Algorithms for Stationary Distributions of Fluid Queues: Interpretations and Re-interpretations

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Algorithms for solving for the return probability matrix Ψ

Ψ is the minimal nonnegative solution to

$$C_+^{-1}T_{+-} + X|C_-|^{-1}T_{--} + C_+^{-1}T_{++}X + X|C_-|^{-1}T_{-+}X = 0.$$

- Asmussen (1995): three iterative schemes
- Guo (2001): fixed-point iterations and Newton's method
- Ramaswami (1999), da Silva Soares and Latouche (2002):

  - approximating fluids using QBDs

  - thus allowing algorithms originally developed for QBDs — such as Logarithmic Reduction (Latouche and Ramaswami, 1993) and Cyclic Reduction (Bini and Meini, 2009) — to be used for solving for Ψ
- Bean, O'Reilly, Taylor (2005): First-Exit, Last-Entrance, etc

Doubling Algorithms: for solving NARE

A nonsymmetric algebraic Riccati equation (NARE) has the form

$$B - AX - XD + XCX = 0.$$

- Guo, Lin, Xu (2006): Structure-Preserving DA (SDA)
- Bini, Meini, Poloni (2010): SDA Shrink-and-Shift (SDA-ss)
- Wang, Wang, Li (2012): Alternating-Directional DA (ADDA)

These algorithms are known to be more computationally efficient than all the previously mentioned algorithms.

Doubling Algorithms: for solving NARE

- An initial matrix

$$P_0 := \begin{array}{c} \\ + \\ - \end{array}\!\!\begin{array}{cc} + & - \\ \left[\begin{matrix} E & G \\ H & F \end{matrix}\right. & \left.\right] \end{array}$$

- A doubling map

$$\mathcal{F}(P_0) := \begin{bmatrix} E(I-GH)^{-1}E & G+E(I-GH)^{-1}GF \\ H+F(I-HG)^{-1}HE & F(I-HG)^{-1}F \end{bmatrix}$$

- For $k \geq 0$, let

$$P_k := \begin{bmatrix} E_k & G_k \\ H_k & F_k \end{bmatrix} := \mathcal{F}^k(P_0) > 0$$

then

$$\lim_{k\to\infty} \begin{bmatrix} E_k & G_k \\ H_k & F_k \end{bmatrix} = \begin{bmatrix} 0 & \Psi \\ \widehat{\Psi} & F_\infty \end{bmatrix}$$

Initial matrix P_0

- Let

$$\alpha_{\rm opt} := \min_{i \in \mathcal{S}_-} \left| \frac{C_{ii}}{T_{ii}} \right| \quad \text{and} \quad \beta_{\rm opt} := \min_{i \in \mathcal{S}_+} \left| \frac{C_{ii}}{T_{ii}} \right|.$$

- Choose $0 \le \alpha \le \alpha_{\rm opt}$ and $0 \le \beta \le \beta_{\rm opt}$, not both being zero.

Initial matrix P_0

- Let

$$\alpha_{\rm opt} := \min_{i\in\mathcal{S}_-}\left|\frac{C_{ii}}{T_{ii}}\right| \quad \text{and} \quad \beta_{\rm opt} := \min_{i\in\mathcal{S}_+}\left|\frac{C_{ii}}{T_{ii}}\right|.$$

- Choose $0 \le \alpha \le \alpha_{\rm opt}$ and $0 \le \beta \le \beta_{\rm opt}$, not both being zero.

$$P_0 := \begin{bmatrix} C_+ - \alpha T_{++} & -\beta T_{+-} \\ -\alpha T_{-+} & |C_-| - \beta T_{--} \end{bmatrix}^{-1} \begin{bmatrix} C_+ + \beta T_{++} & \alpha T_{+-} \\ \beta T_{-+} & |C_-| + \alpha T_{--} \end{bmatrix}$$

ADDA: P_0

Initial matrix P_0

- Let

$$\alpha_{\text{opt}} := \min_{i \in \mathcal{S}_-} \left| \frac{C_{ii}}{T_{ii}} \right| \quad \text{and} \quad \beta_{\text{opt}} := \min_{i \in \mathcal{S}_+} \left| \frac{C_{ii}}{T_{ii}} \right|.$$

- Choose $0 \leq \alpha \leq \alpha_{\text{opt}}$ and $0 \leq \beta \leq \beta_{\text{opt}}$, not both being zero.

$$P_0 := \begin{bmatrix} C_+ - \alpha T_{++} & -\beta T_{+-} \\ -\alpha T_{-+} & |C_-| - \beta T_{--} \end{bmatrix}^{-1} \begin{bmatrix} C_+ + \beta T_{++} & \alpha T_{+-} \\ \beta T_{-+} & |C_-| + \alpha T_{--} \end{bmatrix}$$

ADDA: P_0

SDA: P_0 with $\alpha = \beta := \min(\alpha_{\text{opt}}, \beta_{\text{opt}})$

SDA-ss: P_0 with $\alpha := 0$

Probabilistically speaking, why do they work?

- For simplicity, assume that the fluid has unit rates:

$$P_0 = \begin{bmatrix} I - \alpha T_{++} & -\beta T_{+-} \\ -\alpha T_{-+} & I - \beta T_{--} \end{bmatrix}^{-1} \begin{bmatrix} I + \beta T_{++} & \alpha T_{+-} \\ \beta T_{-+} & I + \alpha T_{--} \end{bmatrix}.$$

- Applying a doubling algorithm to

$$P_0 = \begin{bmatrix} E & G \\ H & F \end{bmatrix}$$

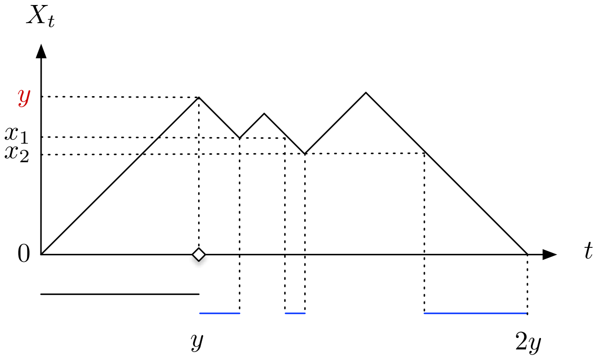
is equivalent to applying the Cyclic Reduction algorithm to the QBD

$$D_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},$$

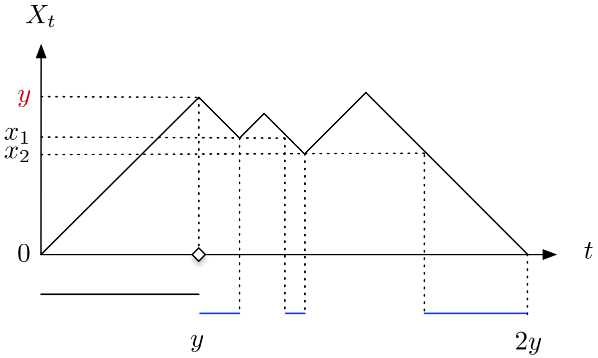
with

$$G_D = \begin{bmatrix} 0 & \Psi \\ 0 & W \end{bmatrix}.$$

The return probability matrix Ψ



The return probability matrix Ψ



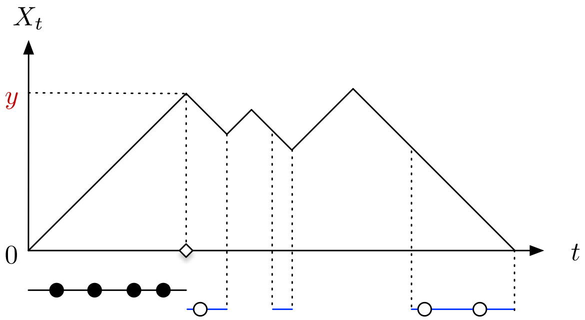
- If we decompose sample paths using y, then

$$\Psi = \int_0^\infty \exp(T_{++}y)T_{+-}\exp(Uy)\,\mathrm{d}y,$$

where $U := T_{--} + T_{-+}\Psi$ = generator of downward record process.

da Silva Soares and Latouche (2002) (almost)

- Uniformize fluid process over $[0, y]$ — $P_\lambda := I + \lambda^{-1}T$
- Uniformize downward record process over $[y, 2y]$ — $V_\mu := I + \mu^{-1}U$



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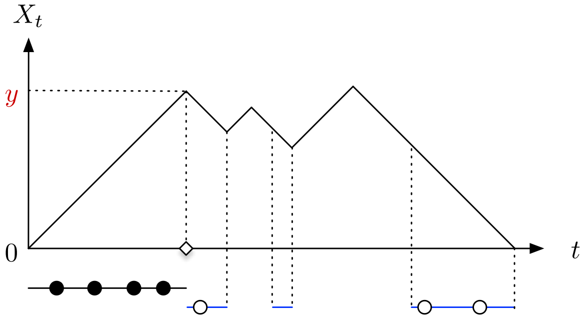


$$\Psi = \sum_{k,n=0}^{\infty} \gamma_{k,n} P_{\lambda++}^{k} P_{\lambda+-} V_{\mu}^{n} \quad \text{where} \quad \gamma_{kn} := \frac{(k+n)!}{k!n!} \frac{\lambda^{k+1}\mu^{n}}{(\lambda+\mu)^{k+n+1}}$$

Other expressions for Ψ

Theorem (Bean, N., Poloni (2016))

$$\Psi = \sum_{k=0}^{\infty} P_{\lambda++}^{k} P_{\lambda+-} \left(I - \lambda^{-1} U\right)^{-k-1}$$

$$= \sum_{n=0}^{\infty} \left(I - \mu^{-1} T_{++}\right)^{-n-1} P_{\mu+-} V_{\mu}^{n}$$

$$= \sum_{m=0}^{\infty} \left(I + \lambda^{-1} T_{++}\right)^{m} \left(I - \mu^{-1} T_{++}\right)^{-m-1} \left(P_{\lambda+-} W + P_{\mu+-}\right) W^{m}$$

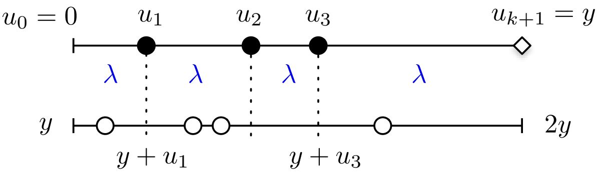
where

$$W := \left(I + \mu^{-1} U\right) \left(I - \lambda^{-1} U\right)^{-1}.$$

First Sum

$$\Psi = \sum_{k=0}^{\infty} P_{\lambda++}^{k} P_{\lambda+-} \left(I - \lambda^{-1} U\right)^{-k-1}$$

Let k be the number of uniformization steps in $[0, y)$.



We

a. make a uniformization step with $P_{\lambda++}$ at u_i, for $i = 1, \ldots, k$, and then do a uniformization step with $P_{\lambda+-}$ at u_{k+1}

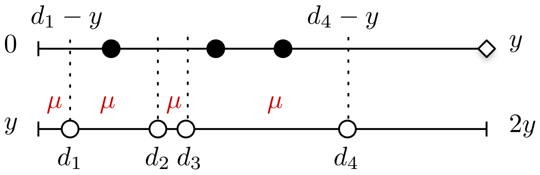
b. observe the downward record process at $y + u_i$, for $i = 1, \ldots, k+1$:

$$(I - \lambda^{-1} U)^{-1} = \int_0^{\infty} \lambda e^{-\lambda x} e^{Ux} \, dx.$$

Second Sum

$$\Psi = \sum_{n=0}^{\infty} \left(I - \mu^{-1} T_{++}\right)^{-n} \left(I - \mu^{-1} T_{++}\right)^{-1} P_{\mu+-} V_{\mu}^{n}$$

Let n be the number of uniformization steps in $[y, 2y)$.



We

a. observe the process at $d_i - y$, $i = 1, \ldots, n$, with $(I - \mu^{-1} T_{++})^{-1}$, perform some magic, and then

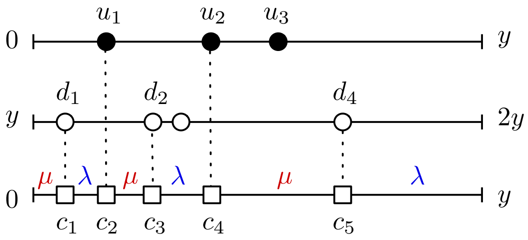
b. make a uniformization step at each d_i with V_μ.

Third Sum

$$\Psi = \sum_{m=0}^{\infty} \left(I + \lambda^{-1} T_{++}\right)^m \left(I - \mu^{-1} T_{++}\right)^{-m-1} \left(P_{\lambda+-} W + P_{\mu+-}\right) W^m$$

where $W := \left(I + \mu^{-1} U\right) \left(I - \lambda^{-1} U\right)^{-1}$.

Define a new sequence $\{c_i\}$ based on $\{u_i\}$ and $\{d_i\}$:

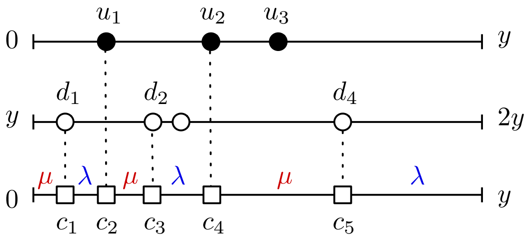


Third Sum

$$\Psi = \sum_{m=0}^{\infty} \left(I + \lambda^{-1} T_{++}\right)^m \left(I - \mu^{-1} T_{++}\right)^{-m-1} \left(P_{\lambda+-} W + P_{\mu+-}\right) W^m$$

where $W := \left(I + \mu^{-1} U\right) \left(I - \lambda^{-1} U\right)^{-1}$.

Define a new sequence $\{c_i\}$ based on $\{u_i\}$ and $\{d_i\}$:



We alternate between observing and doing a uniformization step, both on the way up and on the way down.

QBD #1

$$\Psi = \sum_{m=0}^{\infty} \left(I + \lambda^{-1} T_{++}\right)^{m} \left(I - \mu^{-1} T_{++}\right)^{-m-1} \left(P_{\lambda+-} W + P_{\mu+-}\right) W^{m}$$

$$= \sum_{m=0}^{\infty} \left(I - \mu^{-1} T_{++}\right)^{-m} P_{\lambda++}^{m} \left(I - \mu^{-1} T_{++}\right)^{-1} P_{\lambda+-} W^{m+1}$$

$$+ \sum_{m=0}^{\infty} \left(I - \mu^{-1} T_{++}\right)^{-m} P_{\lambda++}^{m} \left(I - \mu^{-1} T_{++}\right)^{-1} P_{\mu+-} W^{m}.$$

$$C'_{-1} := \begin{bmatrix} 0 & \left(I - \mu^{-1} T_{++}\right)^{-1} P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C'_{0} := \begin{bmatrix} 0 & (I - \mu^{-1} T_{++})^{-1} P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \; C'_{1} := \begin{bmatrix} \left(I - \mu^{-1} T_{++}\right)^{-1} P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad G_{C'} = \begin{bmatrix} 0 & \Psi \\ 0 & W \end{bmatrix}.$$

From QBD #1 to QBD #2

$$C'_{-1} := \begin{bmatrix} 0 & \left(I - \mu^{-1}T_{++}\right)^{-1} P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C'_0 := \begin{bmatrix} 0 & (I - \mu^{-1}T_{++})^{-1}P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C'_1 := \begin{bmatrix} \left(I - \mu^{-1}T_{++}\right)^{-1} P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left(I - \mu^{-1}T_{++}\right)^{-1} P_{\lambda++} = \sum_{n=0}^{\infty} \left(\frac{1}{2}P_{\mu++}\right)^n \frac{1}{2}P_{\lambda++}$$

$$C''_{-1} := \begin{bmatrix} 0 & \left(I - \mu^{-1}T_{++}\right)^{-1} P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C''_0 := \begin{bmatrix} \frac{1}{2}P_{\mu++} & (I - \mu^{-1}T_{++})^{-1}P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C''_1 := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

From QBD $\#2$ to QBDs $\#3$, $\#4, \#5, \#6, \#7$

From QBD #7 to QBD #8

$$C_{-1}^{(7)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & \frac{1}{2}P_{\mu--} \end{bmatrix}$$

$$C_{0}^{(7)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ \frac{1}{2}P_{\mu-+} & \frac{1}{2}P_{\lambda--} \end{bmatrix}, \quad C_{1}^{(7)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ \frac{1}{2}P_{\lambda-+} & 0 \end{bmatrix}$$

- Observe the QBD only at times τ_i where there are level changes

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$$C_{-1}^{(8)} := (I - C_0^{(7)})^{-1} C_{-1}^{(7)} = \begin{bmatrix} 0 & G \\ 0 & F \end{bmatrix}$$

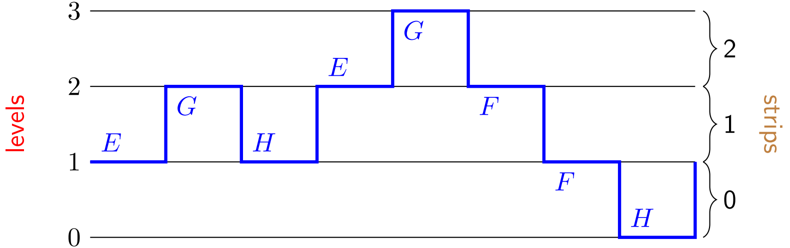
$$C_0^{(8)} := 0, \quad C_1^{(8)} := (I - C_0^{(7)})^{-1} C_1^{(7)} = \begin{bmatrix} E & 0 \\ H & 0 \end{bmatrix}$$

- What we really need:

$$D_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

Finally, from QBD $\#8$ to QBD $\#9$

$$C_{-1}^{(8)} := \begin{bmatrix} 0 & G \\ 0 & F \end{bmatrix}, \quad C_0^{(8)} := 0, \quad C_1^{(8)} := \begin{bmatrix} E & 0 \\ H & 0 \end{bmatrix}$$



$X_{C^{(8)}}(\tau_i)$	1	2	1	2	3	2	1	0
$X_D(\tau_i)$	1	1	1	2	2	1	0	0

$$D_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

In a nutshell

- We have a probablistic proof/interpretation for Doubling Algorithms.

From QBD #2 to QBD #3

$$C''_{-1} := \begin{bmatrix} 0 & \left(I - \mu^{-1} T_{++}\right)^{-1} P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C''_0 := \begin{bmatrix} \frac{1}{2} P_{\mu++} & (I - \mu^{-1} T_{++})^{-1} P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C''_1 := \begin{bmatrix} \frac{1}{2} P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left(I - \mu^{-1} T_{++}\right)^{-1} P_{\mu+-} = \sum_{n=0}^{\infty} \left(\frac{1}{2} P_{\mu++}\right)^n \frac{1}{2} P_{\mu+-}$$

$$C'''_{-1} := \begin{bmatrix} 0 & \frac{1}{2} P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C'''_0 := \begin{bmatrix} \frac{1}{2} P_{\mu++} & (I - \mu^{-1} T_{++})^{-1} P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C'''_1 := \begin{bmatrix} \frac{1}{2} P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

From QBD #3 to QBD #4

$$C'''_{-1} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & W \end{bmatrix},$$

$$C'''_0 := \begin{bmatrix} \frac{1}{2}P_{\mu++} & (I-\mu^{-1}T_{++})^{-1}P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C'''_1 := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left(I-\mu^{-1}T_{++}\right)^{-1}P_{\lambda+-} = \sum_{n=0}^{\infty}\left(\frac{1}{2}P_{\mu++}\right)^n \frac{1}{2}P_{\lambda+-}$$

$$C^{(4)}_{-1} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C^{(4)}_0 := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C^{(4)}_1 := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

From QBD #4 to QBD #5

$$C_{-1}^{(4)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & W \end{bmatrix}$$

$$C_{0}^{(4)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ 0 & 0 \end{bmatrix}, \quad C_{1}^{(4)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$W := (I - \lambda^{-1}U)^{-1}V_{\mu} = \sum_{n=0}^{\infty} \left(\frac{1}{2}V_{\lambda}\right)^{n} \frac{1}{2}V_{\mu}$$

$$C_{-1}^{(5)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & \frac{1}{2}V_{\mu} \end{bmatrix}$$

$$C_{0}^{(5)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ 0 & \frac{1}{2}V_{\lambda} \end{bmatrix}, \quad C_{1}^{(5)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

From QBD #5 to QBD #6

$$C_{-1}^{(5)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & \frac{1}{2}V_{\mu} \end{bmatrix}$$

$$C_{0}^{(5)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ 0 & \frac{1}{2}V_{\lambda} \end{bmatrix}, \quad C_{1}^{(5)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}V_{\mu} = \frac{1}{2}P_{\mu--} + \frac{1}{2}P_{\mu-+}\Psi$$

$$C_{-1}^{(6)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & \frac{1}{2}P_{\mu--} \end{bmatrix}$$

$$C_{0}^{(6)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ \frac{1}{2}P_{\mu-+} & \frac{1}{2}V_{\lambda} \end{bmatrix}, \quad C_{1}^{(6)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

From QBD #6 to QBD #7

$$C_{-1}^{(6)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & \frac{1}{2}P_{\mu--} \end{bmatrix}$$

$$C_{0}^{(6)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ \frac{1}{2}P_{\mu-+} & \frac{1}{2}V_{\lambda} \end{bmatrix}, \quad C_{1}^{(6)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}V_{\lambda} = \frac{1}{2}P_{\lambda--} + \frac{1}{2}P_{\lambda-+}\Psi$$

$$C_{-1}^{(7)} := \begin{bmatrix} 0 & \frac{1}{2}P_{\mu+-} \\ 0 & \frac{1}{2}P_{\mu--} \end{bmatrix}$$

$$C_{0}^{(7)} := \begin{bmatrix} \frac{1}{2}P_{\mu++} & \frac{1}{2}P_{\lambda+-} \\ \frac{1}{2}P_{\mu-+} & \frac{1}{2}P_{\lambda--} \end{bmatrix}, \quad C_{1}^{(7)} := \begin{bmatrix} \frac{1}{2}P_{\lambda++} & 0 \\ \frac{1}{2}P_{\lambda-+} & 0 \end{bmatrix}.$$