Shift techniques for Quasi-Birth-and-Death processes: canonical factorizations and matrix equations

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QBD processes

Let

$$P = \begin{bmatrix} A_0' & A_1' & & 0 \\ A_{-1} & A_0 & A_1 & \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

be the transition matrix of a QBD with space state $\mathbb{N} \times S$, $S = \{1, \ldots, n\}$.

Here A_{-1}, A_0, A_1 are $n \times n$ nonnegative matrices such that $A_{-1} + A_0 + A_1$ is stochastic and irreducible

Define the matrix polynomial

$$A(z) = A_{-1} + z(A_0 - I) + z^2 A_1$$

We call eigenvalues of the matrix polynomial $A(z)$ the roots of $a(z) = \det A(z)$

Remark: Since $A(1)\mathbf{1} = 0$ then $z = 1$ is an eigenvalue of $A(z)$

Quadratic matrix equations and canonical factorizations

Let G, R, $\widehat{G}$ and $\widehat{R}$ be the minimal nonnegative solutions of the matrix equations

$$
\begin{aligned}
A_{-1} + A_0 X + A_1 X^2 &= X \\
X^2 A_{-1} + X A_0 + A_1 &= X \\
A_{-1} X^2 + A_0 X + A_1 &= X \\
X^2 A_1 + X A_0 + A_{-1} &= X.
\end{aligned}
$$

Then $A(z)$ and the reversed matrix polynomial $\widehat{A}(z) = z^2 A_{-1} + z(A_0 - I) + A_1$ have the weak canonical factorizations

$$
\begin{aligned}
A(z) &= (I - zR)K(zI - G) \\
\widehat{A}(z) &= (I - z\widehat{R})\widehat{K}(zI - \widehat{G})
\end{aligned}
$$

with $K = A_0 - I + A_1 G$ and $\widehat{K} = A_0 - I + A_{-1}\widehat{G}$.

Roots of the matrix polynomial $A(z)$

The roots ξ_i, $i = 1, \ldots, 2n$ of $a(z)$ are such that

$$|\xi_1| \le \cdots \le |\xi_{n-1}| \le \xi_n \le 1 \le \xi_{n+1} \le |\xi_{n+2}| \le \cdots \le |\xi_{2n}|$$

where we have introduced $2n - \deg a(z)$ roots at infinity if $\deg a(z) < 2n$

More specifically, we have the following scenario:

- $\xi_n = 1 < \xi_{n+1}$ positive recurrent
- $\xi_n = 1 = \xi_{n+1}$ null recurrent
- $\xi_n < 1 = \xi_{n+1}$ transient

Remark. If $Gu = \lambda u$ then $A(\lambda)u = 0$; if $v^T R = \mu v^T$ then $v^T A(\mu^{-1}) = 0$. That is, the eigenvalues of G and the reciprocals of the eigenvalues of R are eigenvalues of $A(z)$. In particular:

- G has eigenvalues $\xi_1, \ldots, \xi_n$
- R has eigenvalues $\xi_{n+1}^{-1}, \ldots, \xi_{2n}^{-1}$

Assumption 1 : the process is recurrent, i.e., $\xi_n = 1$

Assumption 2 : $|\xi_{n-1}| < \xi_n$ and $\xi_{n+1} < |\xi_{n+2}|$

Motivation of the shift

- There exist algorithms for computing the minimal nonnegative solution G; their efficiency deteriorates as ξ_n/ξ_{n+1} gets close to 1
- In the null recurrent case where $\xi_n = \xi_{n+1}$, the convergence speed turns from linear to sublinear, or from superlinear to linear, according to the used algorithm

Here we provide a tool for getting rid of this drawback

The idea is an elaboration of the Brauer theorem and of the shift technique for matrix polynomials [He, Meini, Rhee 2001]

It relies on transforming the matrix polynomial $A(z)$ into a new one $\widetilde{A}(z)$ in such a way that $\widetilde{a}(z) = \det \widetilde{A}(z)$ has the same roots of $a(z)$ except for $\xi_n = 1$ which is shifted to 0, and/or $\xi_{n+1} = 1$ which is shifted to infinity

Brauer's theorem on eigenvalues

Theorem (Brauer 1956)

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let x_k be an eigenvector of A associated with the eigenvalue λ_k, $1 \le k \le n$, and let q be any n-dimensional vector. Then the matrix $A + x_k q^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + q^T x_k, \lambda_{k+1}, \ldots, \lambda_n$.

Remark: if q is such that $q^T x_k = -\lambda_k$, then $A + x_k q^T$ has eigenvalues $0, \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n$, i.e., the eigenvalue λ_k is shifted to 0.

Question: can we generalize this shifting to the eigenvalues of matrix polynomials?

Functional interpretation of Brauer's theorem

Let A be an $n \times n$ matrix and let $Au = \lambda u$, $u \neq 0$.
Choose any vector v such that $v^T u = 1$ and define $\widetilde{A} = A - \lambda u v^T$.

Remark: $\widetilde{A}u = Au - \lambda u v^T u = \lambda u - \lambda u = 0$.

Functional interpretation: by direct inspection, one has

$$\widetilde{A} - zI = (A - zI)\left(I + \frac{\lambda}{z-\lambda} u v^T\right)$$

Taking determinants:

$$\det(\widetilde{A} - zI) = \det(A - zI)\frac{z}{z-\lambda}$$

Therefore:

- $\widetilde{A}$ has the same eigenvalues of A except for λ which is shifted to zero
- A and $\widetilde{A}$ share the right eigenvector u and the left eigenvectors not corresponding to λ

Question: can we do anything similar for $A(z)$?

YES! Shift to the right

Let $u_G \neq 0$ such that $A(\xi_n)u_G = 0$, and let v be any vector such that $v^T u_G = 1$.
Define:

$$\widetilde{A}_r(z) = A(z)\left(I + \frac{\xi_n}{z - \xi_n} Q\right), \quad Q = u_G v^T$$

Remark: similarly to the matrix case, $\det \widetilde{A}_r(z) = \det A(z) \frac{z}{z-\xi_n}$

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Theorem

The function $\widetilde{A}_r(z)$ coincides with the quadratic matrix polynomial $\widetilde{A}_r(z) = \widetilde{A}_{-1} + z(\widetilde{A}_0 - I) + z^2 \widetilde{A}_1$ with matrix coefficients

$$\widetilde{A}_{-1} = A_{-1}(I - Q), \quad \widetilde{A}_0 = A_0 + \xi_n A_1 Q, \quad \widetilde{A}_1 = A_1.$$

Moreover, the eigenvalues of $\widetilde{A}_r(z)$ are $0, \xi_1, \ldots, \xi_{n-1}, \xi_{n+1}, \ldots, \xi_{2n}$.

Shift to the left

Let $v_R \neq 0$ such that $v_R^T A(\xi_{n+1}) = 0$, and let w be any vector such that $w^T v_R = 1$.
Define

$$\widetilde{A}_\ell(z) = \left(I - \frac{z}{z - \xi_{n+1}} S \right) A(z), \quad S = w v_R^T$$

Remark: similarly to the right shift, $\det \widetilde{A}_\ell(z) = \det A(z) \frac{1}{z - \xi_n}$

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Theorem

The function $\widetilde{A}_\ell(z)$ coincides with the quadratic matrix polynomial $\widetilde{A}_\ell(z) = \widetilde{A}_{-1} + z(\widetilde{A}_0 - I) + z^2 \widetilde{A}_1$ with matrix coefficients

$$\widetilde{A}_{-1} = A_{-1}, \quad \widetilde{A}_0 = A_0 + \xi_{n+1}^{-1} S A_{-1}, \quad \widetilde{A}_1 = (I - S) A_1.$$

Moreover, the eigenvalues of $\widetilde{A}_\ell(z)$ are $\xi_1, \ldots, \xi_n, \xi_{n+2}, \ldots, \xi_{2n}, \infty$.

Double shift

The right and left shifts can be combined together.
Define the matrix function

$$\widetilde{A}_d(z) = \left(I - \frac{z}{z-\xi_{n+1}} S\right) A(z) \left(I + \frac{\xi_n}{z-\xi_n} Q\right).$$

Theorem

The function $\widetilde{A}_d(z)$ coincides with the quadratic matrix polynomial $\widetilde{A}_d(z) = \widetilde{A}_{-1} + z(\widetilde{A}_0 - I) + z^2\widetilde{A}_1$ with matrix coefficients

$$\begin{aligned}
\widetilde{A}_{-1} &= A_{-1}(I - Q),\\
\widetilde{A}_0 &= A_0 + \xi_n A_1 Q + \xi_{n+1}^{-1} S A_{-1} - \xi_{n+1}^{-1} S A_{-1} Q,\\
\widetilde{A}_1 &= (I - S)A_1.
\end{aligned}$$

Moreover, the eigenvalues of $\widetilde{A}_d(z)$ are 0, $\xi_1, \ldots, \xi_{n-1}$, $\xi_{n+2}, \ldots, \xi_{2n}$, ∞. In particular, $\widetilde{A}_d(z)$ is nonsingular on the unit circle and on the annulus $|\xi_{n-1}| < |z| < |\xi_{n+2}|$.

Shifts and canonical factorizations

Question: Under which conditions both the polynomials $\widetilde{A}_s(z)$ and $z^2\widetilde{A}_s(z^{-1})$ for $s \in \{r, \ell, d\}$ obtained after applying the shift have a (weak) canonical factorization?

In different words:

Question: Under which conditions there exist the four minimal solutions to the matrix equations associated with the polynomial $\widetilde{A}_s(z)$ obtained after applying the shift?

These matrix solutions will be denoted by G_s, R_s, $\widehat{G}_s$, $\widehat{R}_s$, with $s \in \{r, \ell, d\}$. They are the analogous of the solutions G, R, $\widehat{G}$, $\widehat{R}$ to the original equations.

We examine the case of the shift to the right

The shift to the left and the double shift can be treated similarly.

Right shift: the polynomial $\widetilde{A}_r(z)$

Recall that

$$\begin{aligned}\widetilde{A}_r(z) &= A(z)\left(I + \frac{\xi_n}{z-\xi_n}Q\right) = \\ &= (I - zR)K(zI - G)\left(I + \frac{\xi_n}{z-\xi_n}Q\right)\end{aligned}$$

with $Q = u_G v^T$.

Right shift: the polynomial $\widetilde{A}_r(z)$

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with $Q = u_G v^T$.
By a direct computation we obtain

$$(zI - G)\left(I + \frac{\xi_n}{z-\xi_n}Q\right) = zI - G_r$$

with $G_r = G - \xi_n Q$. Therefore

$$\widetilde{A}_r(z) = (I - zR)K(zI - G_r)$$

Right shift: the polynomial $\widetilde{A}_r(z)$

Theorem

- *The polynomial $\widetilde{A}_r(z)$ has the following factorization*

$$\widetilde{A}_r(z) = (I - zR)K(zI - G_r), \quad G_r = G - \xi_n Q$$

This factorization is canonical in the positive recurrent case, and weak canonical otherwise.

- *The eigenvalues of G_r are those of G, except for the eigenvalue ξ_n which is replaced by zero*

- *$X = G_r$ and $Y = R$ are the solutions with minimal spectral radius of the equations*

$$\widetilde{A}_1 X^2 + \widetilde{A}_0 X + \widetilde{A}_{-1} = X, \quad Y^2 \widetilde{A}_{-1} + Y \widetilde{A}_0 + \widetilde{A}_1 = Y$$

Right shift: the reversed polynomial $z^2\widetilde{A}_r(z^{-1})$

Recall that

$$z^2\widetilde{A}_r(z^{-1}) = z^2A(z^{-1})\left(I + \frac{z\xi_n}{1 - z\xi_n}Q\right) = \\ = (I - z\widehat{R})\widehat{K}(zI - \widehat{G})\left(I + \frac{z\xi_n}{1 - z\xi_n}Q\right)$$

with $Q = u_G v^T$.

Right shift: the reversed polynomial $z^2\widetilde{A}_r(z^{-1})$

Recall that

$$z^2\widetilde{A}_r(z^{-1}) = z^2A(z^{-1})\left(I + \frac{z\xi_n}{1 - z\xi_n}Q\right) =$$
$$= (I - z\widehat{R})\widehat{K}(zI - \widehat{G})\left(I + \frac{z\xi_n}{1 - z\xi_n}Q\right)$$

with $Q = u_G v^T$.
By a direct computation we obtain

$$(zI - \widehat{G})\left(I + \frac{z\xi_n}{1 - z\xi_n}Q\right) = z(I - \xi_n Q) - \widehat{G}$$

and $I - \xi_n Q$ is singular!

Things are more complicated. We need some preliminary results

General properties

Theorem (Bini, Latouche, Meini 2005)

Let $B(z)$ be an $n \times n$ quadratic matrix polynomial with eigenvalues λ_i, such that $|\lambda_i| \leq |\lambda_{i+1}|$, $i = 1, \ldots, 2n-1$. Assume that $|\lambda_n| < 1 < |\lambda_{n+1}|$ and that $B(z)$ has the canonical factorization
$B(z) = (I - zR)K(zI - G)$. *Then:*

1. $B(z)$ *is invertible in* $\mathbb{A} = \{z \in \mathbb{C} : |\xi_n| < z < |\xi_{n+1}|\}$ *and* $H(z) = (z^{-1}B(z))^{-1} = \sum_{i=-\infty}^{+\infty} z^i H_i$ *is convergent for* $z \in \mathbb{A}$, *where*

$$H_i = \begin{cases} G^{-i}H_0, & i < 0, \\ \sum_{j=0}^{+\infty} G^j K^{-1} R^j, & i = 0, \\ H_0 R^i, & i > 0. \end{cases}$$

2. *If H_0 is nonsingular, then* $\widehat{B}(z) = z^2 B(z^{-1})$ *has the canonical factorization*

$$\widehat{B}(z) = (I - z\widehat{R})\widehat{K}(zI - \widehat{G}),$$

where $\widehat{G} = H_0 R H_0^{-1}$, $\widehat{R} = H_0^{-1} G H_0$.

Right shift: the reversed polynomial $z^2\widetilde{A}_r(z^{-1})$

Theorem (Positive recurrent case)

Assume that $1 = \xi_n < \xi_{n+1}$. *Let* $Q = u_G v^T$, *with* v *any vector such that* $u_G^T v = 1$ *and* $v^T W R W^{-1} u_G \neq 1$, *with* $W = \sum_{i=0}^{+\infty} G^i K^{-1} R^i$. *Then* $z^2\widetilde{A}_r(z^{-1})$ *has the canonical factorization*

$$z^2\widetilde{A}_r(z^{-1}) = (I - z\widetilde{R}_r)\widetilde{K}_r(zI - \widetilde{G}_r)$$

$$\widetilde{R}_r = W_r^{-1} G_r W_r, \quad \widetilde{G}_r = W_r R W_r^{-1}, \quad G_r = G - \xi_n Q,$$

$$W_r = W - \xi_n Q W R, \quad \widetilde{K}_r = \widetilde{A}_0 + \widetilde{A}_{-1}\widetilde{G}_r.$$

Moreover, $\widetilde{G}_r$ *and* $\widetilde{R}_r$ *are the solutions with minimal spectral radius of the matrix equations*

$$\widetilde{A}_{-1}X^2 + \widetilde{A}_0 X + \widetilde{A}_1 = X, \qquad X^2\widetilde{A}_1 + X\widetilde{A}_0 + \widetilde{A}_{-1} = X$$

Right shift: the reversed polynomial $z^2\widetilde{A}_r(z^{-1})$

Theorem (Null recurrent case)

Assume that $\xi_n = \xi_{n+1} = 1$ *and let* $Q = u_G v_{\widehat{G}}^T$, *where* $u_G^T v_{\widehat{G}} = 1$ *and* $v_{\widehat{G}}^T \widehat{K}^{-1} u_{\widehat{R}} = 1$. *Then* $z^2\widetilde{A}_r(z^{-1})$ *has the weak canonical factorization*

$$z^2\widetilde{A}_r(z^{-1}) = (I - z\widetilde{R}_r)\widetilde{K}_r(zI - \widetilde{G}_r)$$

$$\widetilde{R}_r = \widehat{R} - u_{\widehat{R}} v_{\widehat{G}}^T \widehat{K}^{-1}, \qquad \widetilde{K}_r = \widehat{K} - (u_{\widehat{R}} - \widehat{K} u_G) v_{\widehat{G}}^T,$$

$$\widetilde{G}_r = \widehat{G} + (u_G - \widehat{K}^{-1} u_{\widehat{R}}) v_{\widehat{G}}^T$$

The eigenvalues of $\widetilde{R}_r$ *are those of* $\widehat{R}$ *except for 1 which is replaced by 0; the eigevalues of* $\widetilde{G}_r$ *are the same as those of* $\widehat{G}$. *Moreover,* $\widetilde{G}_r$ *and* $\widetilde{R}_r$ *are solutions of minimum spectral radius of the quadratic matrix equations*

$$\widetilde{A}_{-1}X^2 + \widetilde{A}_0 X + \widetilde{A}_1 = X, \quad X^2\widetilde{A}_1 + X\widetilde{A}_0 + \widetilde{A}_{-1} = X$$

Application to the Poisson problem

Bini, Dendievel, Latouche, Meini, 2016

The Poisson problem for a QBD consists in solving the equation

$$(I - P)z = q,$$

where q is an infinite vector, z is the unknown and

$$P = \begin{bmatrix} A_0 + A_{-1} & A_1 & & & \\ A_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where A_{-1}, A_0, A_1 are nonnegative and $A_{-1} + A_0 + A_1$ is stochastic.

If $\xi_n \neq \xi_{n+1}$, the series $W = \sum_{i=0}^{\infty} G^i K^{-1} R^i$ is convergent and $\det W \neq 0$. Through W we may construct a resolvent triple for $A(z)$, and provide the general expression of the solution.

Application to the Poisson problem

This is not possible in the null recurrent case, where $\xi_n = \xi_{n+1}$

Solution :

- represent the Poisson problem in functional form
- apply the shift to the right to move ξ_n to zero
- construct a new matrix difference equation and solve it by using resolvent triples
- recover the solution of the original problem

Generalizations

The shift technique can be generalized in order to shift to zero or to infinity a set of selected eigenvalues, leaving unchanged the remaining eigenvalues.

Potential applications:

- Shifting a pair of conjugate complex eigenvalues to zero or to infinity still maintaining real arithmetic.
- Deflation of already approximated roots within a polynomial rootfinder
- Solution of matrix difference equation where resolvent triples cannot be explicitly constructed for the presence of multiple eigenvalues