An interaction between queing and change detection

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THE DYNAMICS of a QUEUE

Consider a single server queue. Waiting time of the n-th customer: W_n.

The dynamics of W_n is given by a *non-linear system*:

$$W_n = (W_{n-1} + X_n)_+ \quad \text{with} \quad W_0 = 0, \tag{1}$$

where $X_n = V_{n-1} - U_n =$ service time minus interarrival time.

A system-theoretic point of view: (8) is not a stable system.

A similar non-linear dynamics arises in the theory of risk processes:

$$W_n^- = (W_{n-1}^- + X_n^-)_- \quad \text{with} \quad W_0^- = K > 0. \tag{2}$$

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PROBABILISTIC STABILITY of a QUEUE

A standard assumption: assume i.i.d. inputs (X_n), with $\mathrm{E}(X_n) < 0$.

Markovian techniques: establish geometric ergodicity assuming

$$\mathbb{E}(\exp c' X_1) < 1 \quad \text{for some} \quad c' > 0. \tag{3}$$

Strong LLN follows for functions of W_n. See Meyn & Tweedie.

PROBLEM STATEMENT

Under what conditions for the inputs (X_n) can we ensure:

1. A strong LLN for the empirical tail probabilities:

$$\limsup_N \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}_{\{W_n > K\}} \leq \limsup_n P(W_n > K) \qquad \text{a.s.}$$

2. Exponential decay of tail probabilities:

$$P(W_n > K) < Ce^{-cK}.$$

Technical kinship of the two problems.

CHANGE DETECTION

The problem: detect changes of statistical patterns of signals in real time.

Example: monitoring EEG signals for epileptic patients

See: V.Poor and O.Hadjiliadis (2009): Quickest Detection.

A CLASSIC PROBLEM

Given a sequence of i.i.d. r.v.-s Y_n with prob. density functions

$$f(y, \theta_0) \quad \text{for} \quad n < \tau \quad \text{and} \quad f(y, \theta_1) \quad \text{for} \quad n \geq \tau.$$

Estimate

$$\text{change point} \quad : \quad \tau$$

using observations (y_n).

The Cumulative Sum (CUSUM) test or Page-Hinkley detector:

E.S. Page, Biometrika, 1954

D.V. Hinkley, J.Amer. Statist. Assoc., 1971

STATISTICS and IT

A modern interpretation, following J.Rissanen, 1989:

Encode data using the two possible models, following Inf.Thy.:

The quasi-optimal code-lengths are

$$-\log f(y_n, \theta_0) \quad \text{and} \quad -\log f(y_n, \theta_1).$$

The differences in code-lengths define the score

$$X_n = -\log f(y_n, \theta_0) + \log f(y_n, \theta_1).$$

Now the information inequality gives

$$\mathrm{E}X_n < 0 \quad \text{for} \quad n < \tau \quad \text{and} \quad \mathrm{E}X_n > 0 \quad \text{for} \quad n \geq \tau.$$

THE CUSUM TEST for I.I.D. DATA

Let $S_0 = 0$ and let

$$S_n = \sum_{k=1}^{n} X_k \quad \text{for} \quad n \geq 1.$$

Then $\mathrm{E}S_n$ has a minimum at $\tau - 1$.

Task: approximate on-line minimization of S_n.

The CUSUM statistics or Page-Hinkley detector: define

$$g_n = S_n - \min_{0 \leq k \leq n} S_k.$$

Generate an alarm if $g_n > \delta$, with some fixed threshold $\delta > 0$.

FALSE ALARM RATE

Apply the Page-Hinkley detector to a process with no change at all.

A key performance characteristics: false alarm probability

$$\limsup_n P_{\theta_0}(g_n > \delta).$$

Practical relevance: false alarm rate (FAR) defined as a.s.

$$\limsup_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}_{\{g_n > \delta\}}.$$

Problem: find an upper bound for the FAR.

THE DYNAMICS of CUSUM

The dynamics of g_n is easily obtained as follows:

$$g_n = (g_{n-1} + X_n)_+ \quad \text{with} \quad g_0 = 0.$$

This establishes the link between queuing, W_n, and change detection, g_n.

Objective: bounding the empirical tail probabilities of g_n.

Two related technical problems:

- Exponential bounds for the tail probabilities of g_n.
- Mixing properties of g_n.

MOTIVATION for MIXING

Let (ν_n) be an $\mathbb{R}^s$-valued i.i.d. sequence of r.v.-s such that

$$\sup_{n \geq 0} \mathbb{E} |\nu_n|^q < +\infty \qquad \text{for all} \quad 1 \leq q < \infty.$$

Let the $s \times s$ matrix A be stable, and define the filtered process

$$X_n = AX_{n-1} + \nu_n \quad \text{with} \quad X_0 = 0.$$

Decompose (X_n) as

$$X_n = A^\tau X_{n-\tau} + \sum_{k=0}^{\tau-1} A^k \nu_{n-k}.$$

L-MIXING, I.

Definition

Let $X = (X_n)$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. X is M-bounded if for all $1 \leq q < +\infty$

$$M_q(X) := \sup_{n \geq 0} \| X_n \|_q < +\infty.$$

Let $\nu = (\nu_n)$ be an i.i.d. sequence, and define its past and future as

$$\mathcal{F}_n = \sigma(\nu_k : k \leq n) \quad \text{and} \quad \mathcal{F}_n^+ = \sigma(\nu_k : k \geq n+1).$$

Let $\tau > 0$ be an integer, a fixed memory length, and defined for $1 \leq q < +\infty$ the error of approximation by the near past as

$$\gamma_q(\tau, X) = \gamma_q(\tau) := \sup_{n \geq \tau} \| X_n - \mathrm{E}(X_n | \mathcal{F}_{n-\tau}^+) \|_q .$$

L-MIXING, II.

Definition

A stochastic process $X = (X_n)$ is L-mixing w.r.t. $(\mathcal{F}_n, \mathcal{F}_n^+)$ if it is adapted i.e. X_n is $\mathcal{F}_n$-measurable for all $n \geq 1$, X is M-bounded, and

$$\Gamma_q(X) := \sum_{\tau=0}^{+\infty} \gamma_q(\tau) < +\infty \quad \text{for all} \quad 1 \leq q < +\infty.$$

Remark: we can also have $q = \infty$.

References:

Ljung, L.: Math. Programming, 1976.

LG: Stochastics, 1989.

THM: for I.I.D. SCORES g_n is L-MIXING

Assume $\mathbb{E}(X_1) < 0$, and also $\mathbb{E}(\exp cX_1) < \infty$ for some $c > 0$. Then

$$\mu := \mathbb{E}(\exp c' X_1) < 1 \quad \text{for some} \quad c' > 0. \tag{4}$$

Let $\mathcal{F}_n := \sigma(X_i \,|\, i \leq n)$ and $\mathcal{F}_n^+ := \sigma(X_i \,|\, i \geq n+1)$.

Theorem

Let (X_n) be a sequence of i.i.d. random variables such that (4) holds. Then the Page-Hinkley detector (g_n) is L-mixing with respect to $(\mathcal{F}_n, \mathcal{F}_n^+)$.

LG and Prosdocimi, C. Systems & Control Letters, 2011.

EMPIRICAL TAIL PROBABILITIES

A known result in risk theory: for any c'' such that $0 < c'' < c'$, we have

$$\sup_n \ \mathbb{E}\left(\exp c''\, g_n\right) < \infty. \tag{5}$$

Hence $P(g_n > \delta) \leq Ce^{-c''\delta}$ with some $0 < C < \infty$. Equivalently,

$$\mathbb{E}\ \mathbb{I}_{\{g_n > \delta\}} \leq Ce^{-c''\delta}.$$

Since (g_n) is L-mixing, by a strong LLN it follows that

$$\limsup_N \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}_{\{g_n > \delta\}} \leq C' e^{-c''\delta}.$$

DEPENDENT SCORES

The key technical condition to be established above: for any $c' > 0$

$$\mathbb{E} \exp \sum_{k=1}^{n} c' X_k \leq C e^{-c'' n}$$

with some $C, c'' > 0$. Writing the l.h.s. as

$$\mathbb{E} \exp \left(\sum_{k=1}^{n} c'(X_k - \mathbb{E} X_k) \right) \cdot \exp \left(\sum_{k=1}^{n} c' \, \mathbb{E} X_k \right),$$

assuming $\mathbb{E} X_k \leq -\varepsilon$, it is sufficient to show that for small c'-s and all n:

$$\mathbb{E} \exp \left(\sum_{k=1}^{n} c'(X_k - \mathbb{E} X_k) \right) \leq e^{\kappa (c')^2 n},$$

with some $\kappa > 0$.

AN EXPONENTIAL MOMENT CONDITION

Let $X = (X_n)$ be a sequence of real-valued random variables.

Condition E. *There exist* $c > 0$ *and* $\kappa > 0$ *such that for* $0 \leq c' < c$ *and all* $1 \leq m \leq n$ *we have*

$$\mathbb{E} \exp \left(c' \sum_{k=m}^{n} (X_k - \mathbb{E}X_k) \right) \leq \exp \left(\kappa \, (c')^2 (n - m + 1) \right) .$$

Objective: find useful conditions for the above inequality to hold.

I.I.D. REVISITED

Lemma

Let (X_n) be a zero-mean, i.i.d. sequence such that $\mathbb{E}\, e^{c|X_n|} < \infty$. Then Condition E is satisfied.

The proof is trivial, noting:

$$\mathbb{E} e^{c'X} \leq e^{\kappa(c')^2}.$$

for $|c'| \leq c$ with some $\kappa > 0$.

We get even more.

ANOTHER EXPONENTIAL MOMENT INEQUALITY

Let X be a two-sided i.i.d. sequence as above. Let $h = (h_k), k = 0, 1, \ldots$ be an l_1 - sequence and define

$$Y_n = \sum_{k=0}^{\infty} h_k X_{n-k} \quad \text{in short} \quad Y = h \star X.$$

Write $\| h \|_2^2 = \sum_{k=0}^{\infty} h_k^2$.

Theorem

Let $Y = (Y_n)$ be a as above. Then for $\| h \|_2 \leq c$

$$\mathbb{E} \exp\left(h \star X\right) \leq \exp\left(\kappa \| h \|_2^2\right).$$

A SECOND EXPONENTIAL MOMENT CONDITION

Let $X = (X_n)$ be a two-sided sequence of real-valued r.v.-s, $\mathbb{E}X_n = 0$.

Condition SE. There exist $c > 0$ and $\kappa > 0$ such that for any $h \in l_1$ with $\| h \|_2 \leq c$ we have

$$\mathbb{E} \exp \left(\sum_{k=0}^{\infty} h_k X_{n-k} \right) \leq \exp \left(\kappa \| h \|_2^2 \right). \tag{6}$$

Note: if $X = (X_n)$ satisfies Condition SE and $g \in l_1$ then

$$Y = g \star X$$

also satisfies Condition SE. (Trivial). Example: $|X_n| \leq K$.

A NON-CONSTRUCTIVE EXAMPLE

Theorem

Let (X_n) be a zero-mean L-mixing process such that we have

$$M_\infty(X) < +\infty \qquad \textit{and} \qquad \Gamma_\infty(X) < +\infty. \tag{7}$$

Then for any deterministic sequence f_n

$$\mathbb{E} \exp \left(\sum_{k=1}^{n} f_n X_n - 2 M_\infty(X) \Gamma_\infty(X) \sum_{k=1}^{n} f_n^2 \right) \leq 1.$$

It follows that Condition SE is satisfied.

LG: Stochastics and Stochastic Reports, 1991.

AN OPEN PROBLEM: SERIAL INTERCONNECTION

Let $X = (X_n)$ be a two-sided sequence of real-valued r.v.-s, satisfying $\mathbb{E}X_n \leq -\epsilon < 0$ and in addition Condition SE.

Consider a single server queue with waiting time denoted by W_n.

Let the dynamics of W_n is given by the non-linear system:

$$W_n = (W_{n-1} + X_n)_+ \quad \text{with} \quad W_0 = 0. \tag{8}$$

Under what conditions does (W_n) satisfy Condition SE ?

RECREATIONAL MATH

Establish stability properties of the Page-Hinkley-detector for *deterministic inputs*: assume that (X_n) is a deterministic sequence satisfying

$$\limsup_{N \longrightarrow +\infty} \frac{1}{N} \sum_{n=1}^{N} X_n < 0. \tag{9}$$

Let (g_n) be the response of the Page-Hinkley-detector driven by (X_n). Does it follow that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g_n < \infty? \tag{10}$$

THANK YOU for YOUR ATTENTION !