Parisian ruin for fluid flow risk processes

Oscar Peralta Gutiérrez[1],
Mogens Bladt[2],
Bo Friis Nielsen[1].

[1]Technical University of Denmark
Department of Applied Mathematics and Compute Science.

[2]Autonomous National University of Mexico.

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Fluid flow model.

Consider a fluid flow model with Brownian noise (started at level $u \in \mathbb{R}$) on the form

$$V_t = u + \int_0^t r_{J_s} ds + \int_0^t \sigma_{J_s} dB_s, \quad (t \geq 0), \tag{1}$$

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Suppose that $V_t \to +\infty$ as $t \to \infty$ a.s..

Notation for the fluid flow model.

- E is partitioned and ordered into

$$E^\sigma := \{i \in E : \sigma_i > 0\},$$
$$E^+ := \{i \in E : \sigma_i = 0, r_i > 0\}, \text{ and}$$
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- The infinitesimal generator of $\{J_t\}_{t\geq 0}$ is written as

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}^{\sigma\sigma} & \mathbf{\Lambda}^{\sigma+} & \mathbf{\Lambda}^{\sigma-} \\ \mathbf{\Lambda}^{+\sigma} & \mathbf{\Lambda}^{++} & \mathbf{\Lambda}^{+-} \\ \mathbf{\Lambda}^{-\sigma} & \mathbf{\Lambda}^{-+} & \mathbf{\Lambda}^{--} \end{pmatrix}, \tag{2}$$

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- Define the row vectors

$$\boldsymbol{r}_\sigma := \{r_i : i \in E^\sigma\}, \quad \boldsymbol{r}_+ := \{r_i : i \in E^+\},$$
$$\boldsymbol{r}_- := \{r_i : i \in E^-\}, \quad \boldsymbol{\sigma} := \{\sigma_i : i \in E^\sigma\}.$$

Which is the connection with risk theory?

We define the **fluid flow risk process** $\{R_t\}_{t\geq 0}$ by regarding the linear downward segments of $\{V_t\}_{t\geq 0}$ as downward jumps of the same height.

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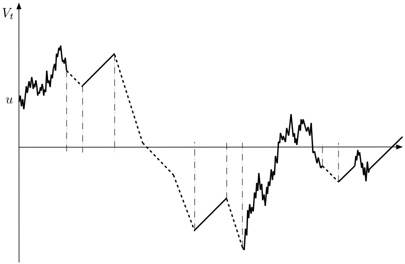

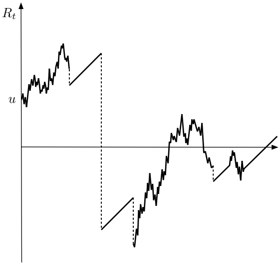

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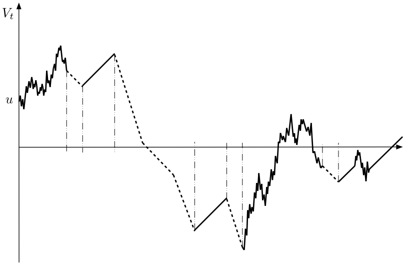

Classic task: Compute $\psi(u) := \mathbb{P}(\inf_{s\geq 0} R_s < 0 | R_0 = u)$, the **classic probability of ruin**.

Classic risk processes as fluid flow risk processes.

Example (Cramér-Lundberg process)

The classic **Cramér-Lundberg process** with linear drift $p > 0$, Poisson arrival rate β, and PH$(\boldsymbol{\alpha}, \boldsymbol{S})$-distributed claims can be represented as a fluid flow risk process $\{R_t\}_{t\geq 0}$ with characteristics $E^\sigma = \emptyset$, $E^+ = \{1\}$, $E^- = \{2, 3, \ldots, m+1\}$, $\boldsymbol{r}_+ = (p)$, $\boldsymbol{r}_- = (-1, \ldots, -1)$ and

$$\boldsymbol{\Lambda} = \begin{pmatrix} -\beta & \beta\boldsymbol{\alpha} \\ -\boldsymbol{Se} & \boldsymbol{S} \end{pmatrix}.$$

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Remark

Instead of having upward linear segments, we can opt to have a Brownian component, so that **Lévy risk processes with phase-type jumps** *are an example of a fluid flow risk process.*

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Example (Sparre-Andersen process)

The Sparre-Andersen process with PH($\alpha, \boldsymbol{S}$)-distributed claims and PH($\boldsymbol{\pi}, \boldsymbol{T}$)-distributed interarrival times can be represented as a fluid flow risk process $\{R_t\}_{t\geq 0}$ with characteristics $E^\sigma = \emptyset$, $E^+ = \{1, \ldots, n\}$, $E^- = \{n+1, \ldots, n+m\}$, $\boldsymbol{r}_+ = (1, \ldots, 1)$, $\boldsymbol{r}_- = (-1, \ldots, -1)$ and

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Remark

We can represent risk processes with MAP arrivals and phase type jumps as fluid flow risk processes. This processes are basically **Markov additive risk processes with phase-type jumps**.

Parisian ruin (when $E_\sigma = \emptyset$).

Definition

Suppose that $E_\sigma = \emptyset$, let $\{L_i\}_{i\geq 1}$ be i.i.d. clocks and associate each L_i to the (possible) i-th excursion below zero of $\{R_t\}_{t\geq 0}$.

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$$\psi_p(u) = \mathbb{P}(\text{Parisian ruin happens } | R_0 = u).$$

In other words, each time the reserve from an insurance company gets below 0, the company is given a (random) time window in order to recover.

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In our setting, we let

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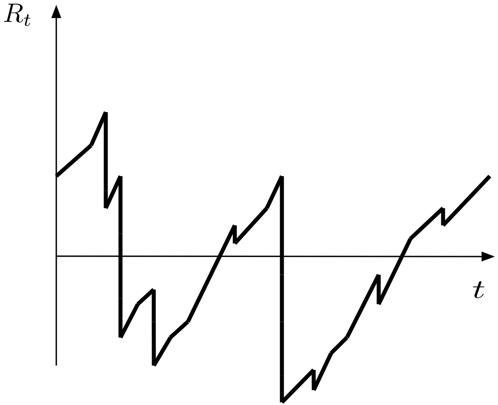
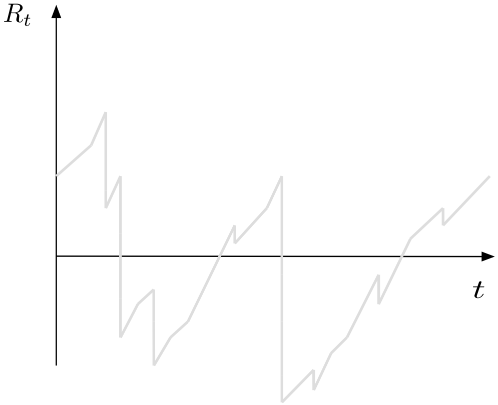
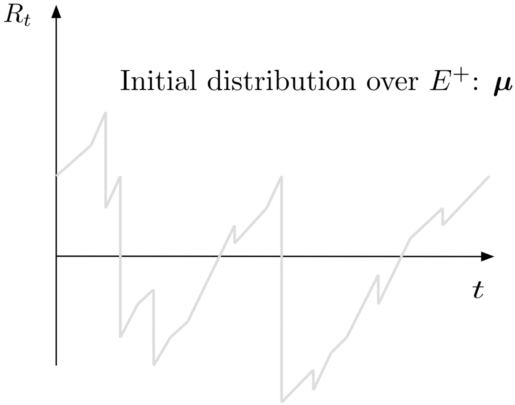
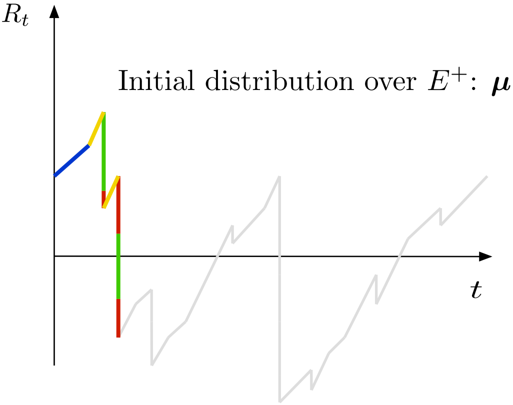
Erlangization arguments can be used to approximate the deterministic clocks case!


R_t
t


R_t
t


R_t
Initial distribution over E^+: $\boldsymbol{\mu}$
t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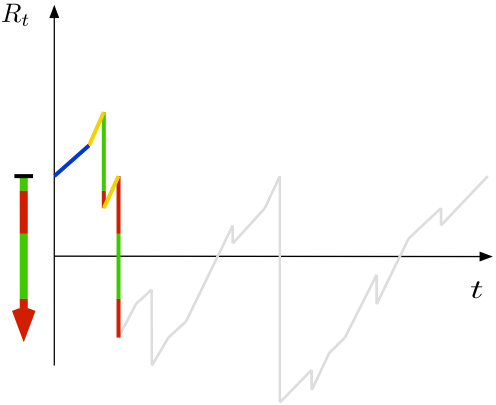
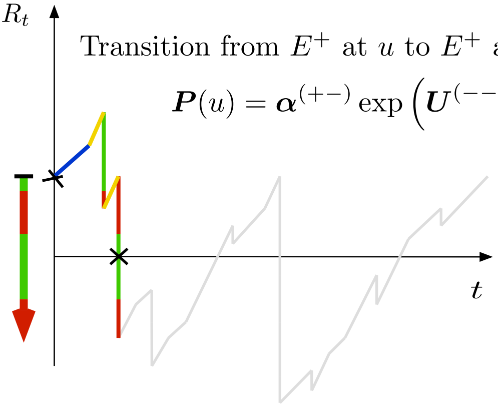
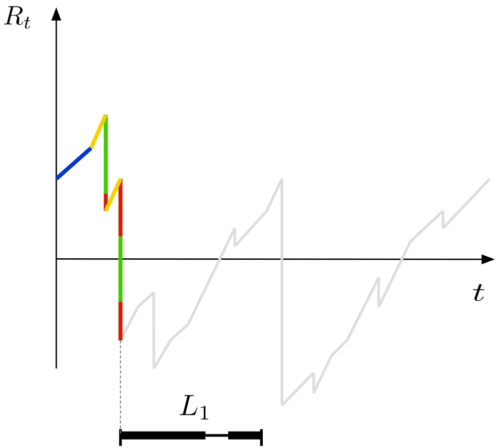
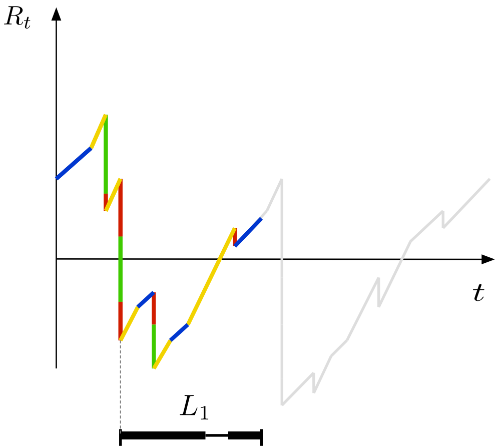

R_t
t

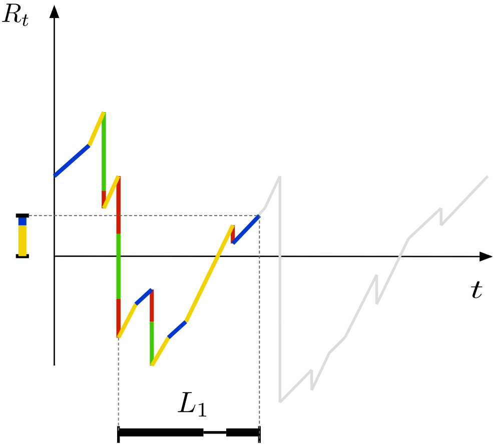
Transition from E^+ at u to E^+ at 0

$$\boldsymbol{P}(u) = \boldsymbol{\alpha}^{(+-)} \exp\left(\boldsymbol{U}^{(--)} u\right)$$



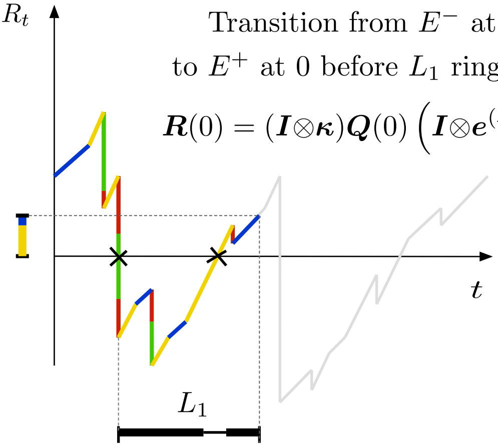

R_t
t
L_1


R_t
t
L_1


R_t
t
L_1

Transition from E^- at 0 to E^+ at 0 before L_1 rings:

$$\boldsymbol{R}(0) = (\boldsymbol{I} \otimes \boldsymbol{\kappa})\boldsymbol{Q}(0)\left(\boldsymbol{I} \otimes \boldsymbol{e}^{(\ell)}\right)$$



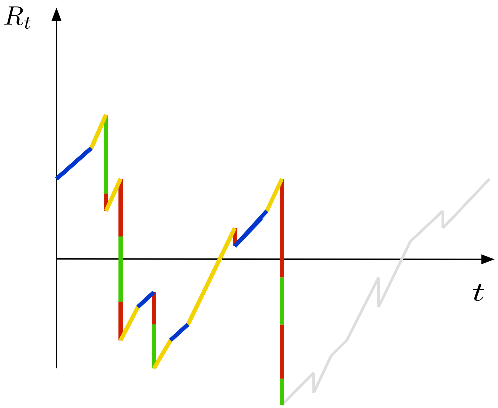
$\boldsymbol{Q}(\cdot)$ is of the form $\boldsymbol{Q}(s) = \boldsymbol{\beta}^{(-+)} \exp\left(\boldsymbol{V}^{(++)} s\right)$

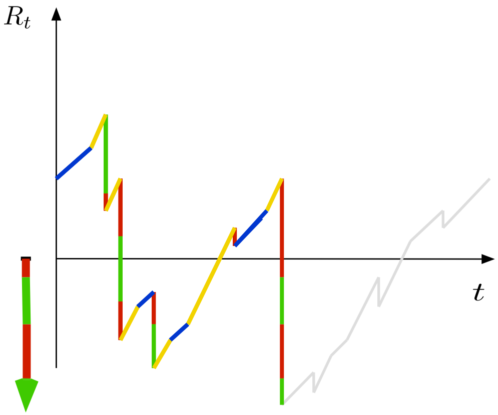
where $\boldsymbol{\beta}^{(-+)}$ and $\boldsymbol{V}^{(++)}$ are the upcrossing probabilities and intensity matrix for the FFM with intensity matrix

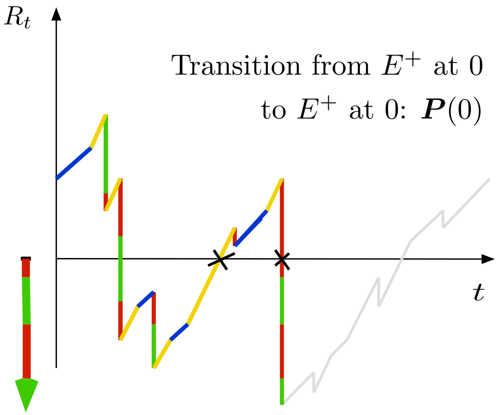
$$\begin{pmatrix} \boldsymbol{\Lambda}^{++} \oplus \boldsymbol{K} & \boldsymbol{\Lambda}^{+-} \otimes \boldsymbol{I} \\ \boldsymbol{\Lambda}^{-+} \otimes \boldsymbol{I} & \boldsymbol{\Lambda}^{--} \otimes \boldsymbol{I}^{(\ell)} \end{pmatrix},$$

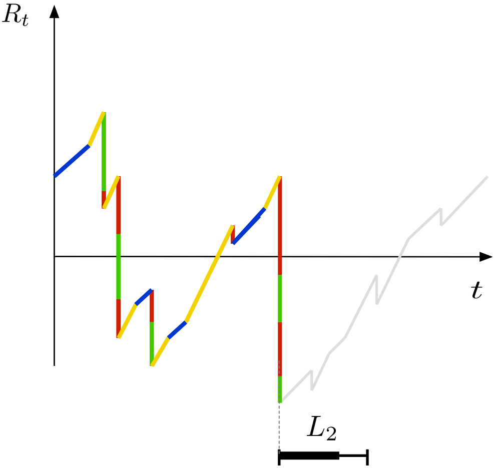
and drifts

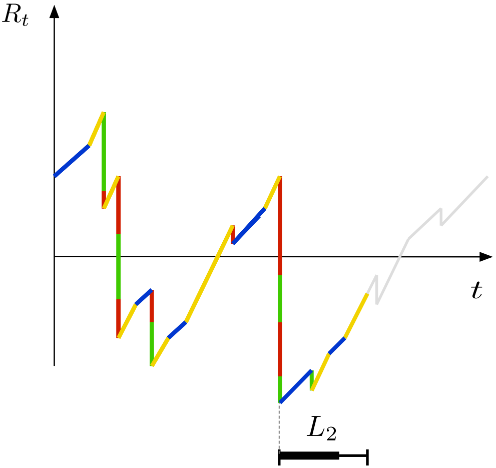
$$\boldsymbol{r}_+ \otimes \boldsymbol{e}^T, \qquad \boldsymbol{r}_- \otimes \boldsymbol{e}^T$$

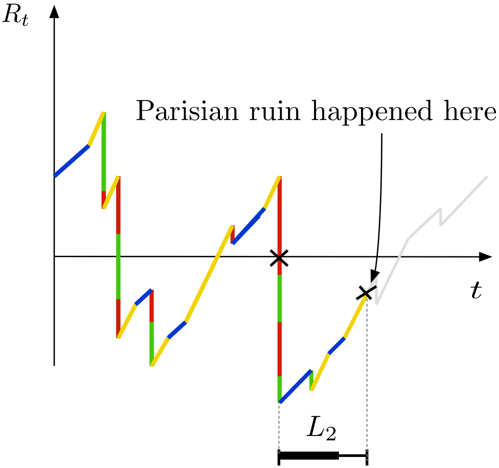

R_t
t

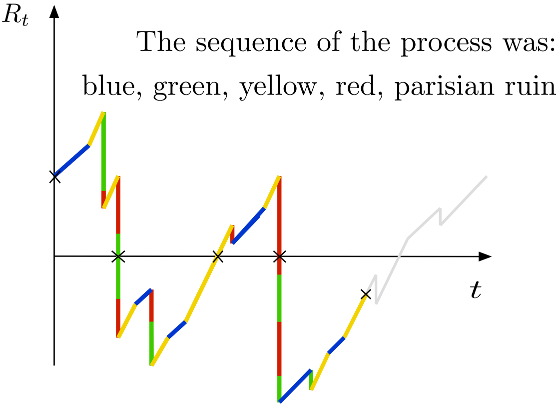

R_t
t


R_t
Transition from E^+ at 0
to E^+ at 0: $\boldsymbol{P}(0)$
t


R_t
t
L_2


R_t
t
L_2


R_t
Parisian ruin happened here
t
L_2


R_t
The sequence of the process was:
blue, green, yellow, red, parisian ruin
t

The transition probability matrix.

We construct a Markov chain whose state space is partitioned into E^+, E^-, ∂_N and ∂_P. Its initial distribution is

$$(\mathbf{0}, \boldsymbol{\mu}\boldsymbol{P}(u), 1 - \boldsymbol{\mu}\boldsymbol{P}(u)\boldsymbol{e}, 0),$$

and its transition matrix is given by

$$\begin{pmatrix} \mathbf{0} & \boldsymbol{P}(0) & \boldsymbol{e} - \boldsymbol{P}(0)\boldsymbol{e} & \mathbf{0} \\ \boldsymbol{R}(0) & \mathbf{0} & \mathbf{0} & \boldsymbol{e} - \boldsymbol{R}(0)\boldsymbol{e} \\ \mathbf{0} & \mathbf{0} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 1 \end{pmatrix}$$

Main result.

Theorem (Probability of parisian ruin for the fluid flow risk process (if $E^\sigma = \emptyset$).)

If $E^\sigma = \emptyset$,

$$\psi_p(u) = \boldsymbol{\mu}\boldsymbol{P}(u)\left(\boldsymbol{I} - \boldsymbol{R}(0)\boldsymbol{P}(0)\right)^{-1}\left(\boldsymbol{e} - \boldsymbol{R}(0)\boldsymbol{e}\right) \tag{3}$$

or alternatively,

$$\psi_p(u) = \boldsymbol{\mu}\boldsymbol{P}(u)\boldsymbol{v}_p,$$

where

$$\boldsymbol{v}_p = \left(\boldsymbol{I} - \boldsymbol{R}(0)\boldsymbol{P}(0)\right)^{-1}\left(\boldsymbol{e} - \boldsymbol{R}(0)\boldsymbol{e}\right).$$

- The interpretation of the vector $\boldsymbol{v}_p$ is

$$\boldsymbol{v}_p(i) = \mathbb{P}\left(\begin{array}{c} \{R_t\}_{t\geq 0} \text{ gets ruined} \\ \text{in a parisian way} \end{array} \middle| \begin{array}{c} \text{The first downcrossing of 0} \\ \text{occured while in state } i \end{array} \right).$$

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- Notice that $\boldsymbol{v}_p$ is independent of the initial reserve.
- Since classic ruin for $\{R_t\}_{t\geq 0}$ is given by $\boldsymbol{\mu}\boldsymbol{P}(u)\boldsymbol{e}$, in order to compare parisian ruin with classical ruin, it is insightful to compare $\boldsymbol{v}_p$ and $\boldsymbol{e}$.

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- Solution: fix $\epsilon > 0$ and start the clocks at the times $\{R_t\}_{t\geq 0}$ downcrosses $-\epsilon$ and declare it recovered once it reaches level 0. Analogously, compute the probability that $\{R_t\}_{t\geq 0}$ is not able to recover before any of these clocks ring. Let $\epsilon \to 0$.

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Theorem (Probability of parisian ruin for the fluid flow risk process (if $E^\sigma \neq \emptyset$).)

If $E^\sigma \neq \emptyset$, then

$$\psi_p(u) = \boldsymbol{\mu} \boldsymbol{P}(u) \boldsymbol{v}_p,$$

where

$$\boldsymbol{v}_p = \lim_{\epsilon \downarrow 0} \left(\boldsymbol{I} - \boldsymbol{R}(\epsilon)\boldsymbol{P}(\epsilon) \right)^{-1} (\boldsymbol{e} - \boldsymbol{R}(\epsilon)\boldsymbol{e}).$$

A Fluid flow risk process example

Take

$$\boldsymbol{r} = (0.15, 1.2, -1, -1, -1, -1), \quad \boldsymbol{\sigma} = (0.2, 0.4, 0, 0, 0, 0), \quad \text{and}$$

$$\boldsymbol{\Lambda} = \left(\begin{array}{cc|cccc} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0 & 0 & 0 \\ \hline 5 & 0 & -6 & 1 & 0 & 0 \\ 3 & 1 & 0 & -6 & 2 & 0 \\ 1 & 2 & 0 & 0 & -6 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right)$$

In this model, the length of its jumps “dictates” in which environmental state we will end up after such jump ends.

Numerical example.

The values of $\boldsymbol{v}_p$ over $\{1, 2, 3, 4, 5, 6\}$ for the 20-stage Erlang-distributed parisian clock case of mean $1, 5, 10$ and 20 are shown below.

	$\mathbb{E}(L) = 1$	$\mathbb{E}(L) = 5$	$\mathbb{E}(L) = 10$	$\mathbb{E}(L) = 20$
σ_1	0.6677	0.4667	0.3915	0.2359
σ_2	0.5532	0.3922	0.3243	0.2022
$-_1$	0.7985	0.5512	0.448	0.2749
$-_2$	0.7737	0.5577	0.4585	0.2914
$-_3$	0.7694	0.5926	0.4964	0.3316
$-_4$	0.8191	0.6654	0.5647	0.391

Direct extensions.

- Penalised parisian ruin: take the rates and variance to be different when the process is below 0.

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