Markov-modulated fluid priority queues

Waiting time and queue length analysis

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Introduction

Model Description

The model studied:

- Continuous (fluid) queue
- Multiple fluid types, priority service
- Fluid arrival process:
  - Common CTMC generator: $\mathbf{Q}$
  - Fluid rates of class k: $\mathbf{R}^{(k)}$
- Fluid service process
  - Constant service rate d

Results:

- Waiting time
  - LST, moments, distribution function (approx.)
- Queue length
  - LST, moments, distribution function (approx.)
- We can't obtain (by this approach):
  - Properties of the joint distribution of the queues

Concept of the Solution

- Based on the tagged customer approach
- For a class k fluid drop:
    - $< k$ priority classes can be omitted
    - Before it can leave:
        - The server has to accomplish the class $k+$ workload present at arrival
        - ...plus the higher pr. workload arrived while waiting
    - Performance measures:
        - Waiting time: this duration
        - Queue length at fluid drop departures: The amount of class k fluid arriving over this duration
- Ingredients of the solution:
    - Construction of special fluid flows for the perf. measures
    - Waiting time & queue length at dep. $\rightarrow$ related to a reward accumulation problem during a busy period of this fluid model
    - Queue length at arbitrary time $\rightarrow$ from the one at departures

The Busy Period of Markovian Fluid Models

Stationary Solution of Markovian Fluid Models

- Parameters:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{++} & \mathbf{F}_{+-} & \mathbf{F}_{+0} \\ \mathbf{F}_{-+} & \mathbf{F}_{--} & \mathbf{F}_{-0} \\ \mathbf{F}_{0+} & \mathbf{F}_{0-} & \mathbf{F}_{00} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{+} & & \\ & \mathbf{C}_{-} & \\ & & \mathbf{0} \end{bmatrix},$$

- Differential equation:

$$\frac{d}{dx}\pi(x)\mathbf{C} = \pi(x)\mathbf{F}, \qquad \pi(0)\mathbf{C} = p\,\mathbf{F}, \qquad p_i = 0, \quad \forall i : c_i > 0.$$

- Matrix-analytic solution:

$$\pi(x) = \kappa\, e^{\mathbf{K}x} \begin{bmatrix} \mathbf{I} & \mathbf{\Psi} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{+} & \\ & |\mathbf{C}_{-}| \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{F}_{+0}(\mathbf{F}_{00})^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{F}_{-0}(\mathbf{F}_{00})^{-1} \end{bmatrix},$$

- Two important matrices:
  - $\mathbf{\Psi}$: phase transition probs. over the busy period
  - $\mathbf{K}$: $e^{\mathbf{K}x}$ is the expected number of crossings of level x

The properties of the busy period

- Matrix $\boldsymbol{\Psi}$: the solution of a Riccati equation

$$\boldsymbol{\Psi}|\mathbf{C}_{-}|^{-1}\mathbf{F}^{\bullet}_{-+}\boldsymbol{\Psi} + \boldsymbol{\Psi}|\mathbf{C}_{-}|^{-1}\mathbf{F}^{\bullet}_{--} + \mathbf{C}_{+}{}^{-1}\mathbf{F}^{\bullet}_{++}\boldsymbol{\Psi} + \mathbf{C}_{+}{}^{-1}\mathbf{F}^{\bullet}_{+-} = \mathbf{0},$$

- Matrix $\boldsymbol{\Psi}^{*}(s)$: LST of the busy period distribution + phase transition probs. $\rightarrow$ solution of a Riccati equation
- Moments of the busy period duration: from the derivatives of $\boldsymbol{\Psi}^{*}(s)$ (solution of 1 Riccati and n Sylvester type equations)
- Approximation for $\boldsymbol{\Psi}(t)$: based on Erlangization:

$$\boldsymbol{\Psi}_{n}(t) = \int_{0}^{\infty} f_{\mathcal{E}(n,n/t)}(u) \cdot \boldsymbol{\Psi}(u)\, du$$

  - Interpretation: $\boldsymbol{\Psi}_{n}(t) = P$(the busy period is shorter than an Erlang)
  - Solution: count the number of Erlang phases during the busy period $\rightarrow$ recursions for $\boldsymbol{\Psi}_{n}(t)$ (Riccati+Sylvester equations)

The busy period with non-zero initial fluid

- If the initial fluid level is x, the LST of the busy period duration is:

$$
\begin{aligned}
\mathbf{G}^*_{+-}(s,x) &= \mathbf{\Psi}^*(s)\mathbf{G}^*_{--}(s,x), \\
\mathbf{G}^*_{--}(s,x) &= e^{\mathbf{H}^*_G(s)x}, \\
\mathbf{G}^*_{0-}(s,x) &= (s\mathbf{I}-\mathbf{F}_{00})^{-1}\mathbf{F}_{0+}\mathbf{G}^*_{+-}(s,x) + (s\mathbf{I}-\mathbf{F}_{00})^{-1}\mathbf{F}_{0-}\mathbf{G}^*_{--}(s,x)
\end{aligned}
$$

- Important point: matrix-exponential in x!

Accumulated reward during the busy period

- Besides $\mathbf{F}, \mathbf{C}$ we now have (diagonal) $\mathbf{D}$ as well
- $\mathbf{D}_{ii}$: reward accumulation rate in i
- We are interested in the distribution of the reward accumulated during the busy period $\mathbf{\Phi}(y)$
- LST: $\mathbf{\Phi}^*(v)$, from Riccati equation, very similar to $\mathbf{\Psi}^*(s)$
- Moments: from the derivatives of $\mathbf{\Phi}^*(v)$ (1 Riccati, n Sylvester)
- Approximation for $\mathbf{\Phi}(t)$: Erlangization can be adapted
- Non-zero initial fluid $x \to \mathbf{B}(y, x)$
- The LST $\mathbf{B}^*(v, x)$ is

$$\mathbf{B}^*_{+-}(v,x) = \mathbf{\Phi}^*(v)\mathbf{B}^*_{--}(v,x),$$
$$\mathbf{B}^*_{--}(v,x) = e^{\mathbf{H}^*_B(v)x},$$
$$\mathbf{B}^*_{0-}(v,x) = (v\mathbf{D}_0 - \mathbf{F}_{00})^{-1}\mathbf{F}_{0+}\mathbf{B}^*_{+-}(v,x) + (v\mathbf{D}_0 - \mathbf{F}_{00})^{-1}\mathbf{F}_{0-}\mathbf{B}^*_{--}(v,x)$$

- Again, matrix-exponential in x

Analysis of the Priority Queue

Workload at fluid drop arrival

Question: how much class $k+$ workload is present in the system when a fluid drop arrives?

Workload process analysis:

- Workload process $\neq$ queue length process
- In state i, over Δ amount of time the service requirement brought into the system is: $\Delta r_i^{k+}/d$
- The workload decreases with slope 1
- Resulting diff. equations:

$$\frac{\partial}{\partial t} v(t,x) + \frac{\partial}{\partial x} v(t,x)(\mathbf{R}^{(k+)}/d - \mathbf{I}) = v(t,x)\mathbf{Q}$$

- $\rightarrow$ Markovian fluid flow model!
- Stationary solution: $v(x) = \kappa e^{\mathbf{K}x}\mathbf{A}$
- Solution at drop arrival instants: $v_A(x) = \frac{1}{\lambda^{(k)}}\kappa e^{\mathbf{K}x}\mathbf{A}\mathbf{R}^{(\mathbf{k})}$

Waiting Time of Fluid Drops

- We create a special fluid flow model
- Goal: the accumulated reward over the busy period (initiated at level 0) = waiting time of fluid drops
- Parameters of the special system:

$$\mathbf{F} = \begin{bmatrix} \mathbf{K} & \mathbf{A}\mathbf{R}^{(k)}/\lambda^{(k)} \\ & \mathbf{Q} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{I} & \\ & \mathbf{R}^{((k+1)+)}/d - \mathbf{I} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} & \\ & \mathbf{I} \end{bmatrix}$$

- Purpose of the first state group:
  - To set the workload seen by an arriving drop (recall: $v_A(x) = \frac{1}{\lambda^{(k)}} \kappa e^{\mathbf{K}x} \mathbf{A}\mathbf{R}^{(\mathbf{k})}$)
- Purpose of the second state group:
  - Fluid drop joined the queue, server is processing the workload
  - Time spent here is measured by the reward rate
- Properties of the waiting time: from $\mathbf{\Phi}^*(v)$, its derivatives, and its Erlangization based approximation

Queue Length at Departure Instants

- An other similar special fluid model with reward is created:

$$\mathbf{F} = \begin{bmatrix} \mathbf{K} & \mathbf{A}\mathbf{R}^{(k)}/\lambda^{(k)} \\ & \mathbf{Q} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \mathbf{I} & \\ & \mathbf{R}^{((k+1)+)}/d - \mathbf{I} \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} \mathbf{0} & \\ & \mathbf{R}^{(k)} \end{bmatrix}$$

- Purpose of the first state group:
  - To set the workload seen by an arriving drop
- Purpose of the second state group:
  - Fluid drop joined the queue, server is processing the workload
  - The amount of arriving class k fluid is measured by the reward rate
- Properties of the queue length at departures: from $\mathbf{\Phi}^*(v)$, its derivatives, and its Erlangization based approximation

Queue Length at Random Point in Time

- Known: queue length at departures $\mathcal{X}$,
  Question: at random point in time $\mathcal{Y}$
- Relation is well known for discrete queues, but not for fluid
- Relation (f: pdf, F: cdf, p mass at 0):

$$\underline{f_{\mathcal{Y}}}(u)\mathbf{R}^{(k)} - \underline{F_{\mathcal{Y}}}(u)\mathbf{Q} = \lambda^{(k)}\underline{f_{\mathcal{X}}}(u), \qquad \lambda^{(k)}\underline{p_{\mathcal{X}}} = \underline{p_{\mathcal{Y}}}\mathbf{R}^{(k)}$$

- Proof: by simple balance equations
- the rate at which state $(\mathcal{Y}(t) = x, \mathcal{J}(t) = j)$ is left (for $x > 0$):

$$\lambda^{(k)}/\delta \cdot \underline{f_{\mathcal{X}}}_j(x-\delta) + \underline{f_{\mathcal{Y}}}_j(x)r_j^{(k)}/\delta + \underline{f_{\mathcal{Y}}}_j(x)(-q_{jj})$$

- the rate at which the system enters state $(\mathcal{Y}(t) = x, \mathcal{J}(t) = j)$ is

$$\lambda^{(k)}/\delta \cdot \underline{f_{\mathcal{X}}}_j(x) + \underline{f_{\mathcal{Y}}}_j(x-\delta)r_j^{(k)}/\delta + \sum_{i \neq j} \underline{f_{\mathcal{Y}}}_i(x)q_{ij}$$

- Equating the two and $\delta \to 0$ provides the theorem
- The relation in LST domain is: $\underline{f_{\mathcal{Y}}^*}(s)(s\,\mathbf{R}^{(k)} - \mathbf{Q}) = \lambda^{(k)}\,s\,\underline{f_{\mathcal{X}}^*}(s)$
- From the moments of $\mathcal{X}$ $\to$ the moments of $\mathcal{Y}$: easy
- From the Erlangization of $\mathcal{X}$ $\to$ the one of $\mathcal{Y}$: less easy

Numerical Example

Example 1

- MATLAB implementation
- Solving Riccati: ADDA, solving Sylvester: lyap (Hessenberg-Schur)

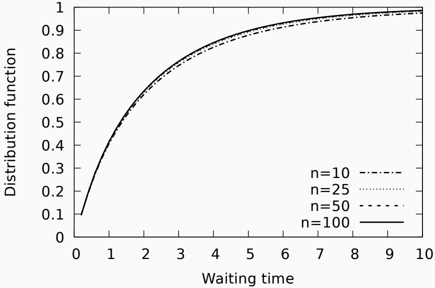
$$\mathbf{Q} = \begin{bmatrix} -8 & 5 & 0 & 3 \\ 3 & -4 & 0 & 1 \\ 4 & 6 & -10 & 0 \\ 2 & 3 & 10 & -15 \end{bmatrix}, \qquad \mathbf{R}^{(1)} = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 2 & \\ & & & 1 \end{bmatrix},$$

$$\mathbf{R}^{(2)} = \begin{bmatrix} 2 & & & \\ & 0 & & \\ & & 4 & \\ & & & 1 \end{bmatrix}, \qquad \mathbf{R}^{(3)} = \begin{bmatrix} 4.5 & & & \\ & 1 & & \\ & & 0 & \\ & & & 2 \end{bmatrix},$$

- $d = 4 \rightarrow$ utilization $= 0.875$
- Computing 3 moments of the waiting time and queue length
    - Prompt response

Example 1

- Accuracy of the Erlangization based approximation:



- Computation times of class 1 waiting time distribution in 50 points:

	$r_3^{(2)} = 4$	$r_3^{(2)} = 4.1$
$n = 10$	$0.523\,s$	$0.189\,s$
$n = 25$	$4.771\,s$	$0.543\,s$
$n = 50$	$33.255\,s$	$1.427\,s$
$n = 100$	$249.72\,s$	$4.616\,s$

Example 2

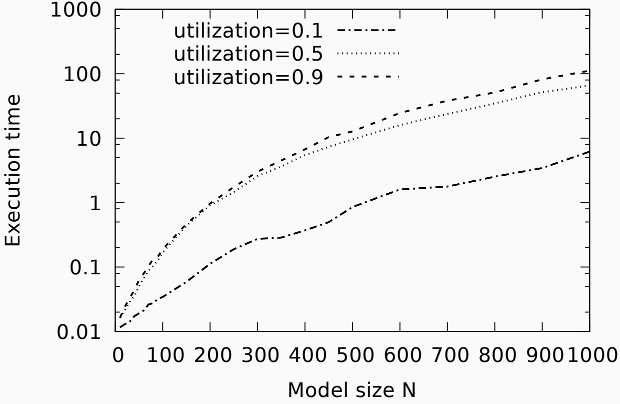
- N determines the size of the model

$$\mathbf{Q}_N = \begin{bmatrix} \bullet & N\nu & & & & \\ \gamma & \bullet & (N-1)\nu & & & \\ & 2\gamma & \bullet & (N-2)\nu & & \\ & & \ddots & \ddots & \ddots & \\ & & & (N-1)\gamma & \bullet & \nu \\ & & & & N\gamma & \bullet \end{bmatrix}, \mathbf{R}_N = \begin{bmatrix} 0 & & & & \\ & \rho/N & & & \\ & & 2\rho/N & & \\ & & & \ddots & \\ & & & & \rho \end{bmatrix},$$

- Per-class fluid rates:
  $\mathbf{R}_N^{(1)} = 0.1 \cdot \mathbf{R}_N, \mathbf{R}_N^{(2)} = 0.3 \cdot \mathbf{R}_N$ and $\mathbf{R}_N^{(3)} = 0.6 \cdot \mathbf{R}_N$
- 5 moments are computed, N varies, utilization varies

Example 1

- Execution time vs. model size:



- Size 1000 models in 10 − −100 seconds!

Conclusion

Conclusion

- Some new results are presented on fluid queues
- They are glued together with existing ones to enable the efficient analysis of fluid priority queues
- Priority queues with huge background process can be analyzed
  - Using standard math frameworks (MATLAB)
  - Without any numerical difficulties
  - With reasonable computation times