On a 2-class Polling Model with Class-dependent Reneging, Switchover Times, and Phase-type Service

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Polling Models

- A typical polling model consists of multiple queues attended by a single server in cyclic order.
- Due to its wide use in the areas of public health systems, transportation, and communication and computer networks, polling models have drawn considerable attention over the past fifty years.

Notable References

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- H. Takagi, "Queueing analysis of polling models". *ACM Computing Surveys*, Vol. **20**, No. 1, pp. 5-28, 1988.
- V.M. Vishnevskii & O.V. Semenova, "Mathematical methods to study the polling systems". *Automation and Remote Control*, Vol. **67**, No. 2, pp. 173-220, 2006.
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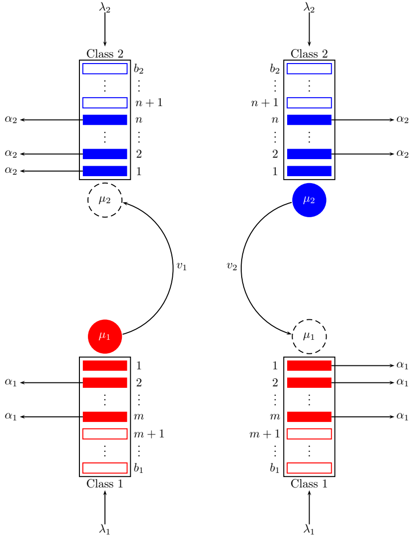
Proposed Queueing Model



- Poisson arrivals with rates λ_i
- Continuous phase-type service times
- FCFS, k_i-limited service discipline
- Exponential switchover times with rates v_i
- Exponential reneging times with rates α_i
- All distributions are independent
- Finite buffer sizes $b_i < \infty$

Proposed Queueing Model

- Service times for class-i customers, $i = 1, 2$, are assumed to follow a continuous phase-type distribution (of dimension s_i), with probability density function of the form

$$f_i(\omega) = \underline{\beta}_i \exp\{S_i\omega\}\underline{S}'_{0,i}, \; \omega > 0.$$

- Initial probability row vector is $\underline{\beta}_i = (\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,\mathrm{s}_i})$, $\sum_{j=1}^{\mathrm{s}_i} \beta_{i,j} = 1$.
- S_i is an $\mathrm{s}_i \times \mathrm{s}_i$ rate matrix and $\underline{S}'_{0,i} = -S_i\underline{e}'_{\mathrm{s}_i}$, where $\underline{e}'_{\mathrm{s}_i}$ is a column vector of s_i ones.

Steady-state Probabilities

- For $i = 1, 2$, let X_i represent the number of class-i customers present in the system, so that $0 \leq X_i \leq b_i$.
- Our first objective is to determine

$$\{P_{m,n}\,;\;\; m = 0, 1, \ldots, b_1,\;\; n = 0, 1, \ldots, b_2\},$$

where $P_{m,n}$ denotes the steady-state joint probability that $X_1 = m$ and $X_2 = n$.

- Define an associated quantity $\pi_{m,n,l,y}$ representing the steady-state joint probability that $X_1 = m$, $X_2 = n$, the server being in *position* l, and the current phase of service being y (with $y = 0$ indicating that the system is in switchover mode).

Steady-state Probabilities

- Dependent on m and n, component l takes on the following values:

$$\begin{array}{rcl} m = n = 0 & \Longrightarrow & l = k_1 + k_2 + 1, k_1 + k_2 + 2, \\ m \neq 0 \text{ and } n = 0 & \Longrightarrow & l = 1, 2, \ldots, k_1, k_1 + k_2 + 1, k_1 + k_2 + 2, \\ m = 0 \text{ and } n \neq 0 & \Longrightarrow & l = k_1 + 1, k_1 + 2, \ldots, k_1 + k_2, k_1 + k_2 + 1, \\ & & \quad k_1 + k_2 + 2, \\ m \neq 0 \text{ and } n \neq 0 & \Longrightarrow & l = 1, 2, \ldots, k_1, k_1 + 1, k_1 + 2, \ldots, k_1 + k_2, \\ & & \quad k_1 + k_2 + 1, k_1 + k_2 + 2. \end{array} \tag{1}$$

- When $l = 1, 2, \ldots, k_1$, the server is serving its l^{th} customer from the class-1 queue.
- When $l = k_1 + 1, k_1 + 2, \ldots, k_1 + k_2$, the server is serving its $(l - k_1)^{th}$ customer from the class-2 queue.
- When $l = k_1 + k_2 + i$, the server is conducting a switchover out of the class-i queue, $i = 1, 2$.

Steady-state Probabilities

- Similarly, component y depends on l in the following way:

$$\begin{aligned} l = 1, 2, \ldots, k_1 \quad &\Longrightarrow \quad y = \quad 1, 2, \ldots, s_1, \\ l = k_1 + 1, k_1 + 2, \ldots, k_1 + k_2 \quad &\Longrightarrow \quad y = \quad 1, 2, \ldots, s_2, \\ l = k_1 + k_2 + 1, k_1 + k_2 + 2 \quad &\Longrightarrow \quad y = \quad 0. \end{aligned} \tag{2}$$

Steady-state Probabilities

- When $m = n = 0$ (i.e., the queue is empty), the system can only be in one of two kinds of switchover modes (as there are no customers to serve in either queue) and so $P_{0,0} = \pi_{0,0,k_1+k_2+1,0} + \pi_{0,0,k_1+k_2+2,0}$.
- It follows that

$$P_{0,n} = \sum_{l=k_1+1}^{k_1+k_2} \sum_{y=1}^{s_2} \pi_{0,n,l,y} + \sum_{l=k_1+k_2+1}^{k_1+k_2+2} \pi_{0,n,l,0} \ , \ \ n \geq 1,$$

$$P_{m,0} = \sum_{l=1}^{k_1} \sum_{y=1}^{s_1} \pi_{m,0,l,y} + \sum_{l=k_1+k_2+1}^{k_1+k_2+2} \pi_{m,0,l,0} \ , \ \ m \geq 1,$$

and

$$P_{m,n} = \sum_{l=1}^{k_1} \sum_{y=1}^{s_1} \pi_{m,n,l,y} + \sum_{l=k_1+1}^{k_1+k_2} \sum_{y=1}^{s_2} \pi_{m,n,l,y} + \sum_{l=k_1+k_2+1}^{k_1+k_2+2} \pi_{m,n,l,0} \ , \ \ m,n \geq 1.$$

Steady-state Probabilities

- Define the 0^{th} steady-state probability row vector to be $\underline{\pi}_0 = (\underline{\pi}_{0,0}, \underline{\pi}_{0,1}, \ldots, \underline{\pi}_{0,b_2})$, where $\underline{\pi}_{0,0} = (\pi_{0,0,k_1+k_2+1,0}, \pi_{0,0,k_2+k_2+2,0})$, and $\underline{\pi}_{0,n}$, $n = 1, 2, \ldots, b_2$, is a row vector of size $z_1 = k_2 s_2 + 2$.
- For $m \geq 1$, the m^{th} steady-state probability row vector is defined as $\underline{\pi}_m = (\underline{\pi}_{m,0}, \underline{\pi}_{m,1}, \ldots, \underline{\pi}_{m,b_2})$, where $\underline{\pi}_{m,0}$ is a row vector of size $k_1 s_1 + 2$ and $\underline{\pi}_{m,n}$, $n = 1, 2, \ldots, b_2$, is a row vector of size $z_2 = k_1 s_1 + z_1$.
- Referring to X_1 as the *level* of the process, we remark that level 0 is comprised of $n_1 = b_2 z_1 + 2$ sub-levels, whereas each non-zero level consists of a total of $n_2 = b_2 z_2 + k_1 s_1 + 2$ sub-levels.

Steady-state Probabilities

- Let $\underline{\pi} = (\underline{\pi}_0, \underline{\pi}_1, \ldots, \underline{\pi}_{b_1})$ be the concatenated steady-state probability (row) vector having a total of $b_1 + 1$ levels.
- To determine $\underline{\pi}_m$ for $m \geq 0$, we need to solve $\tilde{\underline{0}} = \underline{\pi} Q$ where Q is the $(n_1 + b_1 n_2)$-dimensioned infinitesimal generator of the process and $\tilde{\underline{0}} = (\underline{0}_{n_1}, \underline{0}_{n_2}, \ldots, \underline{0}_{n_2})$ is an appropriately partitioned row vector (having a total of $b_1 + 1$ levels) such that $\underline{0}_{n_i}$ denotes a $1 \times n_i$ row vector of zeros.
- Q is block-structured as a level-dependent QBD process with blocks $Q_{m,j}$ containing all transitions where X_1 changes from m to j.

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b_1-2 \\ b_1-1 \\ b_1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & b_1-2 & b_1-1 & b_1 \end{array} \\ \begin{pmatrix} Q_{0,0} & Q_{0,1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_{2,1} & Q_{2,2} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & Q_{b_1-2,b_1-2} & Q_{b_1-2,b_1-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & Q_{b_1-1,b_1-2} & Q_{b_1-1,b_1-1} & Q_{b_1-1,b_1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & Q_{b_1,b_1-1} & Q_{b_1,b_1} \end{pmatrix} \end{array}$$

Building Q

- Note that $Q_{1,2} = Q_{2,3} = \cdots = Q_{b_1-1,b_1} = \lambda_1 I_{n_2}$ where I_{n_2} denotes the $n_2 \times n_2$ identity matrix.
- Define $\lambda = \lambda_1 + \lambda_2$.
- Define $\underline{e}_{i,j}$ to be a row vector of length i, with 1 as the j^{th} entry and zeros everywhere else.
- $\otimes$ denotes the Kronecker product operator, $\delta_{i,j}$ denotes the Kronecker delta function, and the prime symbol, $'$, denotes vector transpose.
- Define $\underline{v} = (v_1, v_2)$, $V = \text{diag}(\underline{v})$, $V_1 = v_1 \underline{e}'_{2,1} \underline{e}_{2,2}$, $V_2 = v_2 \underline{e}'_{2,2} \underline{e}_{2,1}$.
- Some select blocks of Q are as follows:

Building Q

$$Q_{0,0} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b_2-1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & b_2-1 & b_2 \end{array} \\ \begin{pmatrix} -(\lambda I_2 + V - V_1 - V_2) & \begin{bmatrix} \mathbf{0} & \lambda_2 I_2 \end{bmatrix} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix} \underline{e}'_{k_2}\underline{e}_{2,2} \otimes \underline{S}'_{0,2} \\ \alpha_2 I_2 \end{bmatrix} & \Delta_1 & \lambda_2 I_{z_1} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma_2 & \Delta_2 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \Delta_{b_2-1} & \lambda_2 I_{z_1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \Gamma_{b_2} & \Delta_{b_2} \end{pmatrix} \end{array},$$

$$\Gamma_j = \begin{bmatrix} (j-1)\alpha_2 I_{k_2 s_2} + U_2 & \underline{e}'_{k_2,k_2}\underline{e}_{2,2} \otimes \underline{S}'_{0,2} \\ \mathbf{0} & j\alpha_2 I_2 \end{bmatrix}, \quad U_i = \begin{cases} \mathbf{0} & \text{if } k_i = 1, \\ \begin{bmatrix} \underline{0}'_{k_i-1} & I_{k_i-1} \\ 0 & \underline{0}_{k_i-1} \end{bmatrix} \otimes \underline{S}'_{0,i}\underline{\beta}_i & \text{if } k_i \geq 2, \end{cases}$$

$$\Delta_j = \begin{bmatrix} -I_{k_2} \otimes ((\lambda - \lambda_2\delta_{j,b_2} + (j-1)\alpha_2)I_{s_2} - S_2) & \mathbf{0} \\ \underline{e}'_{2,1}\underline{e}_{k_2,1} \otimes v_1\underline{\beta}_2 & -((\lambda - \lambda_2\delta_{j,b_2} + j\alpha_2)I_2 + V - V_2) \end{bmatrix},$$

Building Q

$$Q_{i,i} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b_2-1 \\ b_2 \end{array} \begin{array}{cccccc} 0 & 1 & 2 & \dots & b_2-1 & b_2 \\ \hline C_{i,0} & \left[\begin{array}{cc} [\lambda_2 I_{k_1 s_1}, \underline{0}'_{k_1 s_1} \underline{0}_{k_2 s_2}] & \mathbf{0} \\ \mathbf{0} & \lambda_2 I_2 \end{array}\right] & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \left[\begin{array}{cc} \alpha_2 I_{k_1 s_1} & \mathbf{0} \\ \mathbf{0} & \underline{e}'_{k_2} \underline{e}_{2,2} \otimes \underline{S}'_{0,2} \\ \mathbf{0} & \alpha_2 I_2 \end{array}\right] & C_{i,1} & \lambda_2 I_{z_2} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & C_{i,2} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{i,b_2-1} & \lambda_2 I_{z_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & B_{b_2} & C_{i,b_2} \end{array}$$

for $i = 1, 2, \dots, b_1$,

$$B_j = \left[\begin{array}{cc} j\alpha_2 I_{k_1 s_1} & \mathbf{0} \\ \mathbf{0} & \Gamma_j \end{array}\right],$$

Building Q

$$C_{i,j} = \begin{cases} \begin{bmatrix} -I_{k_1} \otimes ((\lambda - \lambda_1 \delta_{i,b_1} + (i-1)\alpha_1) I_{s_1} - S_1) & \mathbf{0} \\ \underline{e}'_{2,2}\underline{e}_{k_1,1} \otimes v_2 \underline{\beta}_1 & -((\lambda - \lambda_1 \delta_{i,b_1} + i\alpha_1) I_2 + V - V_1) \end{bmatrix} & \text{if } j = 0, \\ \begin{bmatrix} \zeta_{1,i,j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \zeta_{2,i,j} & \mathbf{0} \\ \underline{e}'_{2,2}\underline{e}_{k_1,1} \otimes v_2 \underline{\beta}_1 & \underline{e}'_{2,1}\underline{e}_{k_2,1} \otimes v_1 \underline{\beta}_2 & -((\lambda - \lambda_1 \delta_{i,b_1} - \lambda_2 \delta_{j,b_2} + i\alpha_1 + j\alpha_2) I_2 + V) \end{bmatrix} & \text{if } j = 1, 2, \ldots, b_2, \end{cases}$$

$$\zeta_{x,i,j} = -I_{k_x} \otimes ((\lambda - \lambda_1 \delta_{i,b_1} - \lambda_2 \delta_{j,b_2} + (i - \delta_{x,1})\alpha_1 + (j - \delta_{x,2})\alpha_2) I_{s_x} - S_x),$$

Building Q

$$Q_{i,i-1} = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ b_2-1 \\ b_2 \end{array} \overset{\begin{array}{cccccc} 0 & 1 & 2 & \cdots & b_2-1 & b_2 \end{array}}{\begin{pmatrix} A_{i,0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{i,1} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{i,1} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{i,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & A_{i,1} \end{pmatrix}}, \quad i = 2, 3, \ldots, b_1,$$

$$A_{i,j} = \begin{bmatrix} (i-1)\alpha_1 I_{k_1 s_1} + U_1 & \underline{e}'_{k_1,k_1} \underline{e}_{k_2 s_2 \delta_{j,1}+2,\, k_2 s_2 \delta_{j,1}+1} \otimes \underline{S}'_{0,1} \\ \mathbf{0} & i\alpha_1 I_{k_2 s_2 \delta_{j,1}+2} \end{bmatrix}.$$

Calculating $\underline{\pi}$

- Level-dependent QBD processes are well-studied in the literature, and it is possible to develop a computational procedure for calculating the steady-state probabilities associated with our model.
- From $\underline{\tilde{0}} = \underline{\pi} Q$, the equilibrium equations in block form are obtained:

$$\underline{0}_{n_1} = \underline{\pi}_0 Q_{0,0} + \underline{\pi}_1 Q_{1,0}, \tag{3}$$

$$\underline{0}_{n_2} = \underline{\pi}_0 Q_{0,1} + \underline{\pi}_1 Q_{1,1} + \underline{\pi}_2 Q_{2,1}, \tag{4}$$

$$\underline{0}_{n_2} = \lambda_1 \underline{\pi}_{m-1} + \underline{\pi}_m Q_{m,m} + \underline{\pi}_{m+1} Q_{m+1,m}, \;\; m = 2, 3, \ldots, b_1 - 1, \tag{5}$$

$$\underline{0}_{n_2} = \lambda_1 \underline{\pi}_{b_1 - 1} + \underline{\pi}_{b_1} Q_{b_1, b_1}. \tag{6}$$

Calculating $\underline{\pi}$

- Solving equations (4) through (6) inductively yields

$$\underline{\pi}_m = \underline{\pi}_0 \prod_{j=1}^{m} \mathcal{S}_j, \;\; m = 1, 2, \ldots, b_1, \tag{7}$$

where the set of matrices $\{\mathcal{S}_j \, ; \; j = 1, 2, \ldots, b_1\}$ satisfy the recursive relation

$$\mathcal{S}_j = -\lambda_1 (Q_{j,j} + \mathcal{S}_{j+1} Q_{j+1,j})^{-1}, \; j = 2, 3, \ldots, b_1 - 1,$$

with

$$\mathcal{S}_{b_1} = -\lambda_1 Q_{b_1,b_1}^{-1} \;\; \text{and} \;\; \mathcal{S}_1 = -Q_{0,1}(Q_{1,1} + \mathcal{S}_2 Q_{2,1})^{-1}.$$

- Defining $\mathcal{S}_0 = Q_{0,0} + \mathcal{S}_1 Q_{1,0}$, equation (3) becomes

$$\underline{\pi}_0 \mathcal{S}_0 = \underline{0}_{n_1}. \tag{8}$$

Calculating $\underline{\pi}$

- Since all probabilities sum to 1, we must have that

$$\underline{\pi}_0 \underline{e}'_{n_1} + \underline{\pi}_0 \mathcal{S}_1 \underline{e}'_{n_2} + \underline{\pi}_0 \mathcal{S}_1 \mathcal{S}_2 \underline{e}'_{n_2} + \cdots + \underline{\pi}_0 \mathcal{S}_1 \mathcal{S}_2 \cdots \mathcal{S}_{b_1} \underline{e}'_{n_2} = 1. \tag{9}$$

- Factoring out $\underline{\pi}_0$ and defining the column vector

$$\underline{u}' = \underline{e}'_{n_1} + \sum_{m=1}^{b_1} \prod_{j=1}^{m} \mathcal{S}_j \underline{e}'_{n_2},$$

equations (8) and (9) give rise to the following system of linear equations which must be solved to determine $\underline{\pi}_0$:

$$\underline{\pi}_0 \left[\begin{array}{cc} \mathcal{S}_0 & \underline{u}' \end{array} \right] = (\underline{0}_{n_1}, 1). \tag{10}$$

- In equation (10), $(\underline{0}_{n_1}, 1)$ represents the concatenated row vector of size $n_1 + 1$.

Calculating $\underline{\pi}$

- Once $\underline{\pi}_0$ is determined, $\underline{\pi}_m, m \geq 1$, is obtained via equation (7).
- Having calculated the steady-state probabilities, the blocking probabilities for each class can be defined:

$$P_{b_1,\bullet} = \sum_{j=0}^{b_2} P_{b_1,j}$$

$$P_{\bullet,b_2} = \sum_{m=0}^{b_1} P_{m,b_2}$$

- These correspond to the probabilities of a class-1 or class-2 customer being turned away at entry (and subsequently lost) due to their class queue being full.
- These values are particularly useful in selecting buffer sizes b_1 and b_2 so as to ensure negligible blocking probabilities are obtained for both queues.

Nominal Waiting Time

- For $i = 1, 2$, let W_i represent the duration of time from the (successful) arrival of an arbitrary class-i customer to the system until the server is reached, referred to as the *nominal* class-i waiting time.
- Without loss of generality, we focus our analysis only on W_1 as the characteristics of the two queues are essentially indifferent.
- Define the modified steady-state probabilities

$$\phi_{0,0,l,0} = \frac{\pi_{0,0,l,0}}{1 - P_{b_1,\bullet}} \quad \text{and} \quad \phi_{m,n,l,y} = \frac{\pi_{m,n,l,y}}{1 - P_{b_1,\bullet}},$$

where $m = 1, 2, \ldots, b_1 - 1$, $n = 1, 2, \ldots, b_2$, and the components l and y are as defined in equations (1) and (2), respectively.

Nominal Waiting Time

- Several row vectors are required in the subsequent analysis such as:

$$\underline{\phi}_{0,n} = \frac{\underline{\pi}_{0,n}}{1 - P_{b_1,\bullet}},\ 1 \leq n \leq b_2\,,$$

$$\underline{\phi}_{m,0} = \frac{\underline{\pi}_{m,0}}{1 - P_{b_1,\bullet}},\ 1 \leq m \leq b_1 - 1\,,$$

$$\underline{\phi}_{m,n} = \frac{\underline{\pi}_{m,n}}{1 - P_{b_1,\bullet}},\ 1 \leq m \leq b_1 - 1, 1 \leq n \leq b_2\,.$$

- Furthermore, let

$$\underline{\phi}_0 = (\phi_{0,0,k_1+k_2+1,0}, \phi_{0,0,k_1+k_2+2,0}, \underline{\phi}_{0,1}, \underline{\phi}_{0,2}, \ldots, \underline{\phi}_{0,b_2})$$

and

$$\underline{\phi}_m = (\underline{\phi}_{m,0}, \underline{\phi}_{m,1}, \ldots, \underline{\phi}_{m,b_2}),\ m = 1, 2, \ldots, b_1 - 1\,.$$

Nominal Waiting Time

- By constructing

$$\underline{\Phi} = (\underline{\phi}_{b_1-1}, \underline{\phi}_{b_1-2}, \ldots, \underline{\phi}_1, \underline{\phi}_0) \tag{11}$$

to be the concatenated row vector of dimension

$$\ell = (b_1 - 1)n_2 + n_1, \tag{12}$$

we note that $\underline{\Phi}\, \underline{e}'_\ell = 1$.
- Upon successful entry into one of the ℓ possible busy states, the *PASTA property* ensures that our Poisson-arriving class-1 customer finds the system in state (m, n, l, y) with probability $\phi_{m,n,l,y}$.
- For now, we assume that the target class-1 customer is **not subject to reneging**.
- While waiting in the class-1 queue, the number of customers in the class-2 queue potentially changes, as well as the indicator on the server which identifies how many customers have completed service in the active serving queue.

Nominal Waiting Time

- As the number of customers in the class-1 queue changes, the ones arriving later have no impact on the waiting time of the target class-1 customer.
- If we effectively think of the arrival rate for the class-1 queue to be equal to 0, the distribution of W_1 can be modelled as the distribution of the time to absorption in a Markov chain with infinitesimal generator of the form

$$\begin{bmatrix} \mathcal{R} & -\mathcal{R}\underline{e}_\ell' \\ \underline{0}_\ell & 0 \end{bmatrix},$$

where

$$\mathcal{R} = \begin{array}{c} \\ b_1-1 \\ b_1-2 \\ b_1-3 \\ \vdots \\ 2 \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{ccccccc} b_1-1 & b_1-2 & b_1-3 & \cdots & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} \widetilde{Q}_{b_1-1,b_1-1} & Q_{b_1-1,b_1-2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{Q}_{b_1-2,b_1-2} & Q_{b_1-2,b_1-3} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{Q}_{b_1-3,b_1-3} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \widetilde{Q}_{2,2} & Q_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \widetilde{Q}_{1,1} & \widetilde{Q}_{1,0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \widetilde{Q}_{0,0} \end{pmatrix} \end{array}. \tag{13}$$

Nominal Waiting Time

- In equation (13), $Q_{2,1}, Q_{3,2}, \ldots, Q_{b_1-1,b_1-2}$ are the same matrices defined earlier and $\widetilde{Q}_{m,m} = Q_{m,m} + \lambda_1 I_{n_2}, m = 1, 2, \ldots, b_1 - 1$.
- In addition,

$$\tilde{Q}_{1,0} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b_2 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \ldots & b_2 \end{array} \\ \begin{pmatrix} \begin{bmatrix} \underline{e}'_{k_1,k_1}\underline{e}_{2,1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_2 \end{bmatrix} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \underline{e}'_{k_1,k_1}\underline{e}_{z_1,z_1-1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_{z_1} \end{bmatrix} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix} \underline{e}'_{k_1,k_1}\underline{e}_{z_1,z_1-1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_{z_1} \end{bmatrix} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \begin{bmatrix} \underline{e}'_{k_1,k_1}\underline{e}_{z_1,z_1-1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_{z_1} \end{bmatrix} \end{pmatrix} \end{array}$$

Nominal Waiting Time

- Also,

$$\tilde{Q}_{0,0} = \begin{array}{c} \\ 0 \\ 1 \\ \\ 2 \\ \vdots \\ b_2-1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & b_2-1 & b_2 \end{array} \\ \begin{pmatrix} -(\lambda_2 I_2 + V - V_1) & \left[\begin{array}{cc} \mathbf{0} & \lambda_2 I_2 \end{array}\right] & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \left[\begin{array}{c} \underline{e}'_{k_2}\underline{e}_{2,2} \otimes \underline{S}'_{0,2} \\ \alpha_2 I_2 \end{array}\right] & \tilde{\Delta}_1 & \lambda_2 I_{z_1} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma_2 & \tilde{\Delta}_2 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \tilde{\Delta}_{b_2-1} & \lambda_2 I_{z_1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \Gamma_{b_2} & \tilde{\Delta}_{b_2} \end{pmatrix} \end{array},$$

where $\tilde{\Delta}_i = \Delta_i + \mathrm{diag}(\lambda_1 I_{k_2 s_2}, \lambda_1 I_2 - V_2)$.

Actual Delay Distribution

- The time to absorption distribution of such a Markov chain has received extensive attention in the literature, and it is well-known that the cumulative distribution function of W_1, denoted by $F_1(\omega)$, is given by

$$F_1(\omega) = 1 - \underline{\Phi} \exp\{\mathcal{R}\omega\} \underline{e}'_\ell, \; \omega \geq 0,$$

  which is of *phase-type form*.
- To incorporate the reneging behaviour of our target class-1 customer, define W_1^* to be the *actual* class-1 delay (i.e., the arriving class-1 customer's total time spent in the system prior to *successfully* entering service).
- Clearly, $G_1(\omega) = \Pr(W_1^* \leq \omega) = \Pr(W_1 \leq \omega \mid W_1 \leq R_1)$, where R_1 denotes an exponentially distributed random variable, independent of W_1, with rate α_1.
- Making use of fundamental matrix algebraic techniques, the following expressions for $G_1(\omega)$ and the moments of W_1^* are obtained:

Actual Delay Distribution

$$\begin{aligned} G_1(\omega) &= 1 - \Pr(W_1 > \omega \mid W_1 \leq R_1) \\ &= 1 - \frac{\Pr(\omega < W_1 \leq R_1)}{\Pr(W_1 \leq R_1)} \\ &= 1 - \frac{\int_\omega^\infty \Pr(W_1 > \omega)\alpha_1 e^{-\alpha_1 x} dx - \int_\omega^\infty \Pr(W_1 > x)\alpha_1 e^{-\alpha_1 x} dx}{1 - \int_0^\infty \Pr(W_1 > x)\alpha_1 e^{-\alpha_1 x} dx} \\ &= 1 - \frac{\underline{\Phi}\left[I_\ell - \alpha_1(\alpha_1 I_\ell - \mathcal{R})^{-1}\right] \exp\{\mathcal{R}\omega\} \underline{e}_\ell' e^{-\alpha_1 \omega}}{1 - \alpha_1 \underline{\Phi}(\alpha_1 I_\ell - \mathcal{R})^{-1} \underline{e}_\ell'}, \, \omega \geq 0, \end{aligned}$$

and

$$E[W_1^{*r}] = \frac{r!\underline{\Phi}\left[I_\ell - \alpha_1(\alpha_1 I_\ell - \mathcal{R})^{-1}\right](\alpha_1 I_\ell - \mathcal{R})^{-r} \underline{e}_\ell'}{1 - \alpha_1 \underline{\Phi}(\alpha_1 I_\ell - \mathcal{R})^{-1} \underline{e}_\ell'}, \ r = 1, 2, \ldots .$$

Total Time Spent Waiting In The System

- We investigate the selection of parameters k_1 and k_2 in order to optimize the overall system, by way of minimizing a specific cost function.
- The total time a class-1 customer actually spends waiting in the system is

$$W_1^{\#} = \min\{W_1, R_1\} \sim PH(\underline{\Phi}_1, \mathcal{R}_1 - \alpha_1 I_{\ell_1}),$$

where $\underline{\Phi}_1$, ℓ_1, and $\mathcal{R}_1$ are given by equations (11), (12), and (13), respectively.
- Parameters $\underline{\Phi}_2$, ℓ_2, and $\mathcal{R}_2$ can be obtained in an analogous fashion to be used in the characterization of the distribution of $W_2^{\#} = \min\{W_2, R_2\}$.

The Cost Function

- We generalize the cost function of Borst, Boxma & Levy (1995) to get

$$\text{Cost} = \text{Cost}_1 + \text{Cost}_2,$$

where

$$\text{Cost}_i = c_i \lambda_i \mathrm{E}[W_i^{\#}] + r_i \lambda_i \Pr(R_i < W_i),$$

and cost parameters c_i and r_i are non-negative constants.

- It is straightforward to obtain:

$$\mathrm{E}[W_1^{\#}] = \underline{\Phi}(\alpha_1 I_{\ell_1} - \mathcal{R}_1)^{-1} \underline{e}'_{\ell_1},$$
$$\Pr(R_1 < W_1) = \alpha_1 \underline{\Phi}(\alpha_1 I_{\ell_1} - \mathcal{R}_1)^{-1} \underline{e}'_{\ell_1} = \alpha_1 \mathrm{E}[W_1^{\#}].$$

Optimization Problem

- We consider a constraint on the total number of services in a cycle – namely, $k_1 + k_2 \leq 12$.
- Three possible reneging rates were used for both classes:

$$\alpha_i \in \{0.025, 0.05, 0.25\}.$$

- Three service time distributions were considered: Exponential [Exp], Hyperexponential [H_2], and Erlang [E_3].
- Each distribution has the same mean, but the H_2 distribution has 1000 times the variance of the Exp distribution, which has 3 times the variance of the E_3 distribution.
- Case 1: arrival rates $\lambda_1 = \lambda_2 = 0.75$, switchover rates $v_1 = v_2 = 1/0.1$, and mean service times of 0.9 for class 1 and 0.1 for class 2.
- Case 2: arrival rates $\lambda_1 = 0.5$, $\lambda_2 = 0.25$, switchover rates $v_1 = 1/0.1$, $v_2 = 1/0.2$, and mean service times of 1 for both classes.
- In both cases, we set buffer sizes of $b_1 = b_2 = 20$.

Numerical Analysis

Reneging Rates		Service Time Distributions							
		(Exp, Exp)		(Exp, H_2)		(Exp, E_3)		(Exp, Exp)	
α_1	α_2	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost
0.025	0.025	(3, 9)	4.3398	(3, 9)	6.2866	(3, 9)	4.3281	(3, 9)	6.9361
	0.05	(4, 8)	4.2581	(3, 9)	5.9977	(4, 8)	4.2468	(2, 10)	7.7429
	0.25	(7, 5)	3.7352	(9, 3)	4.8325	(7, 5)	3.7269	(2, 10)	11.9875
0.05	0.025	(3, 9)	3.6482	(3, 9)	5.2460	(3, 9)	3.6386	(3, 9)	7.1422
	0.05	(3, 9)	3.5947	(3, 9)	4.9824	(3, 9)	3.5855	(2, 10)	7.8847
	0.25	(6, 6)	3.2543	(6, 6)	4.0882	(6, 6)	3.2470	(1, 11)	11.9519
0.25	0.025	(2, 10)	2.1520	(2, 10)	3.0167	(2, 10)	2.1464	(3, 9)	9.0264
	0.05	(2, 10)	2.1334	(2, 10)	2.8169	(2, 10)	2.1279	(3, 9)	9.7272
	0.25	(2, 10)	2.0230	(2, 10)	2.2667	(2, 10)	2.0183	(1, 11)	13.3357
		(H_2, Exp)		(H_2, H_2)		(H_2, E_3)		(H_2, H_2)	
α_1	α_2	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost
0.025	0.025	(4, 8)	20.3486	(3, 9)	21.7547	(4, 8)	20.3441	(3, 9)	35.8657
	0.05	(5, 7)	18.2711	(4, 8)	19.5065	(5, 7)	18.2666	(3, 9)	36.3322
	0.25	(8, 4)	14.6158	(9, 3)	15.5758	(8, 4)	14.6114	(2, 10)	33.8738
0.05	0.025	(3, 9)	16.4205	(2, 10)	17.4343	(3, 9)	16.4171	(2, 10)	34.1935
	0.05	(4, 8)	14.3710	(3, 9)	15.2235	(4, 8)	14.3676	(3, 9)	34.6786
	0.25	(7, 5)	10.7526	(8, 4)	11.3615	(7, 5)	10.7490	(2, 10)	32.2619
0.25	0.025	(1, 11)	8.8681	(1, 11)	9.4376	(1, 11)	8.8681	(3, 9)	27.1216
	0.05	(1, 11)	6.9769	(1, 11)	7.3890	(1, 11)	6.9756	(3, 9)	27.6632
	0.25	(6, 6)	3.5293	(5, 7)	3.7245	(6, 6)	3.5263	(2, 10)	25.4136
r_i		$r_1 = 1,\ r_2 = 0.5$						$r_1 = r_2 = 40$	

Table 1: Optimal (k_1, k_2) and minimum cost under Case 1 with $c_1 = 2$, $c_2 = 1$, and $r_1 = 1$, $r_2 = 0.5$ or $r_1 = r_2 = 40$. [$\lambda_1 = \lambda_2 = 0.75$, $v_1 = v_2 = 1/0.1$, service times of 0.9 & 0.1]

Numerical Analysis

Reneging Rates		Service Time Distributions							
		(Exp, Exp)		(Exp, H_2)		(Exp, E_3)		(Exp, Exp)	
α_1	α_2	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost
0.025	0.025	(10, 2)	2.6649	(10, 2)	7.8090	(10, 2)	2.4761	(10, 2)	4.3972
	0.05	(11, 1)	2.4083	(11, 1)	6.8184	(11, 1)	2.2577	(9, 3)	4.6890
	0.25	(11, 1)	1.8357	(11, 1)	5.1854	(11, 1)	1.7432	(8, 4)	5.9054
0.05	0.025	(10, 2)	2.3812	(9, 3)	5.6575	(10, 2)	2.2247	(10, 2)	4.6660
	0.05	(10, 2)	2.2030	(10, 2)	4.8100	(10, 2)	2.0669	(10, 2)	4.9625
	0.25	(11, 1)	1.6934	(11, 1)	3.5890	(11, 1)	1.6127	(8, 4)	6.1953
0.25	0.025	(6, 6)	1.5047	(5, 7)	2.9877	(6, 6)	1.4353	(11, 1)	6.4562
	0.05	(7, 5)	1.4550	(6, 6)	2.2615	(7, 5)	1.3907	(10, 2)	6.7245
	0.25	(11, 1)	1.2323	(10, 2)	1.5454	(11, 1)	1.1879	(8, 4)	7.8861
		(H_2, Exp)		(H_2, H_2)		(H_2, E_3)		(H_2, H_2)	
α_1	α_2	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost	(k_1, k_2)	Cost
0.025	0.025	(11, 1)	11.9085	(10, 2)	15.6149	(11, 1)	11.8306	(10, 2)	24.8349
	0.05	(11, 1)	10.6576	(11, 1)	13.9119	(11, 1)	10.5843	(10, 2)	23.2350
	0.25	(11, 1)	9.5631	(11, 1)	12.3096	(11, 1)	9.5085	(9, 3)	21.9772
0.05	0.025	(10, 2)	8.2620	(9, 3)	10.5575	(10, 2)	8.1927	(9, 3)	20.6821
	0.05	(11, 1)	7.0367	(10, 2)	8.9056	(11, 1)	6.9730	(9, 3)	19.1262
	0.25	(11, 1)	5.9644	(11, 1)	7.4260	(11, 1)	5.9166	(8, 4)	17.9732
0.25	0.025	(4, 8)	3.9129	(4, 8)	5.0239	(4, 8)	3.8846	(8, 4)	15.7298
	0.05	(9, 3)	2.8207	(6, 6)	3.4547	(9, 3)	2.7818	(8, 4)	14.2528
	0.25	(11, 1)	1.8533	(9, 3)	2.1237	(11, 1)	1.8198	(7, 5)	13.2301
r_i		$r_1 = 1,\ r_2 = 0.5$						$r_1 = r_2 = 40$	

Table 2: Optimal (k_1, k_2) and minimum cost under Case 2 with $c_1 = 2$, $c_2 = 1$, and $r_1 = 1$, $r_2 = 0.5$ or $r_1 = r_2 = 40$. $[\lambda_1 = 0.5, \lambda_2 = 0.25,\ v_1 = 1/0.1, v_2 = 1/0.2$, service times of 1]

Numerical Analysis

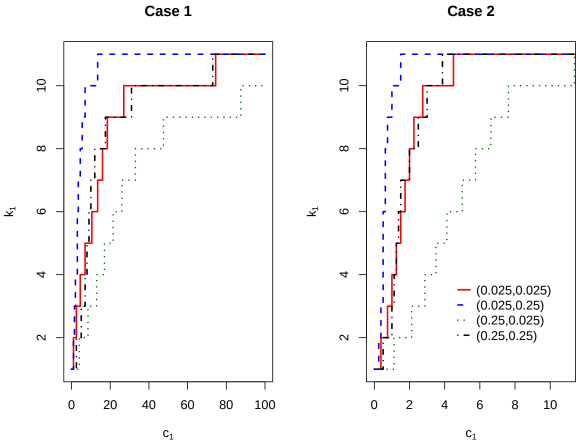


Figure 1: Plots of k_1 vs. c_1 under both cases with Exp service times, $c_2 = 2$, $r_1 = r_2 = 1$, and four combinations of reneging rates.

Numerical Analysis

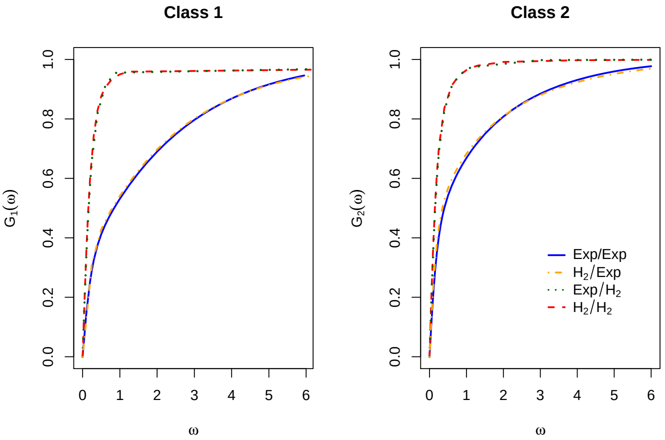


Figure 2: Plots of $G_i(\omega)$ vs. ω for both classes under Case 1 with $\alpha_1 = 0.025$, $\alpha_2 = 0.25$, and either Exp or H_2 service time distributions, at optimal k_i's from Table 1.

Concluding Remarks

- We modelled an $M/PH/1$-type polling model with 2 classes and exponential switchover/reneging times operating under a k_i-limited service discipline.
- We were able to obtain the steady-state probabilities as well as the nominal waiting time and actual delay distributions for each class.
- Under a variety of scenarios, we found optimal (k_1, k_2) values that minimize (subject to a particular constraint) a defined cost function depending on the expected time waiting in the system and the probability of reneging.

Future Extensions

- Queue length dependent reneging rates.
- Phase-type renewal process for arrivals.
- (2-dimensional) phase-type reneging time distributions.
- Different service disciplines.
- Multiple servers.
- A third class of customers.

Questions?

Building Q

$$Q_{1,0} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b_2 \end{array} \begin{array}{ccccc} 0 & 1 & 2 & \dots & b_2 \\ \left(\begin{array}{ccccc} \begin{bmatrix} \underline{e}'_{k_1}\underline{e}_{2,1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_2 \end{bmatrix} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \underline{e}'_{k_1}\underline{e}_{z_1,z_1-1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_{z_1} \end{bmatrix} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix} \underline{e}'_{k_1}\underline{e}_{z_1,z_1-1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_{z_1} \end{bmatrix} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \begin{bmatrix} \underline{e}'_{k_1}\underline{e}_{z_1,z_1-1} \otimes \underline{S}'_{0,1} \\ \alpha_1 I_{z_1} \end{bmatrix} \end{array}\right) \end{array}$$

$$Q_{0,1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b_2 \end{array} \begin{array}{ccccc} 0 & 1 & 2 & \dots & b_2 \\ \left(\begin{array}{ccccc} \begin{bmatrix} \underline{0}'_2\underline{0}_{k_1 s_1} & \lambda_1 I_2 \end{bmatrix} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \underline{0}'_{z_1}\underline{0}_{k_1 s_1} & \lambda_1 I_{z_1} \end{bmatrix} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix} \underline{0}'_{z_1}\underline{0}_{k_1 s_1} & \lambda_1 I_{z_1} \end{bmatrix} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \begin{bmatrix} \underline{0}'_{z_1}\underline{0}_{k_1 s_1} & \lambda_1 I_{z_1} \end{bmatrix} \end{array}\right) \end{array}$$