Dependence patterns related to the *BMAP*

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Dependence? Why?

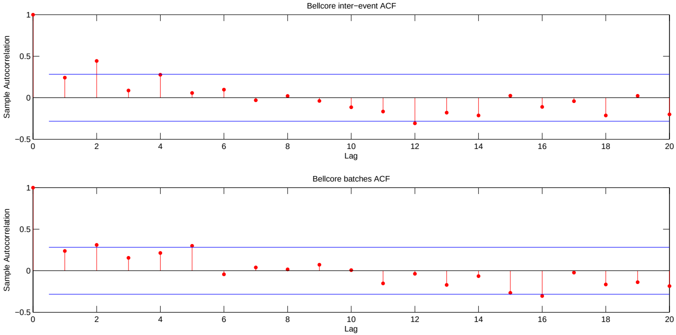
- In real life scenarios, there exist data sets that display significant and complex correlations structures in both **the times** of consecutive events and in **the size** of the consecutive events.
- Events $\equiv$ Failures of a system, arrivals of a packet of bytes, claims in an insurance company, calls in a call center...
- The event ocurrence can be understood as a single event or batch event.
- The models used in the literature to fit these types of data sets ignore the dependence.

Example I: teletraffic data set

Bellcore LAN trace files (named BC-pAug89) found in

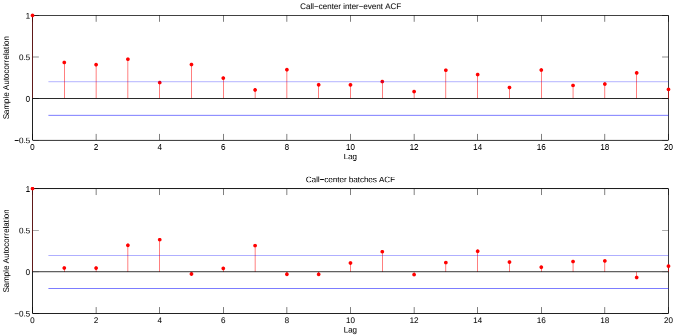
`http://ita.ee.lbl.gov/html/contrib/BC.html`.

The data file consists of the time in seconds of the packet arrival, and the Ethernet data length in bytes.



Example II: call center

The data archive of Mandelbaum (2012), collected daily from March 26, 2001 to October 26, 2003 from an American banking call center.



Our proposal $\Longrightarrow$ The *BMAP*

- *Versatile Markovian point process* (Neuts, 1979).
- *Batch Markovian Arrival process* or *BMAP* (Lucantoni, 1991).
  1. Stationary *BMAP*s are **dense** in the family of stationary point processes.
  2. **Tractability** of the Poisson process.
  3. **Dependent** interarrival times.
  4. **Non-exponential** interarrival times.
  5. **Correlated** batch sizes.
- Special cases:
  1. A *MAP* with i.i.d. batch arrivals.
  2. Batch PH-renewal processes.
  3. Batch Markov-modulated Poisson process.

The *BMAP* as a generalization of the Batch Poisson proccess

Batch Poisson process:

$$Q_{B-POISSON} = \begin{pmatrix} -\lambda & \lambda p_1 & \lambda p_2 & \lambda p_3 & \cdots & \cdots \\ 0 & -\lambda & \lambda p_1 & \lambda p_2 & \cdots & \cdots \\ 0 & 0 & -\lambda & \lambda p_1 & \cdots & \cdots \\ 0 & 0 & 0 & -\lambda & \cdots & \cdots \\ \cdots & . & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Consider now $m \times m$ matrices for the rates instead of numbers....

$$Q_{BMAP} = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ 0 & D_0 & D_1 & D_2 & \cdots \\ 0 & 0 & D_0 & D_1 & \cdots \\ 0 & 0 & 0 & D_0 & \cdots \\ \cdots & . & \cdots & \cdots & \cdots \end{pmatrix},$$

How does a $BMAP_m(k)$ work?

Notation: $\begin{cases} m & \equiv & \text{order of the matrix} \quad D_b, \quad \text{with} \quad 1 \leq b \leq k, \\ k & \equiv & \text{the maximum batch arrival size.} \end{cases}$

The $BMAP_m(k)$ behaves as follows:

- The Initial state $i_0 \in \mathcal{S} = \{1, 2..., m\}$ is given by an initial probability vector $\boldsymbol{\theta} = (\theta_1, ..., \theta_m)$.
- At the end of an exponentially distributed sojourn time in state i, with rate λ_i, two possible state transitions can occur:
  1. With probability p_{ij0}, **no arrival** occurs and the $BMAP_m$ enters in a different state $j \neq i$.
  2. With probability p_{ijb}, with $1 \leq l \leq k$, a transition to state j with a **batch arrival** of size b occurs.

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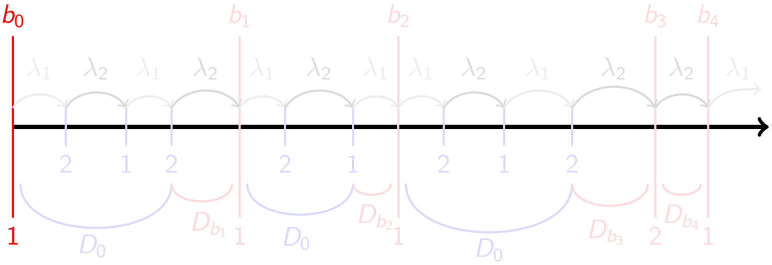
The $BMAP_2(k)$ example



The rate matrices $D_0, D_1, \dots, D_k$ are defined in terms of the transitions probabilities as:

$$
\begin{aligned}
(D_0)_{ii} &= -\lambda_i, \quad i \in S, \\
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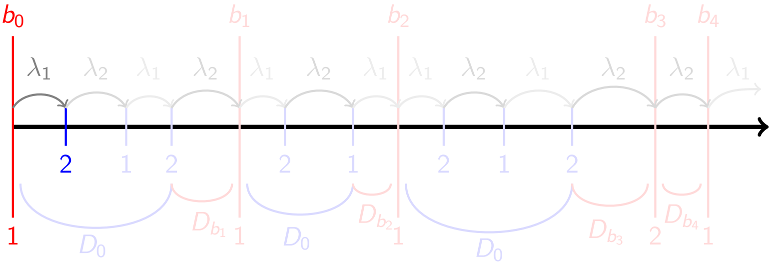

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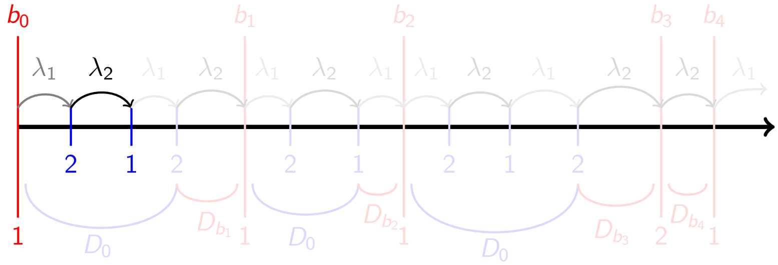

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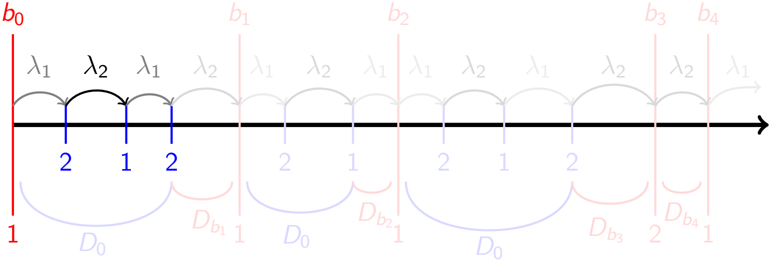

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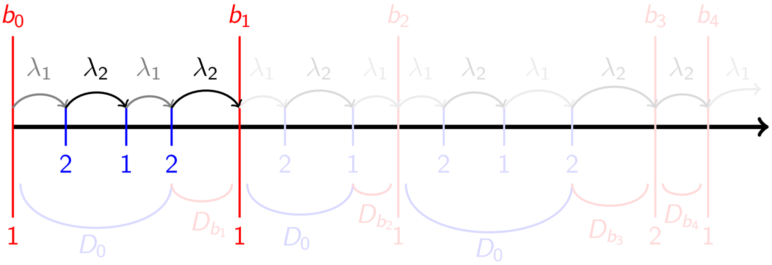

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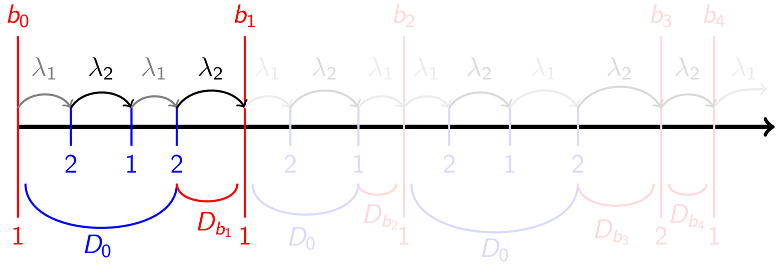

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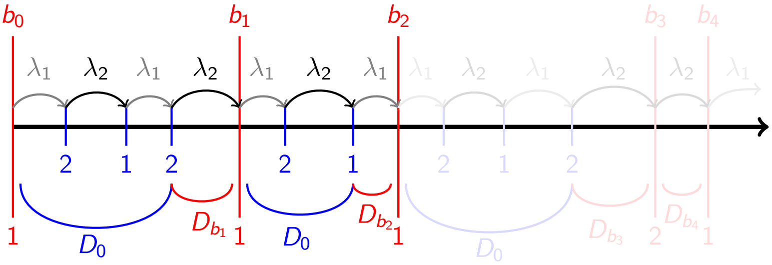

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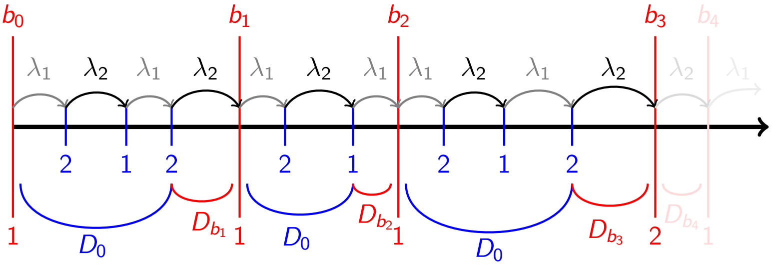

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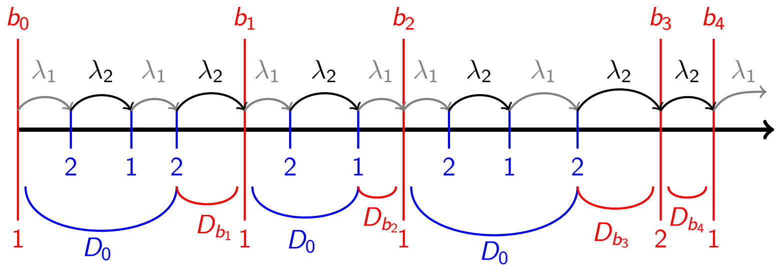

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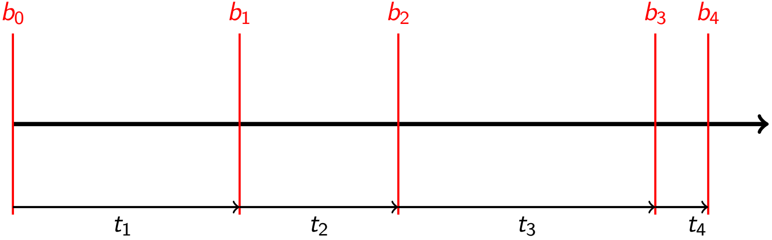
The $BMAP_2(k)$ example in practice



In practice, the *BMAP* is used to fit data where both **the inter-arrival times and batch size are observed**, but not the state of the embedded Markov renewal process. (**Partially observed**).

Properties related to the time between events

- The stationary probability vector ϕ related to $P^\star \equiv$ the transition probability matrix, $\left(P^\star = (-D_0)^{-1}\left(\sum_{b=1}^{k} D_b\right)\right)$ is calculated as

$$\phi = (\boldsymbol{\pi} D\mathbf{e})^{-1}\boldsymbol{\pi}\left(\sum_{b=1}^{k} D_b\right),$$

where $\boldsymbol{\pi}$ is the stationary probability of $D = \sum_{b=0}^{k} D_b$.

- $T =$ time between two successive events (stationary case). $T \sim PH\{\phi, D_0\}$. Then, the moments of T are given by,

$$\mu_n = E\left(T^n\right) = n!\phi\left(-D_0\right)^{-n}\mathbf{e}.$$

- The Autocorrelation Function (ACF) related to the times between events in the stationary version, is given by

$$\rho_T(l) = \rho(T_1, T_{l+1}) = \frac{\left(\boldsymbol{\pi}\left[(-D_0)^{-1}D\right]^{l}(-D_0)^{-1}\mathbf{e} - \mu_T\right)}{2\boldsymbol{\pi}(-D_0)^{-1}\mathbf{e} - \mu_T}.$$

where $\mu_T = E[T]$.

Properties related to the batch sizes

Let B_n, denotes the batch size at the time of the n'th event occurrence.

- The B_ns are distributed according to the random variable B, with probability mass function,
$$P(B=b)=\phi(-D_0)^{-1}D_b\mathbf{e}.$$
- The moments of B are obtained as
$$E[B^n]=\phi(-D_0)^{-1}D_n^\star\mathbf{e},$$
where $D_n^\star=\sum_{b=1}^k b^n D_b$.
- The ACF, $\rho(B_1,B_{l+1})$ is given by
$$\rho_B(l)=\frac{\phi(-D_0)^{-1}D_1^\star\left[(-D_0)^{-1}D\right]^{l-1}(-D_0)^{-1}D_1^\star\mathbf{e}-(\phi(-D_0)^{-1}D_1^\star\mathbf{e})^2}{\phi(-D_0)^{-1}D_2^\star\mathbf{e}-(\phi(-D_0)^{-1}D_1^\star\mathbf{e})^2},$$
where $l\geq 1$ represents the time lag.

Previous works about dependence

- Most works regarding the theoretical aspect of the auto-correlation structure are focused on special cases of the MAP, specifically, MAP_2, see Heindl et al. (2006), Casale et al. (2008) and Ramírez-Cobo and Carrizosa (2013). Hervé and Ledoux (2013) considered the general MAP.

- The auto-correlation function for a sequence of inter-event times in a $BMAP$ is the same as MAP. However, the structure of the auto-correlation of the batch arrivals has not been studied in detail in the literature.

- **Our aim**: Obtain information about the possible dependence structures that the $BMAP$ offers. **(Thinking in data fitting)**

A useful and general result for the $BMAP_m(k)$

An alternative characterization of $\rho_T(l)$ and $\rho_B(l)$, which helps to understand the dependence structure for the inter-event times and the batch sizes of the process is,

$$\begin{aligned} \rho_T(l) &= \sum_{i=2}^{m} p_i(T) q_i^l, \\ \rho_B(l) &= \sum_{i=2}^{m} p_i(B) q_i^{l-1}, \end{aligned}$$

where $\{q_i\}_{i=2}^m$, are the eigenvalues of $P^\star$ less than 1 in absolute value and $\{p_i(T)\}_{i=2}^m$ and $\{p_i(B)\}_{i=2}^m$ are real-value sequences obtained from the Perron-Frobenious decomposition of $P^\star$.

Recall: $P^\star = (-D_0)^{-1} \left(\sum_{b=1}^k D_b \right)$.

ACF for $\rho_T(l)$ in the $BMAP_2(k)$

The auto-correlation function for the inter-event times, $\rho_T(l)$, is the same as for a $MAP_2 \Rightarrow$ the results by Heindl et al. (2006) and Ramírez-Cobo and Carrizosa (2013) are also valid for the $BMAP_2(k)$.

- $\rho_T(l)$, is upper-bounded by 0.5.
- $|\rho_T(l)| \geq |\rho_T(l+1)|$, for all $l \geq 1$ and $\lim_{l\to\infty} \rho_T(l) = 0$ (**Decreases geometrically**).
- Correlation patterns for $\rho_T(l)_{l\geq 1}$.
  - Pattern 1. If $p(T) \geq 0$ and $q \geq 0 \Rightarrow \rho_T(l) \geq 0$.
  - Pattern 2. If $p(T) \leq 0$ and $q \geq 0 \Rightarrow \rho_T(l) \leq 0$.
  - Pattern 3. If $p(T) \geq 0$ and $q \leq 0 \Rightarrow \rho_T(2l) \geq 0$ and $\rho_T(2l+1) \leq 0$.
  - Pattern 4. If $p(T) \leq 0$ and $q < 0 \Rightarrow \rho_T(2l) \leq 0$ and $\rho_T(2l+1) \geq 0$.

ACF for $\rho_B(l)$ in the $BMAP_2(k)$

- For the $BMAP_2(k)$, we obtain

$$|\rho_B(l)| \geq |\rho_B(l+1)|, \quad \text{for all } l \geq 1 \text{ and } \lim_{l\to\infty} \rho_B(l) = 0.$$

  (**Decreases geometrically**)

- The expression for the auto-correlation, $\rho_B(l)$, for the $BMAP_2(2)$, is given by

$$\rho_B(l) = p(B)q^{l-1}.$$

  It can be checked that $P(B)$ and q can be positive or negative $\Rightarrow$ Correlation patterns for $\rho_B(l)$ in for the $BMAP_2(2)$.

  - Pattern 1. If $p(B) \geq 0$ and $q \geq 0 \Rightarrow \rho_B(l) \geq 0$.
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  - Pattern 4. If $p(B) \leq 0$ and $q < 0 \Rightarrow \rho_B(2l) \geq 0$ and $\rho_B(2l+1) \leq 0$.

Open problem

Is $\rho_B(l)$ bounded for the $BMAP_2(k)$?

Empirical evidence shows that $\rho_B(l)$ is unbounded in $[-1, 1]$

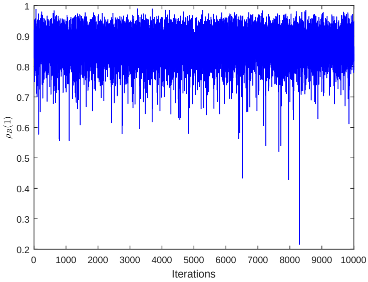


Figure : Values of $\rho_B(1)$ close to 1, for a total 10000 simulated $BMAP_2(2)s$.

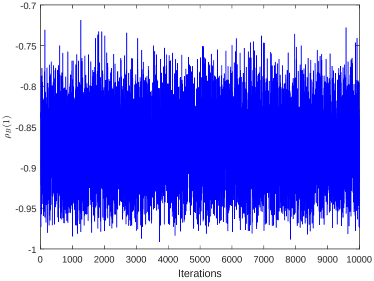


Figure : Values of $\rho_B(1)$ close to -1, for a total 10000 simulated $BMAP_2(2)s$.

Dependence structure of the $BMAP_m(k)$, $m \geq 3$

- The property of both $\rho_T(l)$ and $\rho_B(l)$ decreasing geometrically is very restrictive when you deal with data $\Rightarrow$

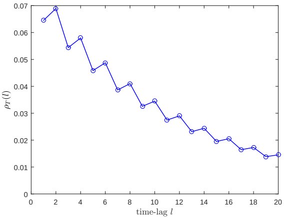
an increase in m leads to new and richer correlation structures for the $BMAP_m(k)$?

- The answer is affirmative although we have not theoretical results for the evidences given by simulations.

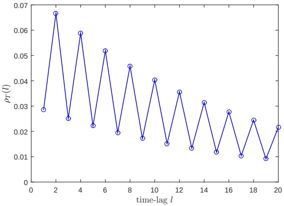
- Let's take a look at some plots!!

Dependence structure of the $BMAP_m(k)$, $m \geq 3$

Examples of $\rho_T(l)$ for $m \geq 3$ where $\rho_T(l)$ does not decrease with the time lag.



Example with $m = 3$



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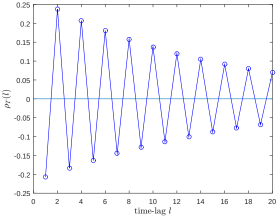
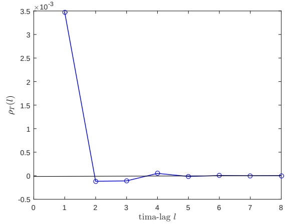


Example with $m = 4$

Examples of $\rho_T(l)$ for $m \geq 3$ where the signs of the autocorrelation coefficients do not alternate or are constant.



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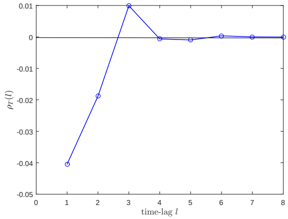

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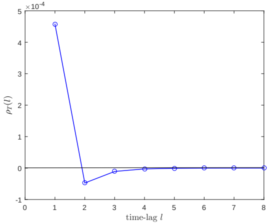

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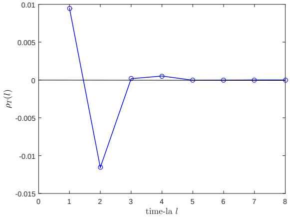

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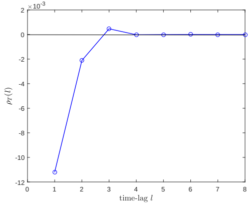

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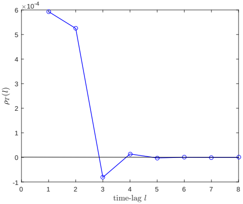

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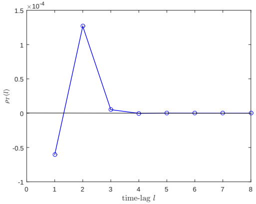
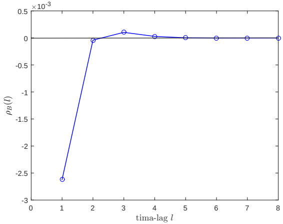


Example with $m = 4$

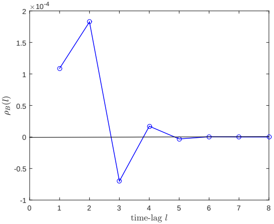
Important remark: We are not able to find any MAP_m such that $|\rho_T(1)| > 0.5$.

Dependence structure of the $BMAP_m(2)$, $m \geq 3$

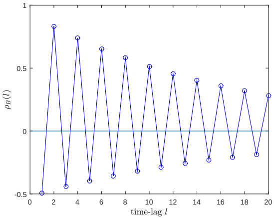
Examples of $\rho_B(l)$ for $m \geq 3$ where $\rho_B(l)$ is not a decreasing function in absolute value, richer pattern that for $m = 2$ are observed and $\rho_B(l)$ is unbounded.

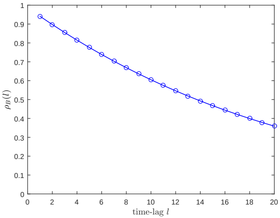


Examples with $m = 3$ and $k = 2$



Examples with $m = 3$ and $k = 2$





...But what is it interesting in applications?

The counting process $\{N(t),\ t \geq 0\}$

- The probability of n event occurrences at time t is given by,

$$P\left(N(t) = n \mid N(0) = 0\right) = \phi P(n, t)\mathbf{e},$$

where the probability of n event occurrences in the interval $(0, t]$ is given by the matrix $P(n, t)$, (cannot be computed in closed-form).
Their numerical computation is based on the uniformization method (Neuts and Li. (1997)).

- The expected number of event occurrences at time t, $E\left(N(t) \mid N(0) = 0\right)$, is computed from,

$$E\left(N(t) \mid N(0) = 0\right) = \lambda^* t,$$

where $\lambda^* = \boldsymbol{\pi} D_1^{\star}\mathbf{e}$, $D_1^{\star} = \sum_{b=1}^{k} b D_b$ and $\boldsymbol{\pi}$ the stationary probability of $D = \sum_{b=0}^{k} D_b$.

Remark: Similar to a Poisson process. $(E(N(t)) = \lambda t)$

...But what is it interesting in applications?

- The variance of $N(t)$ for a $BMAP_m(k)$ is given by,

$$\left(\boldsymbol{\pi} D_2^{\star}\mathbf{e} - 2\left(\lambda^{*}\right)^2 + 2\mathbf{c}D_1^{\star}\mathbf{e}\right) t - 2\mathbf{c}(I - e^{Dt})(\mathbf{e}\boldsymbol{\pi} - D)^{-1} D_1^{\star}\mathbf{e}$$

where $\mathbf{c} = \boldsymbol{\pi} D_1^{\star}(\mathbf{e}\boldsymbol{\pi} - D)^{-1}$ and $D_2^{\star} = \sum_{b=1}^{k} b^2 D_b$.

- The variance of $N(t)$ for a MAP_m is given by,

$$(1 + 2\lambda^{*})E[N(t)] - 2\boldsymbol{\pi} D_1(\mathbf{e}\boldsymbol{\pi} + D)^{-1} D_1\mathbf{e}t - 2\boldsymbol{\pi} D_1\left(I - e^{Dt}\right)(\mathbf{e}\boldsymbol{\pi} + D)^{-2} D_1\mathbf{e}$$

Remark: Very different to a Poisson process. $(V(N(t)) = \lambda t)$

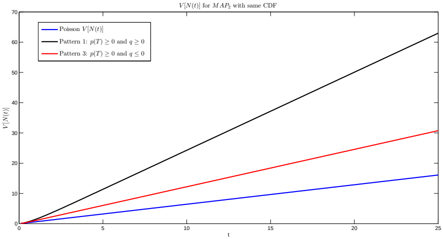
Exploring the influence of the dependence in the counting process

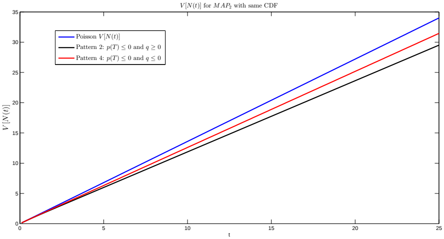
- **Objective**: Identify the influence of the dependence pattern of $\rho_T(l)$ and $\rho_B(l)$ in $E(N(t))$, $V(N(t))$ or $P(n,t)$.

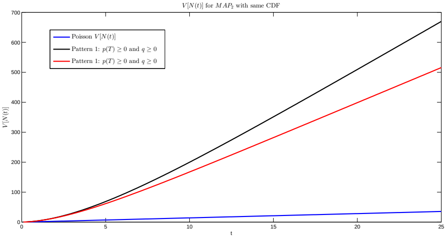
- **First scenario**: Compare MAP_2 with the same CDF of T but with different dependence patterns of $\rho_T(l)$.

  In this case, the MAP_2s have the same $E(N(t))$ and the same $P(n,t)$,...but different $V(N(t))$?

First scenario: MAP_2 with the same CDF of T but with different dependence patterns of $\rho_T(l)$

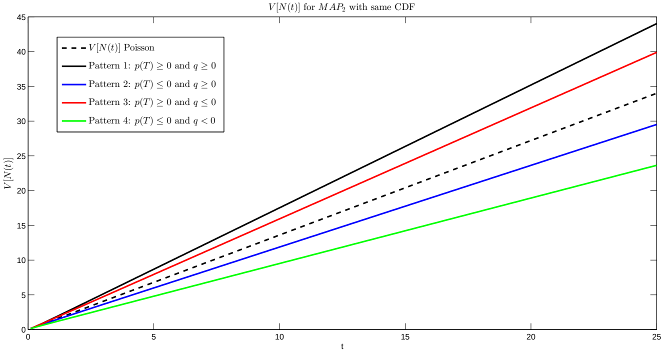




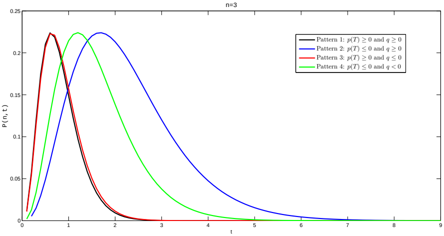


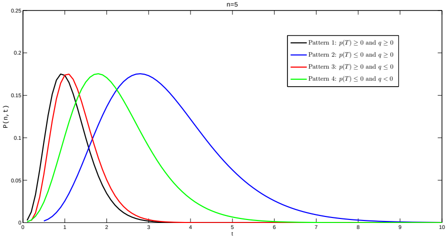
Second scenario: MAP_2 with the same λ^* but with different dependence patterns of $\rho_T(l)$

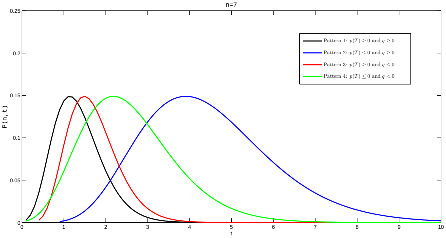
- In this case, the MAP_2s have the same $E(N(t))$,...but different $V(N(t))$ and $P(n,t)$?



Second scenario: MAP_2 with the same λ^* but with different dependence patterns of $\rho_T(l)$

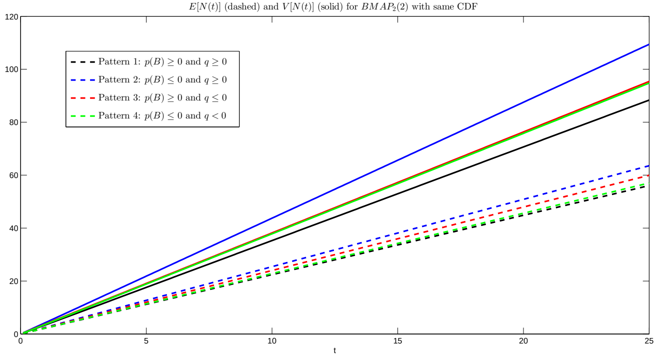




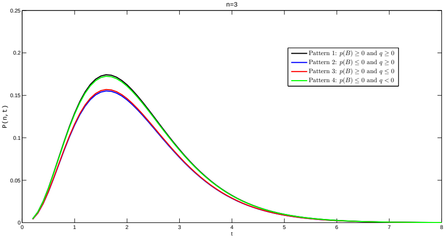


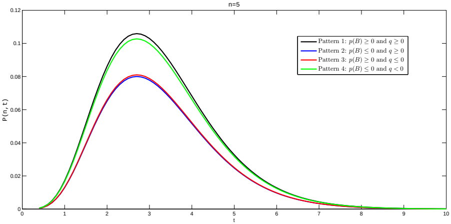
Third scenario: $BMAP_2(2)$ with the same CDF of T but with different dependence patterns of $\rho_B(l)$

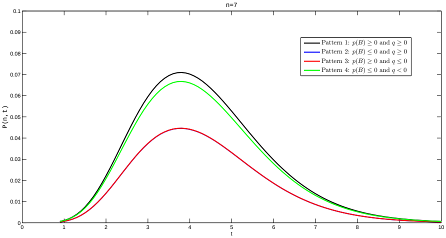
- In this case, the $BMAP_2(2)$s have different $E(N(t))$, $V(N(t))$ and $P(n,t)$?



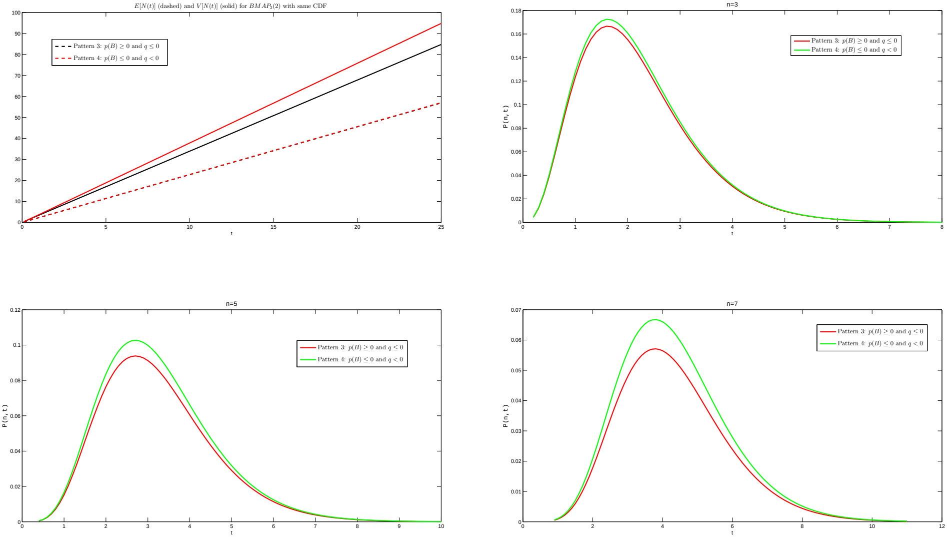
Third scenario: $BMAP_2(2)$ with the same CDF of T but with different dependence patterns of $\rho_B(l)$







Third scenario: $BMAP_2(2)$ with the same CDF of T but with different dependence patterns of $\rho_B(l)$



Conclusions

- We provide a characterization of both ACFs in terms of the eigenvalues of $P^{\star}$ for the general $BMAP_m(k)$.
- We prove that the auto-correlation function for the batch event sizes for the $BMAP_2(k)$, for $k \geq 2$, decreases geometrically as the time lag increases.
- We identify four behavior patterns for ACF for the batch event sizes for the $BMAP_2(2)$.
- Richer dependence structure for the inter-event times and batch sizes are captured with higher order $BMAP$s.
- There are evidences that the dependence patterns have influence in the counting process related to these models.

Work in progress

- Perform a theoretical analysis of the correlation bounds for the inter-event times for $m \geq 3$ and the batch sizes for $m \geq 2$.
- Develop estimation methods to fit properly the correlation pattern of the data to a $BMAP_{m?}(k)$.
- Understand how the autocorrelation functions modify the behavior of the counting process.

The results showed in this talk have been recently accepted for publication in:

Rodríguez, J., Lillo, R.E. and Ramírez-Cobo, P. (2016). Dependence patterns for modeling simultaneous events, *Reliability Engineering and System Safety*, 154, 19-30.

References

Asmussen, S. and Koole, G. (1993). Marked point processes as limits of Markovian arrival streams. *Journal of Applied Probability*, 30, 365-372.

Chakravarthy, SR. (2001). The batch Markovian arrival process: a review and future work. In: Krishnamoorthy A, et al., editors. Advances in probability theory and stochastic processes. NJ: Notable Publications, Inc. pp. 21-49.

Chakravarthy, SR. (2010). The Markovian arrival processes: a review and future work.In J. Cochran, editor. Wiley Encyclopedia of Operations Research and Management Science, John Wiley & Sons, Inc.

Heindl, A., Mitchell, K. and van de Liefvoort, A. (2006). Correlation bounds for second-order *MAP*s with application to queueing network decomposition. *Performance Evaluation*, 63, 553-577.

Hervé, L. and Ledoux, J. (2013). Geometric ρ-mixing property of the interarrival times of a stationary Markovian arrival process. *Journal of Applied Probability*, 50, 598-601.

IEEE Standard Computer Dictionary: A Compilation of IEEE Standard Computer Glossaries. IEEE Press Piscataway, New York, NY; 1990.

Latouche, G. and Ramaswami, V. (1990). Introduction to Matrix Analytic Methods in Stochastic Modeling, ASA-SIAM Series on Statistics and Applied Probability, vol. 5, SIAM, Philadelphia, PA.

References

Lucantoni, D. (1991). New results on the single server queue with a batch Markovian arrival process. *Stochastic Models*, 7, 1-46.

Lucantoni, D. (1993). The $BMAP/G/1$ queue: a tutorial. In: L. Donatiello and R. Nelson (eds.) Models and Techniques for performance Evaluation of Computer and Communications Systems. Springer Verlag, Berlin, pp 330-358.

Mandelbaum, A. (2012). Service Engineering of Stochastic Networks web page: http://iew3.technion.ac.il/serveng/.

Montoro-Cazorla, D. and Pérez-Ocón, R. (2014). Matrix stochastic analysis of the maintainability of a machine under shocks. *Reliability Engineering and System Safety*, 121, 11-17.

Montoro-Cazorla, D. and Pérez-Ocón, R. (2014). A redundant n-system under shocks and repairs following Markovian arrival processes. *Reliability Engineering and System Safety*, 130, 69-75.

Montoro-Cazorla, D. and Pérez-Ocón, R. (2014).A reliability system under cumulative shocks governed by a BMAP. *Applied Mathematical Modelling*, doi:10.1016/j.apm.2015.03.066.

Montoro-Cazorla, D. and Pérez-Ocón, R. (2015).A shock and wear model with dependence between the interarrival failures. *Applied Mathematics and Computation*, 259, 339-352.

References

Neuts, MF. (1979). A versatile Markovian point process. *Journal of Applied Probability*, 16, 764-779.

Ramírez-Cobo, P., Lillo, R.E. and Wiper, M. (2010). Non identifiability of the two-state Markovian arrival process. *Journal of Applied Probability*, 47, 630-649.

Ramírez-Cobo, P. and Carrizosa., E. (2012). A note on the dependence structure of the two-state Markovian arrival process. *Journal of Applied Probability*, 49, 295-302.

Rodríguez, J., Lillo, R.E. and Ramírez-Cobo, P. (2014). Nonidentifiability of the two-state *BMAP*, *Methodology and Computing in Applied Probability*. http://dx.doi.org/10.1007/s11009-014-9401-z.

Rodríguez, J., Lillo, R.E. and Ramírez-Cobo, P. (2015). Failure modeling of an electrical *N*-component framework by the non-stationary Markovian arrival process, *Reliability Engineering and System Safety*, 134, 126-133.