An *MAP/PH/K* Queue with Constant Impatient Time

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1. Introduction

- **Customer abandonment: Customers wait in queue and then leave the system without service**
  - Call center;
  - Supply chain;
  - Supermarket, Restaurant, Healthcare, etc.
- **Customer impatience**
- **System congestion control**

1. Introduction (continued)

- **Queues with constant impatient/abandonment time**
  - Constant approximation of abandonment time
  - System congestion/performance control

  - Choi et al. (2004): $MAP/M/K + \tau$ (τ is constant)
  - Kim and Kim (2014): $M/PH/1 + \tau$
  - He, Zhang, and Ye (2015): $M/PH/K + \tau$
  - He, Cai, and Huang (2016): $MAP/PH/K + \tau$

1. Introduction (continued)

- **Our model, method, and contribution**
  - **Model**: $MAP/PH/K + \tau$
  - **Method**: Matrix-analytic methods (*Explicit*)
  - **Potential contributions of our research**
    - Developed a computational procedure for computing distributions and moments of waiting times and queue lengths, for systems with
      - i) small and moderate *K* (from 1 to 100),
      - ii) phase-type service times, and
      - iii) a Markovian arrival process.

2. The *MAP/PH/K*+τ Queue

- Customers arrive according to a Markovian arrival process with matrix representation (D_0, D_1).
  - There is a underlying (continuous time) Markov chain $\{I_a(t), t>0\}$ with states $\{1, \ldots, m_a\}$ associated with the service time.
- All customers join a single queue waiting for service and are served on a first-come-first-served basis. If a customer's waiting time reaches constant time τ, the customer leaves the system immediately without service.
- There are K identical servers.
- The service time of each customer has a phase-type distribution with *PH*-representation ($\boldsymbol{\beta}$, S) of order m_s.
  - There is a underlying (continuous time) Markov chain $\{I_s(t), t>0\}$ with states $\{0, 1, \ldots, m_s\}$ associated with the service time.

2. The $MAP/PH/K+\tau$ Queue (continued)

- **System state: Track-phase-for-server (TPFS)**
  - $a(t)$: the age of the first customer waiting in the queue at time t, if the (waiting) queue is not empty; otherwise, $a(t) = -\infty$.
  - $I_k(t)$: the phase of the server k at time t, for $k = 1, 2, \ldots, K$.
  - $\{(a(t), I_a(t), I_1(t), \ldots, I_K(t)), t>0\}$ is a continuous time Markov chain.

- **System state: Count-server-for-phase (CSFP)**
  - $a(t)$: the age of the first customer waiting in the queue at time t, if the (waiting) queue is not empty; otherwise, $a(t) = -\infty$.
  - $n_i(t)$: the number of servers whose service phase is i at time t, for $i = 1, 2, \ldots, m_s$.
  - $\{(a(t), I_a(t), n_1(t), \ldots, n_{m\_s}(t)), t > 0\}$ is a continuous time Markov chain.

3. Existing Approach for $M/PH/K+\tau$

- **For states with $a(t) = -\infty$ (no one is waiting)**
  - $p_0 = P\{a(t) = -\infty\}$;
  - for $k = 1, 2, \ldots, K$, and $n_1 + \ldots n_{m\_s} = k$,

    $p_k(n_1, \ldots, n_{m\_s}) = P\{a(t) = -\infty, n_1(t) = n_1, \ldots, n_{m\_s}(t) = n_{m\_s}\}$;

    $\mathbf{p}_k = (p_k(\mathbf{n}), \mathbf{n} \in \Omega(k))$ (*probabilities*)

- **For states with $a(t) > 0$ (at least one is waiting)**

$$p_{K+1}(x, n_1, \ldots, n_{m_s}) = \frac{\mathrm{d}}{\mathrm{d}\,x} P\{a(t) < x, n_i(t) = n_i, i = 1, \ldots, m_s\}, \quad \text{for } 0 \le x < \tau,$$

  - The vector $\mathbf{p}_{K+1}(x)$ (*transition rates*) is defined in a way similar to $\mathbf{p}_K$.

3. Existing Approach for $M/PH/K+\tau$ (continued)

- **Fundamental equations for $\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_K, \mathbf{p}_{K+1}(x)\}$**

$$0 = \mathbf{p}_0(-\lambda I + Q) + \mathbf{p}_1(M \otimes S^-(1, m_s));$$

$$0 = \mathbf{p}_{k-1}(I \otimes S^+(k-1, m_s)) + \mathbf{p}_k\left(-\lambda I + Q \otimes I + M \otimes S(k, m_s)\right) + \mathbf{p}_{k+1}\left(M \otimes S^-(k+1, m_s)\right),$$

$$\text{for} \quad k = 1, 2, \ldots, K-1;$$

$$0 = \mathbf{p}_{K-1}(I \otimes S^+(K-1, m_s)) + \mathbf{p}_K\left(-\lambda I + Q \otimes I + M \otimes S(K, m_s)\right)$$

$$+ \frac{1}{\lambda}\int_0^\tau \mathbf{p}_{K+1}(y) e^{-\lambda y}\, \mathrm{d}y \left(M \otimes \left(S^-(K, m_s) S^+(K-1, m_s)\right)\right) + \mathbf{p}_{K+1}(\tau) e^{-\lambda\tau};$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{p}_{K+1}(x) = \mathbf{p}_{K+1}(x)\left(Q \otimes I + M \otimes S(K, m_s)\right) + \mathbf{p}_{K+1}(\tau)\lambda e^{-\lambda(\tau - x)}$$

$$+ \int_x^\tau \mathbf{p}_{K+1}(y) e^{-\lambda(y-x)}\, \mathrm{d}y \left(M \otimes \left(S^-(K, m_s) S^+(K-1, m_s)\right)\right), \quad 0 \le x < \tau;$$

$$\mathbf{p}_{K+1}(0) = \lambda \mathbf{p}_K.$$

3. Existing Approach for *M/PH/K+τ* (continued)

- **Solution approach for {$\mathbf{p}_0$, $\mathbf{p}_1$, …, $\mathbf{p}_K$, $\mathbf{p}_{K+1}(x)$}**
  - For the *M/PH/K+τ* case:

$$0 = \frac{\mathrm{d}^2 \mathbf{p}_{K+1}(x)}{\mathrm{d}\,x^2} - \frac{\mathrm{d}\,\mathbf{p}_{K+1}(x)}{\mathrm{d}\,x}\left(\lambda I + S(K, m_\mathrm{s})\right) + \lambda \mathbf{p}_{K+1}(x)\left(S(K, m_\mathrm{s}) + S^-(K, m_\mathrm{s}) S^+(K, m_\mathrm{s}) / \lambda\right).$$

3. Existing Approach for *M/PH/K*+τ (continued)

1. **Use a routine method for QBD process**

$$\mathbf{p}_k = \mathbf{p}_K D_K \cdots D_{k+1}, \quad \text{for } k = 0, 1, 2, \ldots, K-1.$$

2. **Following the approach in Choi et al. (2004) or Kim and Kim (2014)**

$$\mathbf{p}_{K+1}(x) = \mathbf{u}_1 \exp\{\lambda(R-I)(\tau - x)\} + \mathbf{u}_2 \exp\{(\lambda G + Q \otimes I + M \otimes S(K, m_s))x\},$$

where $\mathbf{u}_1$, $\mathbf{u}_2$, R, and G are constant vectors/matrices.

3. **Boundary queue length distribution:**

$$\mathbf{p}_K = \frac{1}{\lambda}\mathbf{p}_{K+1}(0) = \frac{1}{\lambda}\left(\mathbf{u}_1 \exp\{\lambda(R-I)\tau\} + \mathbf{u}_2\right).$$

3. Existing Approach for $M/PH/K+\tau$ (continued)

- **Computation procedure for stationary distribution $\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_K, \mathbf{p}_{K+1}(x)\}$:**
  - Computing $\{R, G\}$ (e.g., Logarithmic reduction)
  - Construct ϕ, and compute ξ or ζ
  - Computing $\{\mathbf{u}_1, \mathbf{u}_2\}$
  - Compute $\mathbf{p}_{K+1}(x)$
  - Compute $\mathbf{p}_K = \mathbf{p}_{K+1}(0)/\lambda$
  - Compute $\{D_k, k = 1, 2, \ldots, K\}$
  - Compute $\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{K-1}\}$
  - Performance measures (loss probability, waiting time, queue length, etc.)

4. Proposed Approach for *MAP/PH/K+τ*

- **Fundamental equations for {$\mathbf{p}_0$, $\mathbf{p}_1$, …, $\mathbf{p}_K$, $\mathbf{p}_{K+1}(x)$}**

$$0 = \mathbf{p}_0 A_{0,0} + \mathbf{p}_1 A_{1,0};$$

$$0 = \mathbf{p}_{k-1} A_{k-1,k} + \mathbf{p}_k A_{k,k} + \mathbf{p}_{k+1} A_{k+1,k}, \quad \text{for } k = 1, 2, \ldots, K-1;$$

$$0 = \mathbf{p}_{K-1} A_{K-1,K} + \mathbf{p}_K A_{K,K} + \int_0^\tau \mathbf{p}_{K+1}(y)\left(e^{D_0 y} \otimes \left(Q^-(K, m_s) P^+(K-1, m_s)\right)\right) \mathrm{d}y + \mathbf{p}_{K+1}(\tau)\left(e^{D_0 \tau} \otimes I\right);$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathbf{p}_{K+1}(x) = \mathbf{p}_{K+1}(x)\left(I \otimes Q(K, m_s)\right) + \mathbf{p}_{K+1}(\tau)\left(e^{D_0(\tau - x)} D_1 \otimes I\right) + \int_x^\tau \mathbf{p}_{K+1}(y)\left(e^{D_0(y-x)} D_1 \otimes \left(Q^-(K, m_s) P^+(K-1, m_s)\right)\right) \mathrm{d}y, \quad 0 \le x < \tau;$$

$$\mathbf{p}_{K+1}(0) = \mathbf{p}_K (D_1 \otimes I),$$

4. Proposed Approach for *MAP/PH/K+τ* (continued)

- **Solution approach for {$\mathbf{p}_0$, $\mathbf{p}_1$, …, $\mathbf{p}_K$, $\mathbf{p}_{K+1}(x)$}**
  - For the *M/PH/K+τ* case:

$$0 = \frac{\mathrm{d}^2 \mathbf{p}_{K+1}(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}\mathbf{p}_{K+1}(x)}{\mathrm{d}x}\left(\lambda I + S(K, m_\mathrm{s})\right) + \lambda \mathbf{p}_{K+1}(x)\left(S(K, m_\mathrm{s}) + S^-(K, m_\mathrm{s})S^+(K, m_\mathrm{s})/\lambda\right).$$

  - For the *MAP/PH/K+τ* case, the above second order (vector) differential equation cannot be obtained (due to the commutability of matrices).
  - Our proposed approach: *Laplace-Stieltjes Transform (LST) of the vector function* $\mathbf{p}_{K+1}(x)$.

4. Proposed Approach for *MAP/PH/K*+τ (continued)

- **LST of $\mathbf{p}_{K+1}(x)$**
  - Definition: $\mathbf{f}^*_{K+1}(s) = \int_0^\tau \mathbf{p}_{K+1}(x)\exp\{-sx\}\,\mathrm{d}\,x$
  - The fundamental equation of $\mathbf{p}_{K+1}(x)$ becomes

$$\mathbf{f}^*_{K+1}(s)A^*(s) = C^*(s),$$

where

$$A^*(s) = \left((sI + D_0)^{-1} \otimes I\right)B^*(s);$$

$$B^*(s) = (sI + D_0) \otimes \left(sI - Q(K, m_s)\right) + D_1 \otimes \left(Q^-(K, m_s)P^+(K-1, m_s)\right);$$

$$C^*(s) = \mathbf{p}_{K+1}(0) + \mathbf{f}^*_{K+1}(-D_0)\left(\left((sI + D_0)^{-1}D_1\right) \otimes \left(Q^-(K, m_s)P^+(K-1, m_s)\right)\right)$$
$$+ \mathbf{p}_{K+1}(\tau)\left(\left((sI + D_0)^{-1}\left(e^{D_0\tau}D_1 - e^{-s\tau}(sI + D)\right)\right) \otimes I\right);$$

$$\mathbf{f}^*_{K+1}(-D_0) = \int_0^\tau \mathbf{p}_{K+1}(y)\left(e^{D_0 y} \otimes I\right)\mathrm{d}\,y,$$

4. Proposed Approach for *MAP/PH/K+τ* (continued)

- **Characterization of the roots of det($B^*(s)$)** (conjecture to be shown)
  - Half of the roots with positive real part;
  - Half with negative real part.
- **Linear system for constant vectors** (validity depending on independence of some vectors, which is not guaranteed.)

$$\mathbf{p}_{K+1}(0)U^{+} + \mathbf{p}_{K+1}(\tau)V^{+} + \mathbf{f}^{*}_{K+1}(-D_0)W^{+} = 0;$$
$$\mathbf{p}_{K+1}(0)U^{-} + \mathbf{p}_{K+1}(\tau)V^{-} + \mathbf{f}^{*}_{K+1}(-D_0)W^{-} = 0,$$

  - $\{U^+, V^+, W^+\}$ are associated with roots with positive real part;
  - $\{U^-, V^-, W^-\}$ are associated with roots with positive real part;

4. Proposed Approach for *MAP/PH/K+τ* (continued)

- **A solution**

However, linear independence is not guaranteed in general, and the matrix is invertible only for $m_s \le 2$.
- $Q^-(K, m_s)P^+(K\text{-}1, m_s)$ is singular!!!

$$\left(M_{\mathbf{p}_{K+1}(0)}, M_{\mathbf{p}_{K+1}(\tau)}\right) = -\left(W^+, W^-\right)\begin{pmatrix} U^+ & U^- \\ V^+ & V^- \end{pmatrix}^{-1}$$

$$\mathbf{p}_K = \mathbf{f}^*_{K+1}(-D_0)M_{\mathbf{p}_K}$$

$$M_{\mathbf{p}_K} = -\left(I \otimes \left(Q^-(K,m_s)P^+(K-1,m_s)\right) + M_{\mathbf{p}_{K+1}(\tau)}\left(e^{D_0\tau} \otimes I\right)\right) \cdot \left(M_K(D_1 \otimes P^+(K-1,m_s)) + D_0 \otimes I + I \otimes Q(K,m_s)\right)^{-1}.$$

$$\mathbf{p}_{K+1}(0) = \mathbf{f}^*_{K+1}(-D_0)M_{\mathbf{p}_K}(D_1 \otimes I),$$

$$\mathbf{p}_{K+1}(\tau) = \mathbf{f}^*_{K+1}(-D_0)M_{\mathbf{p}_{K+1}(\tau)},$$

$$\mathbf{f}^*_{K+1}(-D_0)Q_{\mathbf{f}^*_{K+1}(-D_0)} = 0 \qquad Q_{\mathbf{f}^*_{K+1}(-D_0)} = M_{\mathbf{p}_{K+1}(0)} - M_{\mathbf{p}_K}(D_1 \otimes I).$$

5. Numerical Examples

- **Consider an *MAP/PH/K* + τ queue with** $\tau = 1$,

$$m_a = 2, \quad D_0 = \begin{pmatrix} -4 & 2 \\ 3 & -9 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix};$$

$$m_s = 2, \quad \boldsymbol{\beta} = (0.8, 0.2), \quad S = \begin{pmatrix} -3 & 2 \\ 0.5 & -2 \end{pmatrix}.$$

K	1	2	3	4	5	6	7
$E[W_a]$	0.8742	0.6759	0.4258	0.2097	0.0847	0.0303	0.0101
$E[N_{all}]$	4.2154	4.3456	4.0425	3.5505	3.1854	3.0055	2.9341

6. Current Research

- **The** $MAP/PH/K + \tau$ **queue** (undergoing)
  - Characterization of the roots of $\det(B^*(s))$;
  - Find new independent vectors to form a linear system for constant vectors;
  - Try probabilistic approaches for some constant vectors;
  - …

Thank you very much!

Any question and suggestion?