Markov modulated Brownian motion and the flip-flop fluid queue

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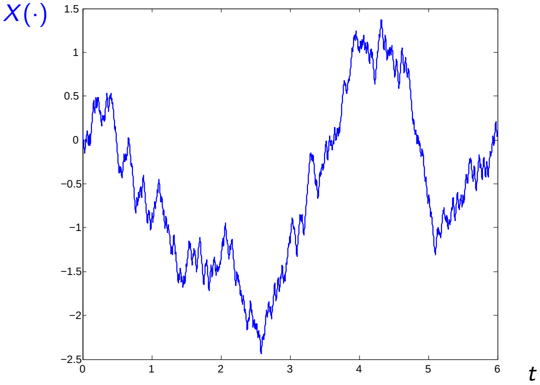
Outline

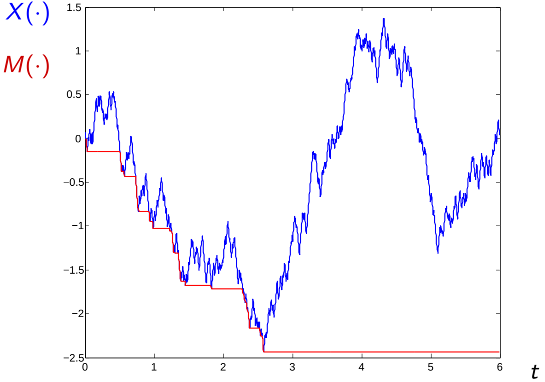
Brownian motion



Approximate simulation of a BM

$\mu = 0$, $\sigma = 1$.

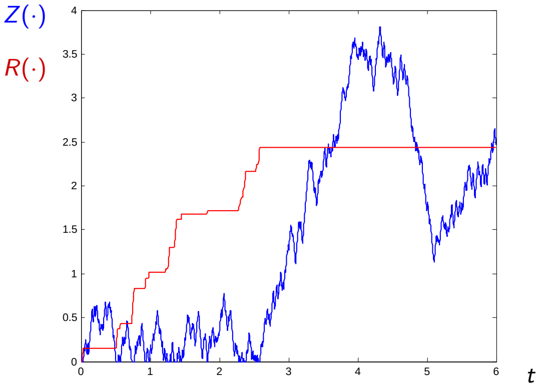
Regulator



$M(t) = \min\{X(s) : 0 \le s \le t\}$

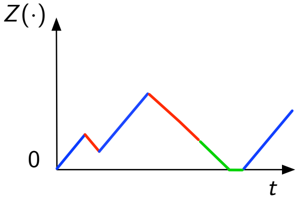
Regulator: $R(t) = |M(t)|$ regulated process: $Z(t) = X(t) + R(t)$

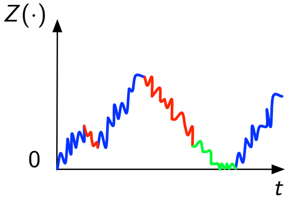
Regulated BM



- R increases only when $Z(t) = 0$

MMBM and FQ

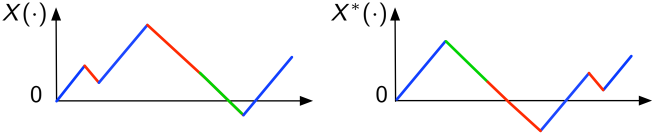




- $\varphi(\cdot)$: Markov process of phases,
- $\varphi(s) = i \quad \rightarrow \quad \text{BM}(\mu_i, \sigma_i$
  $\sigma_1 = \cdots = \sigma_m = 0 \qquad \rightarrow$ Fluid Queue
- Intervals of sojourn at zero for fluid
  No sojourn at zero for BM
- Focus on stationary distribution: drift is negative
  BM: assume $\sigma_i > 0$ for all i

Regenerative Approach

Reversed process



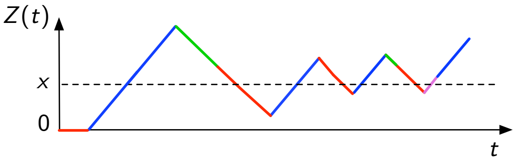
- $X(\cdot)$: Markov modulated process.
- $Z(\cdot)$: regulated process, boundary at zero
- $\varphi(\cdot)$: phase (control) process with stationary distribution $\underline{\alpha}$
- Reversed process: $X^*(t) = -X(-t), \quad \varphi^*(t) = \varphi(-t)$

Rogers '94, Asmussen '95

$$\lim_{t \to \infty} \mathrm{P}[\varphi(t) = i, Z(t) \leq x] = \alpha_i \mathrm{P}[\sup_{u \geq 0} X^*(u) \leq x | \varphi^*(0) = i]$$

Matrix-analytics for fluid queues

Starting with Ram's paper at ITC 1999.



- two subsets of phases: S_+ and S_- such that fluid $\uparrow$ or $\downarrow$

$$\text{mass at } 0: \quad \underline{\gamma}(T_{--} + T_{-+}\Psi) = \underline{0}$$

$$\text{density at } x = \underline{\gamma}\, T_{-+}\; e^{Kx} \left[C_+^{-1} \quad \Psi |C_-|^{-1} \right] \qquad x > 0$$

- · Ψ first return probability to level 0
- · $(e^{Kx} \left[I \quad \Psi \right])_{ij}$ = E[number of crossings] of (x, j), taboo of 0

- physically meaningful

  clean separation between boundary $x = 0$ and interior $x > 0$

Opportunities for extensions

- Finite buffer
- Reactive boundaries

  change of phase upon hitting boundary,
  change of generator while at boundary
- Piecewise level-dependent fluid rates
- Two-dimensional fluid model
- Algorithms to compute the key matrix Ψ
- What about MMBMs? Ψ does not make sense as such.

  By design, MMBMs are about intervals

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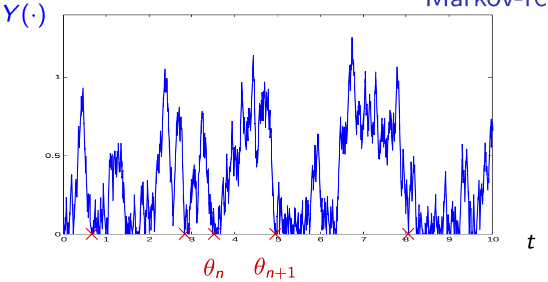
Markov-regenerative approach



Need

- Regenerative epochs $\{\theta_n\}$
- $\underline{\rho}$ = stationary distribution of $\varphi(\theta_n)$
- $M_{ij}(x) = E[\text{time spent in } [0, x] \times j \text{ between } \theta_n \text{ and } \theta_{n+1} \mid \varphi(\theta_n) = i]$

Then

$$\lim_{t\to\infty} P[Y(t) \le x, \varphi(t) = j] = (\gamma\, \underline{\rho}\, M(x))_j$$

The flip-flop

Fluid queues and the BM

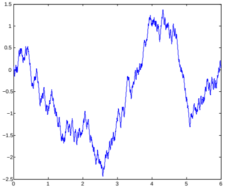
Ram at MAM-in-NY (2011): fluid queue with 2 phases.

Transition matrix: $T = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$

Fluid rates: $c_+ = \mu + \sigma\sqrt{\lambda}$, $c_- = \mu - \sigma\sqrt{\lambda}$.

- Oscillates faster as $\lambda \to \infty$,
- Amplitude increases
- Converges to BM(μ, σ)

Example: $\lambda = 100$, $\mu = 0$, $\sigma = 1$



Markov Modulated flip-flop

Flip-flop parameters:

generator $T_\lambda = \begin{bmatrix} T - \lambda I & \lambda I \\ \lambda I & T - \lambda I \end{bmatrix}$

fluid rates $C^* = \begin{bmatrix} \Delta_\mu + \sqrt{\lambda}\Delta_\sigma & \\ & \Delta_\mu - \sqrt{\lambda}\Delta_\sigma \end{bmatrix}$

with $\Delta_v = \mathrm{diag}(v_1, \ldots, v_m)$

Two copies of the phase Markov process,

κ_λ tells us whether copy $+$ or copy $-$ is active

three-dimensional process $\{L_\lambda(t), \varphi_\lambda(t), \kappa_\lambda(t)\}$ to be projected on $\{L_\lambda(t), \varphi_\lambda(t)\}$

Convergence (G.L. & G.N., 2015)

- Projection $\{L_\lambda(t), \varphi_\lambda(t) : t \geq 0\}$ converges weakly to $\{X(t), \varphi(t) : t \geq 0\}$.
- Weak convergence for process regulated at 0, as well as for finite buffer.
- Convergence of stationary distributions
- Establish connection to earlier results (duality / time and level reversal, spectral decomposition)

Evolution of road map

- Take λ big (and finite) and apply algorithms from fluid queue theory to compute approximations for MMBMs.
- Determine characteristic for the flip-flop and formally take $\lim_{\lambda\to\infty}$

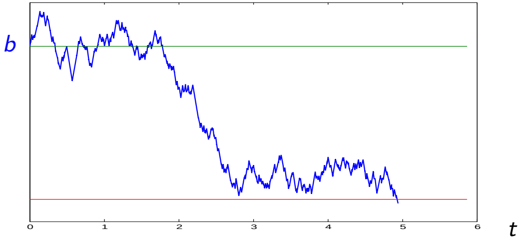
  Use flip-flop to construct building blocks

  example: first passage probability matrix from regulated level 0 to level x
- Work on new processes (two examples later)

Matrix U

For a while: sidetracked by the importance of Ψ for the fluid queues.

For MMBM, fundamental matrix is U:



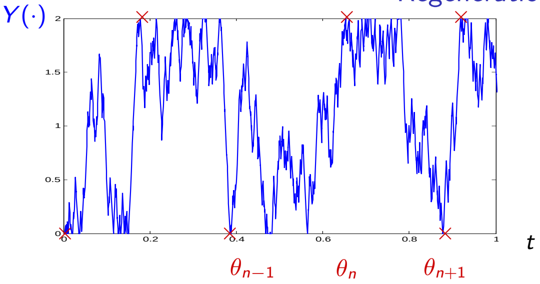
$(e^{Ub})_{ij} = \mathrm{P}[\text{reach 0 in phase } j \,|\, \text{start from } (b, i)]$.

$U(\lambda)$ for flip-flop is analytic function around $1/\lambda = 0$,

converges to U of MMBM

Two boundaries

Regeneration for two boundaries



Alternate first visits to 0 and first visits to upper bound b

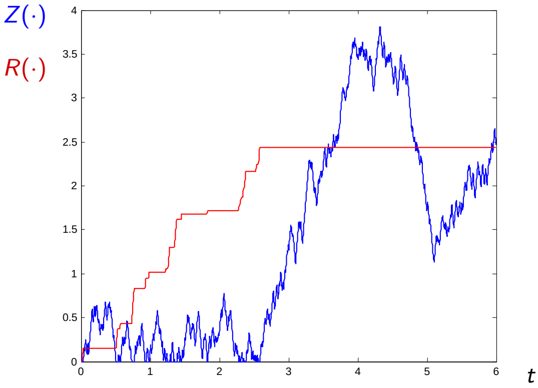
Need transition probabilities $P_{0\rightsquigarrow b}$ and $P_{b\rightsquigarrow 0}$ and expected time in $[0, x] \times j$ during an excursion

Obtained from flip-flop

Reactive boundary for free. Example: one set of parameters between θ_{n-1} and θ_n and another one between θ_n and θ_{n+1}

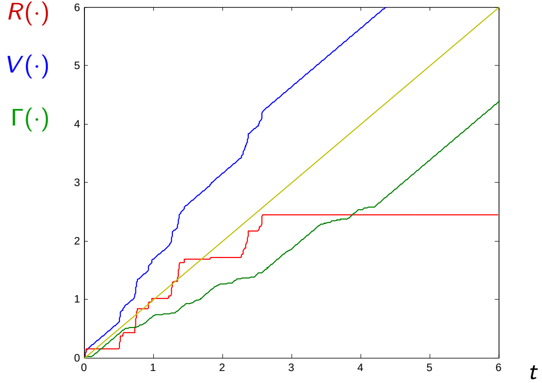
Sticky boundary (BM)

Regulator



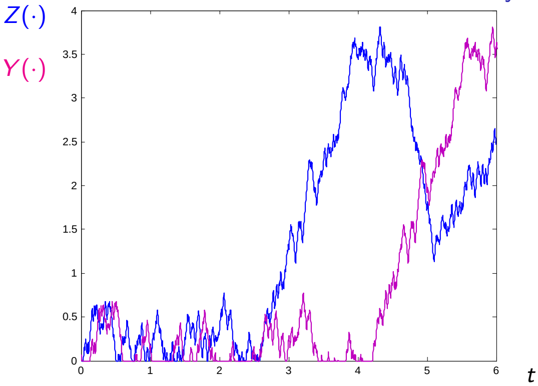
- R increases only when $Z(t) = 0$

Change of clock



- $V(t) = t + R(t)/\omega$ for some $\omega > 0$ — grows faster than t when $Z(t) = 0$
- Γ such that $V(\Gamma(t)) = t$ New clock: slowed down when $Z(t) = 0$
- $Y(t) = Z(\Gamma(t))$ BM with sticky boundary

BM with sticky boundary



- $V(t) = t + R(t)/\omega$ for some $\omega > 0$ here, $\omega = 1.5$
- Γ such that $V(\Gamma(t)) = t$ New clock: slowed down when $Z(t) = 0$
- $Y(t) = Z(\Gamma(t))$ BM with sticky boundary

Sticky boundary (MMBM)

MMBM with sticky boundary

Markov modulation: $\{\varphi(t)\}$ on $\{1, 2, \ldots, m\}$; mean μ_i, variance σ_i^2

$$R(t) = |\min_{0 \le s \le t} X(s)|, \qquad Z(t) = X(t) + R(t)$$

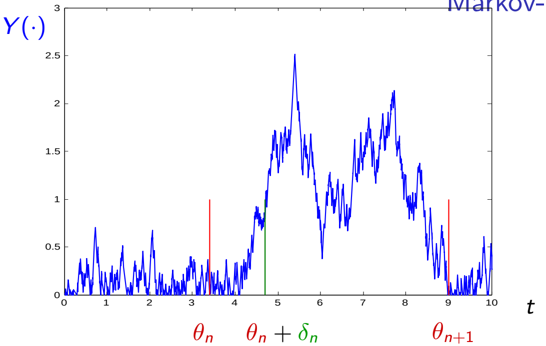
$$r_i(t) = \int_0^t \mathbb{1}[\varphi(s) = i] \, \mathrm{d}R(t)$$

$$V(t) = t + \sum_i r_i(t)/\omega_i$$

ω_i: stretching of time may depend on the phase.

$$\Gamma(t) \text{ such that } V(\Gamma(t)) = t \qquad Y(t) = Z(\Gamma(t))$$

Markov-regenerative process



create artificial intervals between "successive" visits to 0

Use a timer

- δ_n i.i.d. exponential (q)
- $\theta_{n+1} = inf\{t > \theta_n + \delta_{n+1} : Y(t) = 0\} \qquad \theta_0 = 0.$

MM-flip-flop with sticky boundary

MM flip-flop parameters:

$$\text{generator } T_\lambda = \begin{bmatrix} T - \lambda I & \lambda I \\ \lambda I & T - \lambda I \end{bmatrix}$$

$$\text{fluid rates } C^* = \begin{bmatrix} \Delta_\mu + \sqrt{\lambda}\Delta_\sigma & \\ & \Delta_\mu - \sqrt{\lambda}\Delta_\sigma \end{bmatrix}$$

with $\Delta_v = \mathrm{diag}(v_1, \ldots, v_m)$

At level 0:

$$T_0 = \begin{bmatrix} \lambda I & T - \lambda I \end{bmatrix}$$

Stretching effected through

$$T_0^* = (1/\sqrt{\lambda})\Delta_\omega T_0$$

Stationary distribution

$$\lim_{t\to\infty} \mathrm{P}[\varphi(t) = i, Z(t) \leq x] = \gamma \underline{\nu}[\Delta_\omega^{-1} + 2(-K)^{-1}(I - e^{Kx})\Delta_\sigma^{-1}]$$

where $\underline{\nu}$ is solution of $\underline{\nu}\Delta_\sigma U = \underline{0}$, $\underline{\nu 1} = 1$,

U is "minimal" solution of

$$\Delta_\sigma^2 X^2 + 2\Delta_\mu X + 2Q = 0$$

and

$$K = \Delta_\sigma^{-1} U \Delta_\sigma^{-1} + 2\Delta_\sigma^{-2}\Delta_\mu$$

Identical to stationary distribution for MMBM except for probability mass at 0

Conclusion

- Easy to think about physical behaviour of flip-flop fluid queue
  and to take limits.
- We could revisit results obtained from “traditional” approach and improve on them.
- Opens path to analysis of new processes
  and raises new questions for investigation.