Perturbation Analysis of Markov Modulated Fluid Models

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June 28, 2016

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Contents

1. Markov Modulated Fluid Models

Markov process $\{(Z(t), \varphi(t)) : t \in \mathbb{R}^+\}$

- $Z(t) \in \mathbb{R}^+$ is the continuous level
- $\varphi(t) \in \mathcal{S}$ is the phase : state of a discrete Markov chain with state space $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ and infinitesimal generator A,

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$$

where $A_{++} : \mathcal{S}_+ \rightsquigarrow \mathcal{S}_+$, $A_{+-} : \mathcal{S}_+ \rightsquigarrow \mathcal{S}_-$, $A_{-+} : \mathcal{S}_- \rightsquigarrow \mathcal{S}_+$, $A_{--} : \mathcal{S}_- \rightsquigarrow \mathcal{S}_-$

Evolution of the level, varies linearly according to the phase

$$\frac{d}{dt} Z(t) = \begin{cases} c_{\varphi(t)} & \text{if } Z(t) > 0 \\ \max\{0, c_{\varphi(t)}\} & \text{if } Z(t) = 0 \end{cases}$$

1. Markov Modulated Fluid Models

- Matrix of the rates

$$C = \mathrm{diag}(c_i : i \in \mathcal{S}) = \begin{bmatrix} C_+ & \\ & C_- \end{bmatrix}$$

- Joint distribution function of the level and the phase at time t

$$F_i(x, t) = \Pr[Z(t) \leq x, \varphi(t) = i]$$

- Stationary density $\boldsymbol{\pi}(x)$ has components

$$\pi_i(x) = \lim_{t \to \infty} \frac{\partial}{\partial x} F_i(x, t)$$

2. Perturbation Analysis

We perturb :

$$\text{1) the transition matrix : } A(\varepsilon) = A + \varepsilon\tilde{A}$$

$$\text{2) the rate matrix : } C(\varepsilon) = C + \varepsilon\tilde{C}$$

Objective :

$$\boldsymbol{\pi}(x,\varepsilon) = \boldsymbol{\pi}(x,0) + \varepsilon\boldsymbol{\pi}^{(1)}(x,0) + O\left(\varepsilon^2\right)$$

where

$$\boldsymbol{\pi}^{(1)}(x,0) = \lim_{\varepsilon\to 0} \frac{\boldsymbol{\pi}(x,\varepsilon) - \boldsymbol{\pi}(x,0)}{\varepsilon}$$

2. Perturbation Analysis

- For $x > 0$,

$$\pi(x) = \boldsymbol{p}_{-} A_{-+} e^{Kx} \left[\; C_{+}^{-1} \; \middle| \; \Psi \left| C_{-}^{-1} \right| \; \right]$$

where $\boldsymbol{p}_{-}$ is the unique solution of

$$\begin{cases} \boldsymbol{p}_{-} U = 0 \\ \boldsymbol{p}_{-} \mathbf{1} + \boldsymbol{p}_{-} A_{-+} (-K)^{-1} \left(C_{+}^{-1} \mathbf{1} + \Psi \left| C_{-}^{-1} \right| \mathbf{1} \right) = 1 \end{cases}$$

and

$$K = C_{+}^{-1} A_{++} + \Psi \left| C_{-}^{-1} \right| A_{-+}$$

(is such that $\exp(Kx)$ = matrix of expected number of crossings of level x given that the initial level is 0, before returning to 0)

$$U = \left| C_{-}^{-1} \right| A_{--} + \left| C_{-}^{-1} \right| A_{-+} \Psi$$

(is such that $\exp(Ux)$ = matrix of probabilities that the process reaches level 0 given that the initial level is x)

2. Perturbation Analysis

- The matrix of first return probabilities from above to the initial level Ψ has components

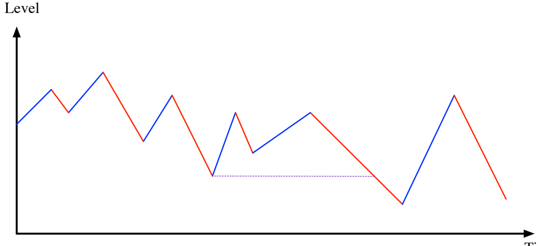
$$\Psi_{ij} = \Pr\left[\varphi(\tau(x)) = j | Z(0) = x, \varphi(0) = i\right]$$

where $i \in \mathcal{S}_+$, $j \in \mathcal{S}_-$, for $x \geq 0$ and

$$\tau(x) = \inf\left\{t > 0 : Z(t) \leq x\right\}$$

- Ψ is the minimal nonnegative solution of the Riccati equation

$$C_+^{-1}A_{+-} + C_+^{-1}A_{++}\Psi + \Psi\left|C_-^{-1}\right|A_{--} + \Psi\left|C_-^{-1}\right|A_{-+}\Psi = 0$$



2. Perturbation Analysis : $A(\varepsilon) = A + \varepsilon\tilde{A}$

Theorem

The matrix $\Psi(\varepsilon)$ is analytic in a neighbourhood of zero. Furthermore,

$$\Psi(\varepsilon) = \bar{\Psi} + \varepsilon\Psi^{(1)} + O(\varepsilon^2)$$

with $\lim_{\varepsilon\to 0}\Psi(\varepsilon) = \bar{\Psi} = \Psi$ *and* $\Psi^{(1)} = \left.\frac{d\Psi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}$ *is the unique solution of*

$$K\Psi^{(1)} + \Psi^{(1)}U = f(\Psi, \tilde{A})$$

where

$$\begin{aligned}
f(\Psi, \tilde{A}) &= -C_+^{-1}\tilde{A}_{+-} - C_+^{-1}\tilde{A}_{++}\Psi \\
&\quad -\Psi\left|C_-^{-1}\right|\tilde{A}_{--} - \Psi\left|C_-^{-1}\right|\tilde{A}_{-+}\Psi \\
K &= C_+^{-1}A_{++} + \Psi\left|C_-^{-1}\right|A_{-+} \\
U &= \left|C_-^{-1}\right|A_{--} + \left|C_-^{-1}\right|A_{-+}\Psi
\end{aligned}$$

2. Perturbation Analysis : $A(\varepsilon) = A + \varepsilon\tilde{A}$

Sketch of Proof

- Key: Implicit Function Theorem
- Define $F(\varepsilon, \mathcal{X})$ the continuous operator as

$$C_+^{-1}A_{+-}(\varepsilon)+C_+^{-1}A_{++}(\varepsilon)\mathcal{X}+\mathcal{X}\left|C_-^{-1}\right|A_{--}(\varepsilon)+\mathcal{X}\left|C_-^{-1}\right|A_{-+}(\varepsilon)\mathcal{X}$$

- One has : $F(0, \Psi(0)) = 0$ and for any Y, H

$$\partial_X F(\varepsilon, \mathcal{X})|_{\varepsilon=0,\mathcal{X}=\Psi(0)}(Y) = H$$

  is equivalently to

$$KY + YU = H$$

- From Rogers (1994) and Govorun *et al.* (2013), we have

$$\begin{aligned} \mathrm{sp}(K) &\in \{z : \mathrm{Re}(z) < 0\} \\ \mathrm{sp}(-U) &\in \{z : \mathrm{Re}(z) \geq 0\} \end{aligned}$$

- Conclusion: $\Psi(\varepsilon)$ is differentiable at 0.

2. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

Theorem

The matrix $\Psi(\varepsilon)$ of first return probabilities from above is analytic near zero and

$$\Psi(\varepsilon) = \bar{\Psi} + \varepsilon \Psi^{(1)} + O(\varepsilon^2)$$

where $\bar{\Psi} = \lim_{\varepsilon \to 0} \Psi(\varepsilon) = \Psi$ and $\Psi^{(1)}$ is the unique solution to

$$K\Psi^{(1)} + \Psi^{(1)} U = g(\Psi, \tilde{C})$$

where

$$\begin{aligned}
\Psi^{(1)} &= \left. \frac{d\Psi_{+-}(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} \\
g(\Psi, \tilde{C}) &= \Psi_{+-} \left| C_-^{-1} \right| \tilde{C}_- U - C_+^{-1} \tilde{C}_+ \Psi U \\
K &= C_+^{-1} A_{++} + \Psi \left| C_-^{-1} \right| A_{-+} \\
U &= \left| C_-^{-1} \right| A_{--} + \left| C_-^{-1} \right| A_{-+} \Psi
\end{aligned}$$

3. Markov Modulated Fluid Models with «Null Phases»

Markov process $\{(Z(t), \varphi(t)) : t \in \mathbb{R}^+\}$

- $Z(t) \in \mathbb{R}^+$ is the continuous level
- $\varphi(t) \in \mathcal{S}$ is the phase : state of a discrete Markov chain with state space $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$ and infinitesimal generator A,

$$A = \left[\begin{array}{c|c|c} A_{++} & A_{+0} & A_{+-} \\ \hline A_{0+} & A_{00} & A_{0-} \\ \hline A_{-+} & A_{-0} & A_{--} \end{array}\right]$$

- Matrix of the rates

$$C = \left[\begin{array}{c|c|c} C_+ & & \\ \hline & C_0 & \\ \hline & & C_- \end{array}\right]$$

3. Markov Modulated Fluid Models with «Null Phases»

Theorem

The matrix of first return probabilities to the initial level Ψ *with dimension* $|\mathcal{S}_+| \times |\mathcal{S}_-|$ *is the minimal nonnegative solution of the Riccati equation*

$$\Psi \left| C_-^{-1} \right| Q_{--} + \Psi \left| C_-^{-1} \right| Q_{-+} \Psi + C_+^{-1} Q_{+-} + C_+^{-1} Q_{++} \Psi = 0$$

where

$$\begin{bmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{bmatrix} = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix} + \begin{bmatrix} A_{+0} \\ A_{-0} \end{bmatrix} (-A_{00})^{-1} \begin{bmatrix} A_{0+} & A_{0-} \end{bmatrix}$$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

Case 1 Migration from S_0 *to* S_-

$$C = \left[\begin{array}{c|c|c} C_+ & & \\ \hline & C_0 & \\ \hline & & C_- \end{array}\right] \Longrightarrow C(\varepsilon) = \left[\begin{array}{c|cc} C_+(\varepsilon) & & \\ \hline & C_\ominus(\varepsilon) & \\ & & C_-(\varepsilon) \end{array}\right]$$

Partition of the *same* transition matrix *before* and *after* perturbation

$$A = \left[\begin{array}{c|c|c} A_{++} & A_{+0} & A_{+-} \\ \hline A_{0+} & A_{00} & A_{0-} \\ \hline A_{-+} & A_{-0} & A_{--} \end{array}\right] = \left[\begin{array}{c|cc} A_{++} & A_{+\ominus} & A_{+-} \\ \hline A_{\ominus+} & A_{\ominus\ominus} & A_{\ominus-} \\ A_{-+} & A_{-\ominus} & A_{--} \end{array}\right]$$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon\tilde{C}$

- For $\varepsilon > 0$, t $\boldsymbol{\Psi}(\varepsilon) = \begin{bmatrix} \Psi_{+\ominus}(\varepsilon) & \Psi_{+-}(\varepsilon) \end{bmatrix}$ may be written as

$$\boldsymbol{\Psi}(\varepsilon) = \bar{\Psi} + \varepsilon\Psi^{(1)} + O(\varepsilon^2)$$

is the minimal nonnegative solution of the Riccati equation

$$\begin{aligned}
&\boldsymbol{\Psi(\varepsilon)} \begin{bmatrix} \left|\varepsilon\tilde{C}_{\ominus}\right| & \\ & \left|C_{-} + \varepsilon\tilde{C}_{-}\right| \end{bmatrix}^{-1} \begin{bmatrix} A_{\ominus\ominus} & A_{\ominus -} \\ A_{-\ominus} & A_{--} \end{bmatrix} \\
&+\boldsymbol{\Psi}(\varepsilon) \begin{bmatrix} \left|\varepsilon\tilde{C}_{\ominus}\right| & \\ & \left|C_{-} + \varepsilon\tilde{C}_{-}\right| \end{bmatrix}^{-1} \begin{bmatrix} A_{\ominus +} \\ A_{-+} \end{bmatrix} \boldsymbol{\Psi}(\varepsilon) \\
&\qquad + \left(C_{+} + \varepsilon\tilde{C}_{+}\right)^{-1} \begin{bmatrix} A_{+\ominus} & A_{+-} \end{bmatrix} \\
&\qquad\qquad + \left(C_{+} + \varepsilon\tilde{C}_{+}\right)^{-1} A_{++}\boldsymbol{\Psi}(\varepsilon) \quad = 0
\end{aligned}$$

We have $\Psi^{(1)}_{+-}$ is the unique solution of $K\Psi^{(1)} + \Psi^{(1)}U = g(\Psi, \tilde{C})$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

Need to compare Ψ with domension $|\mathcal{S}_+| \times |\mathcal{S}_-|$ and $\boldsymbol{\Psi}(\varepsilon)$ with dimension $(|\mathcal{S}_+|) \times (|\mathcal{S}_\ominus| + |\mathcal{S}_-|)$:

$$\bar{\Psi} \quad = \left[\begin{array}{cc} \Psi_{+\ominus}(0) & \Psi_{+-}(0) \end{array} \right] = \quad \left[\begin{array}{cc} 0 & \Psi \end{array} \right]$$

where Ψ with dimension $|\mathcal{S}_+| \times |\mathcal{S}_-|$ is the minimal nonnegative solution of the Riccati equation

$$\Psi \left| C_-^{-1} \right| Q_{--} + \Psi \left| C_-^{-1} \right| Q_{-+} \Psi + C_+^{-1} Q_{+-} + C_+^{-1} Q_{++} \Psi = 0$$

where

$$\left[\begin{array}{cc} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{array} \right] = \left[\begin{array}{cc} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{array} \right] + \left[\begin{array}{c} A_{+0} \\ A_{-0} \end{array} \right] (-A_{00})^{-1} \left[\begin{array}{cc} A_{0+} & A_{0-} \end{array} \right]$$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

Case 1 Migration from S_0 *to* S_+

$$C = \left[\begin{array}{c|c|c} C_+ & & \\ \hline & C_0 & \\ \hline & & C_- \end{array}\right] \Rightarrow C(\varepsilon) = \left[\begin{array}{cc|c} C_+(\varepsilon) & & \\ & C_\oplus(\varepsilon) & \\ \hline & & C_-(\varepsilon) \end{array}\right]$$

Partition of the *same* transition matrix *before* and *after* perturbation

$$A = \left[\begin{array}{c|c|c} A_{++} & A_{+0} & A_{+-} \\ \hline A_{0+} & A_{00} & A_{0-} \\ \hline A_{-+} & A_{-0} & A_{--} \end{array}\right] = \left[\begin{array}{cc|c} A_{++} & A_{+\oplus} & A_{+-} \\ A_{\oplus+} & A_{\oplus\oplus} & A_{\oplus-} \\ \hline A_{-+} & A_{-\oplus} & A_{--} \end{array}\right]$$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon\tilde{C}$

- For $\varepsilon > 0$, $\boldsymbol{\Psi}(\varepsilon) = \begin{bmatrix} \Psi_{+-}(\varepsilon) \\ \Psi_{\oplus -}(\varepsilon) \end{bmatrix}$ may be written as

$$\boldsymbol{\Psi}(\varepsilon) = \bar{\Psi} + \varepsilon\Psi^{(1)} + O(\varepsilon^2)$$

It is the minimal nonnegative solution of

$$\begin{aligned}
&\boldsymbol{\Psi}(\boldsymbol{\varepsilon}) \left| C_- + \varepsilon\tilde{C}_- \right|^{-1} A_{--} \\
&+\boldsymbol{\Psi}(\varepsilon) \left| C_- + \varepsilon\tilde{C}_- \right|^{-1} \begin{bmatrix} A_{-+} & A_{-\oplus} \end{bmatrix} \boldsymbol{\Psi}(\varepsilon) \\
&+ \begin{bmatrix} C_+ + \varepsilon\tilde{C}_+ & \\ & \varepsilon\tilde{C}_\oplus \end{bmatrix}^{-1} \begin{bmatrix} A_{+-} \\ A_{\oplus -} \end{bmatrix} \\
&+ \begin{bmatrix} C_+ + \varepsilon\tilde{C}_+ & \\ & \varepsilon\tilde{C}_\oplus \end{bmatrix}^{-1} \begin{bmatrix} A_{++} & A_{+\oplus} \\ A_{\oplus +} & A_{\oplus\oplus} \end{bmatrix} \boldsymbol{\Psi}(\varepsilon) \quad = 0
\end{aligned}$$

We have : $\Psi^{(1)}_{+-}$ is the unique solution of
$K\Psi^{(1)} + \Psi^{(1)}U = g(\Psi, \tilde{C})$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

Substitute for the matrix of first return probabilities :

$$\bar{\Psi} = \begin{bmatrix} \lim_{\varepsilon \to 0} \Psi_{+-}(\varepsilon) \\ \lim_{\varepsilon \to 0} \Psi_{\oplus -}(\varepsilon) \end{bmatrix}$$

$$= \begin{bmatrix} \Psi \\ (-A_{\oplus\oplus})^{-1} A_{\oplus -} + (-A_{\oplus\oplus})^{-1} A_{\oplus +} \Psi \end{bmatrix}$$

Interpretation

For $i \in \mathcal{S}_\oplus, j \in \mathcal{S}_-$

- $\left((-A_{\oplus\oplus})^{-1} \tilde{C}_\oplus^{-1} A_{\oplus -}\right)_{ij}$ = probability that the phase process goes grom phase i to j, after some time spend in phases of $\mathcal{S}_\oplus$
- $\left((-A_{\oplus\oplus})^{-1} \tilde{C}_\oplus^{-1} A_{\oplus +} \Psi\right)_{ij}$ = probability the phase process leaves i for a phase in S_+ and later returns to the initial level in j

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon\tilde{C}$

General Case

$$C = \left[\begin{array}{c|c|c} C_{+} & & \\ \hline & C_0 & \\ \hline & & C_{-} \end{array}\right] \rightarrow C(\varepsilon) = \left[\begin{array}{cc|cc} C_{+}(\varepsilon) & & & \\ & C_{\oplus}(\varepsilon) & & \\ \hline & & C_{\ominus}(\varepsilon) & \\ & & & C_{-}(\varepsilon) \end{array}\right]$$

Partition of the *same* transition matrix *before* and *after* perturbation

$$A = \left[\begin{array}{c|c|c} A_{++} & A_{+0} & A_{+-} \\ \hline A_{0+} & A_{00} & A_{0-} \\ \hline A_{-+} & A_{-0} & A_{--} \end{array}\right] = \left[\begin{array}{cc|cc} A_{++} & A_{+\oplus} & A_{+\ominus} & A_{+-} \\ A_{\oplus+} & A_{\oplus\oplus} & A_{\oplus\ominus} & A_{\oplus-} \\ \hline A_{\ominus+} & A_{\ominus\oplus} & A_{\ominus\ominus} & A_{\ominus-} \\ A_{-+} & A_{-\oplus} & A_{-\ominus} & A_{--} \end{array}\right]$$

Here:

$$\boldsymbol{\Psi}(\varepsilon) = \begin{bmatrix} \Psi_{+\ominus}(\varepsilon) & \Psi_{+-}(\varepsilon) \\ \Psi_{\oplus\ominus}(\varepsilon) & \Psi_{\oplus-}(\varepsilon) \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} 0 & \Psi \\ \Psi_{\oplus\ominus} & \Psi_{\oplus-} \end{bmatrix}$$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

For $i \in S_\oplus$ and $j \in S_-$

$$
\begin{aligned}
[\Psi_{\oplus -}]_{ij} \; = \; & \left[-K_{\oplus\oplus}^{-1} \tilde{C}_\oplus^{-1} A_{\oplus -} \right]_{ij} \\
& + \left[-K_{\oplus\oplus}^{-1} \tilde{C}_\oplus^{-1} (-A_{\oplus\oplus})^{-1} A_{\oplus +} \Psi \right]_{ij} \\
& + \left[-K_{\oplus\oplus}^{-1} \Psi_{\oplus\ominus} \left| \tilde{C}_\ominus^{-1} \right| A_{\ominus -} \right]_{ij} \\
& + \left[-K_{\oplus\oplus}^{-1} \Psi_{\oplus\ominus} \left| \tilde{C}_\ominus^{-1} \right| A_{\ominus +} \Psi \right]_{ij}
\end{aligned}
$$

with

$$
K_{\oplus\oplus} = \tilde{C}_\oplus^{-1} A_{\oplus\oplus} + \Psi_{\oplus\ominus} \left| \tilde{C}_\ominus^{-1} \right| A_{\ominus\oplus}
$$

4. Perturbation Analysis : $C(\varepsilon) = C + \varepsilon \tilde{C}$

$\Psi_{\oplus\ominus}$ is the unique solution of the Riccati equation

$$\Psi_{\oplus\ominus} \left|C_{\ominus}^{-1}\right| A_{\ominus\ominus} + \Psi_{\oplus\ominus} \left|C_{\ominus}^{-1}\right| A_{\ominus\oplus} \Psi_{\oplus\ominus} + C_{\oplus}^{-1} A_{\oplus\ominus} + C_{\oplus}^{-1} A_{\oplus\oplus} \Psi_{\oplus\ominus} = 0$$

The block matrix $\Psi_{\oplus\ominus}$ is *the matrix of first return probabilities from above* for a Markov modulated fluid model with infinitesimal *sub*-generator

$$\begin{bmatrix} A_{\oplus\oplus} & A_{\oplus\ominus} \\ A_{\ominus\oplus} & A_{\ominus\ominus} \end{bmatrix}$$

and rate matrix

$$\begin{bmatrix} C_{\oplus} & \\ & C_{\ominus} \end{bmatrix}$$

Thanks for your attention !