psi \widetilde{M}_-(x) \\ \widetilde{M}_-(x) = H_-^b(b) + \widehat{\Psi}^b \widetilde{M}_+(x). \end{cases}$$



If $0 < x < b$:

$$\begin{cases} \widetilde{M}_{+}(x) = \Psi(I - \widehat{\Psi}^{b}\Psi)^{-1} H_{-}^{b}(x) \\ \widetilde{M}_{-}(x) = (I - \widehat{\Psi}^{b}\Psi)^{-1} H_{-}^{b}(x). \end{cases}$$

If $x \geq b$

$$\begin{cases} \widetilde{M}_{+}(x) = (I - \Psi\widehat{\Psi}^{b})^{-1} \left(\Gamma(x-b) + \Psi H_{-}^{b}(b)\right) \\ \widetilde{M}_{-}(x) = (I - \widehat{\Psi}^{b}\Psi)^{-1} \left(\widehat{\Psi}^{b}\Gamma(x-b) + H_{-}^{b}(b)\right). \end{cases}$$

Further work:

- random size of the jumps;
- jumps after a random interval of time;
- brownian motion.

Thank you for your attention!