Stationary analysis of MAP/PH/1/r queue with bi-level hysteretic control of arrivals

Rostislav Razumchik

Institute of Informatics Problems
of the Federal Research Center "Computer Science and Control"
of the Russian Academy of Sciences, Moscow, Russia

RUDN University, Moscow, Russia

28-30 June 2016

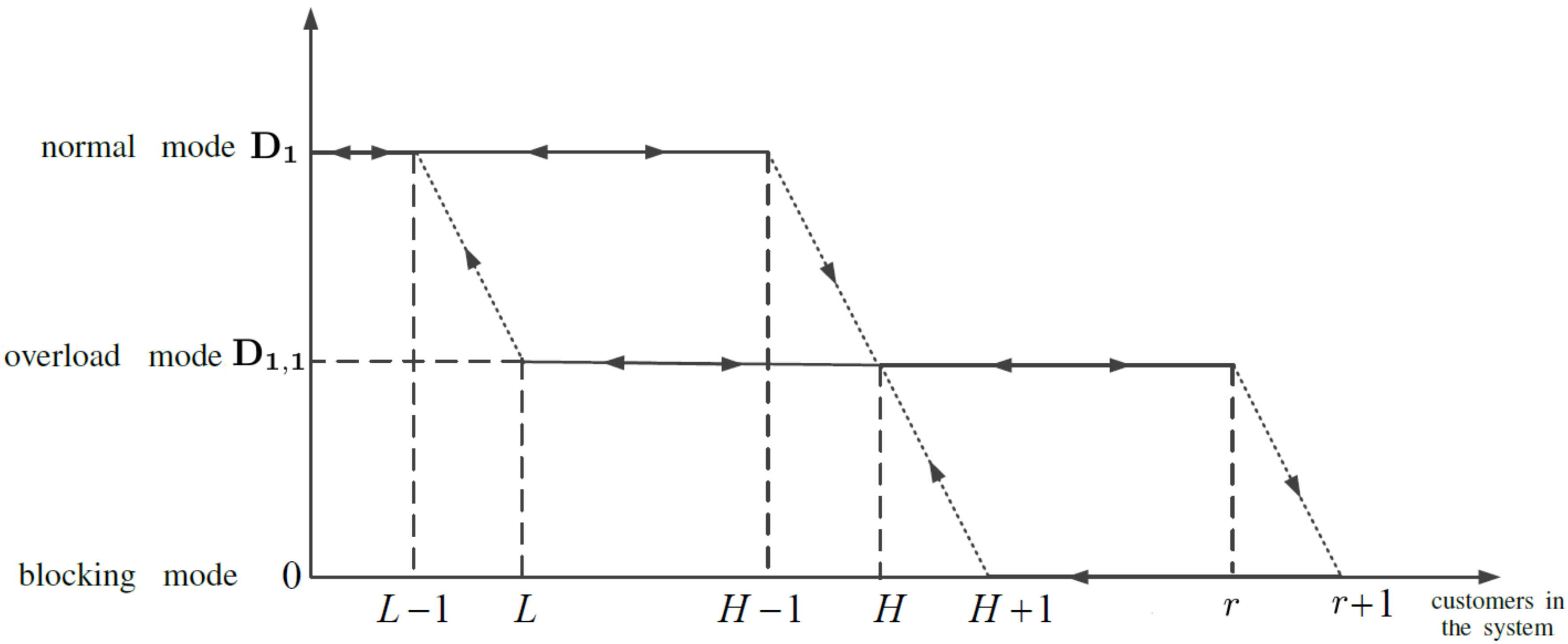
Outline:

- description of the queueing system
- algorithm for the stationary distribution
- stationary sojourn times
- concluding remarks

System description:

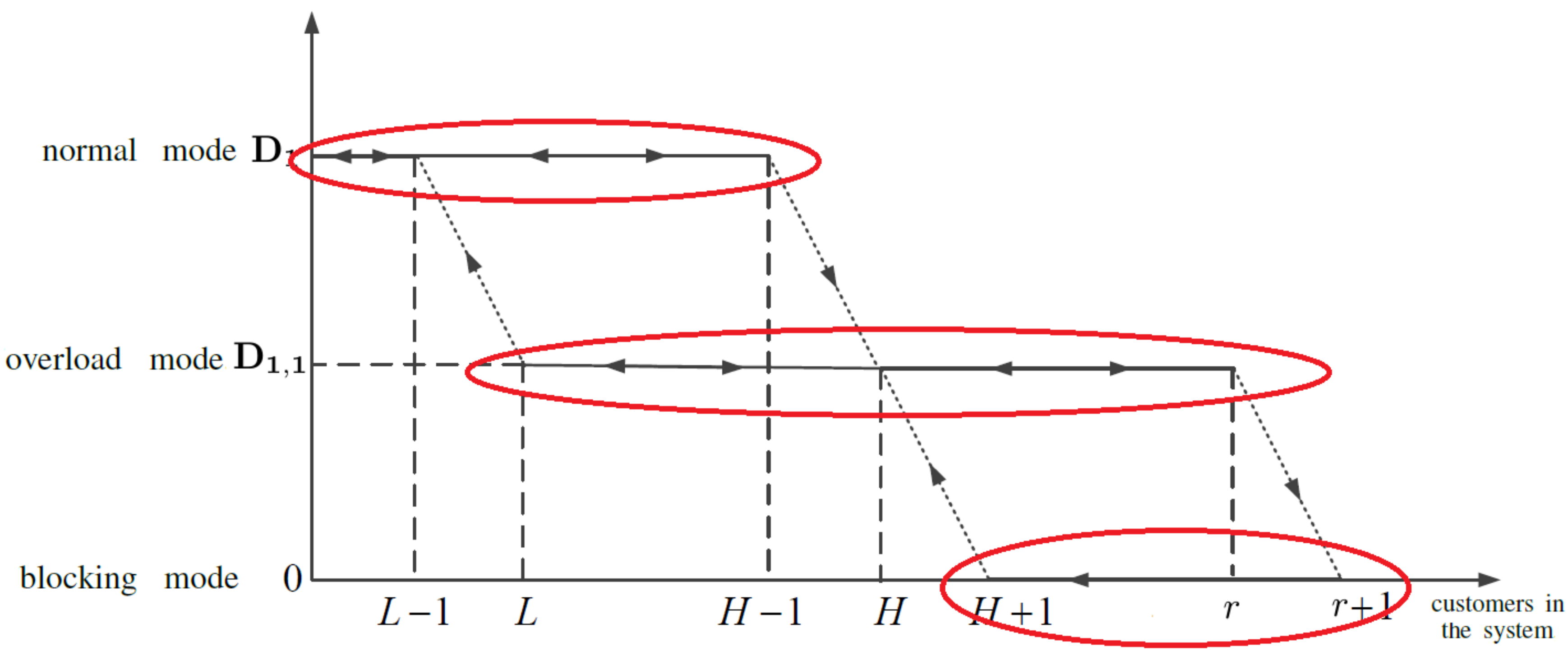
- MAP arrivals, $(\mathbf{D_0}, \mathbf{D_1} = \mathbf{D_{1,1}} + \mathbf{D_{1,2}})$ of order N
- PH service times, $(\vec{f}, \mathbf{G})$ of order M
- queue capacity is r
- bi-level hysteretic control of arrivals is implemented

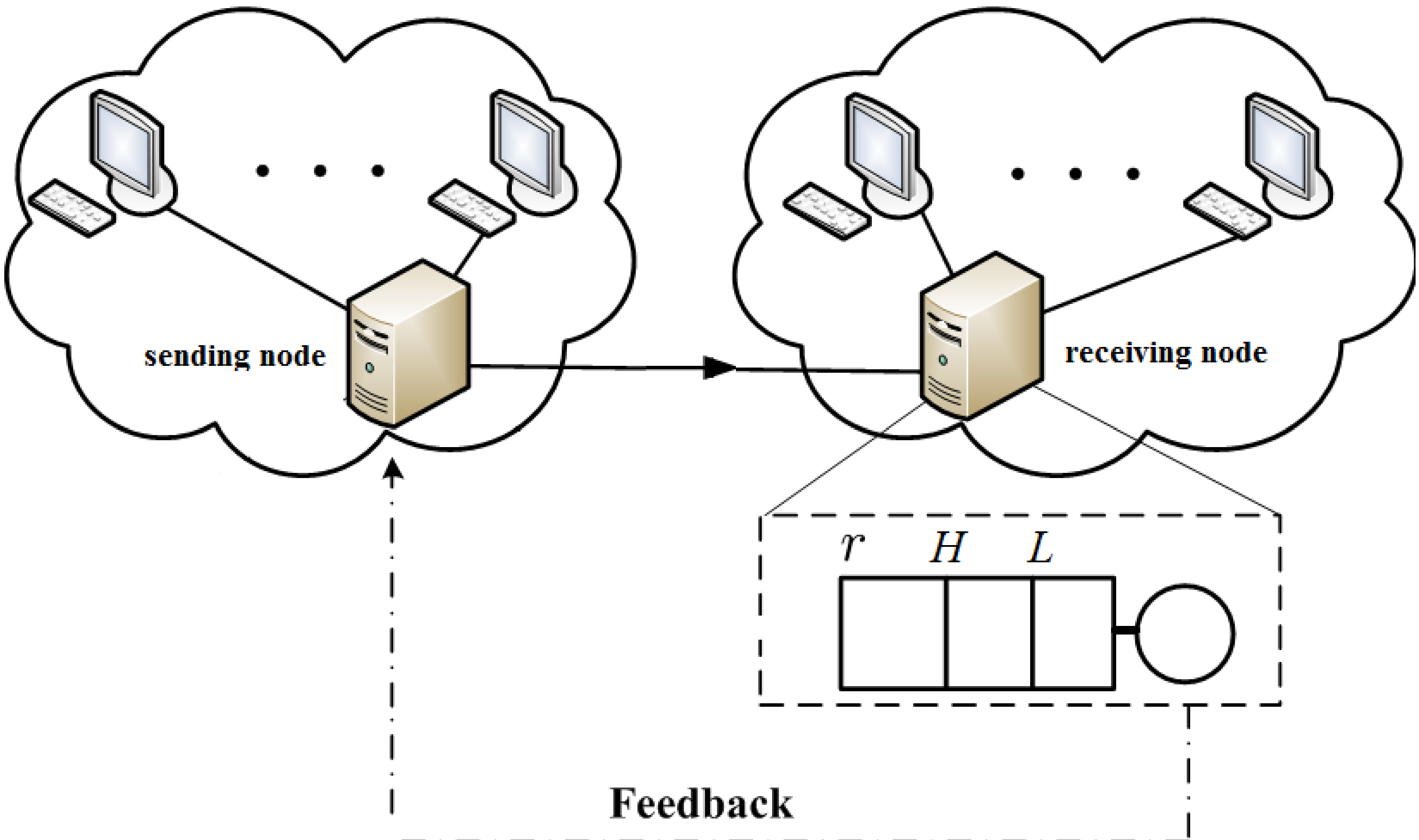
  (L – low threshold, H – high threshold)



Main performance measures of interest:

- **joint stationary distribution of the system size, system mode and the states of the background processes**
- **stationary sojourn times in different modes (moments, distribution)**




sending node
receiving node
r
H
L
Feedback

Some references:

1. Gebhart R. A queuing process with bilevel hysteretic service-rate control. Naval Research Logistics Quarterly, VINITI, 1967. Vol. 14. pp. 55–68.
2. Gortsev A. M. A queueing system with an arbitrary number of standby channels and hysteresis control of their connection and disconnection. Automation and remote control, 1977. no. 10, pp. 30–37.
3. Takagi H. Analysis of a Finite-Capacity M/G/1 Queue with a Resume Level. Performance Evaluation, Vol. 5, 1985, pp. 197–203.
4. Ye J., Li S. Analysis of Multi-Media Traffic Queues with Finite Buffer and Overload Control - Part 1: Algorithm. INFOCOM, 1991. pp. 1464–1474.
5. Dshalalow J.H. Queueing systems with state dependent parameters. In: Frontiers in Queueing: Models and Applications in Science and Engineering, 1997, pp. 61–116.
6. Bekker R., Boxma O.J. An M/G/1 queue with adaptable service speed. Stochastic Models. 2007. Vol. 23. Issue 3. Pp. 373–396.
7. Choi D.I., Kim T.S., Lee S. Analysis of an MMPP/G/1/K queue with queue length dependent arrival rates, and its application to preventive congestion control in telecommunication networks. European Journal of Operational Research, 2008. Vol. 187. Issue 2. Pp. 652–659.
8. Bekker R. Queues with Levy input and hysteretic control. Queueing Systems, 2009. Vol. 63. Issue 1. Pp. 281–299.

Markov process: $\mathbf{X}(t) = (\xi(t); \eta(t); \mu(t); \nu(t))$

- $\xi(t)$ — MAP generation phase at time t,
- $\eta(t)$ — PH service phase at time t,
- $\mu(t)$ — system's mode at time t,
- $\nu(t)$ — number in the system at time t.

The state space: $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \mathcal{X}_2$,

$$\begin{aligned}
\mathcal{X}_0 &= \{(k,0,0) : 1 \le k \le N\} \cup \\
&\qquad \{(k,0,n) : 1 \le k \le NM, 1 \le n \le H-1\}, \\
\mathcal{X}_1 &= \{(k,1,n) : 1 \le k \le NM, L \le n \le r\}, \\
\mathcal{X}_2 &= \{(k,2,n) : 1 \le k \le NM, H+1 \le n \le r+1\}.
\end{aligned}$$

Notation:

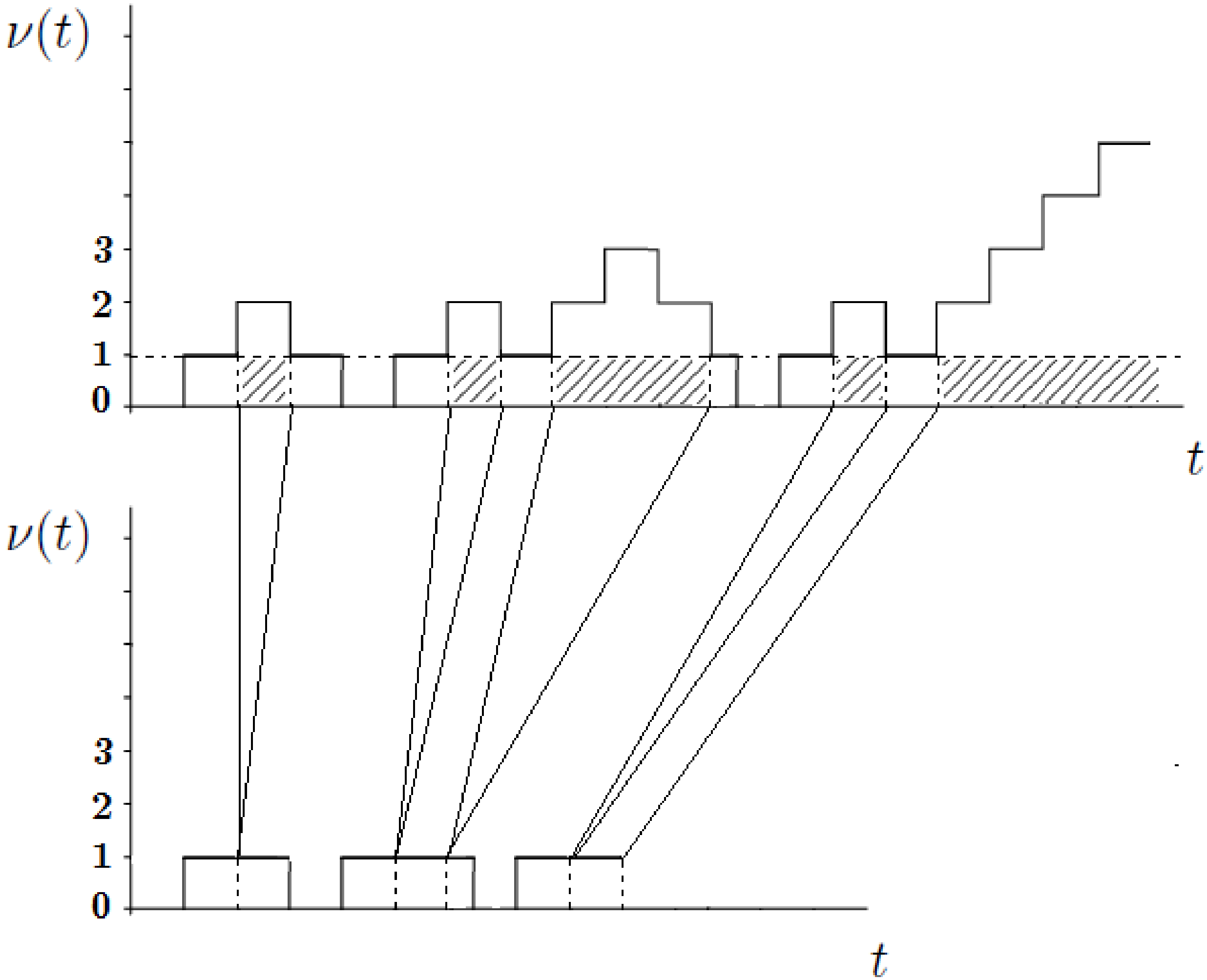
- service of a customer after which the system becomes empty: $\mathbf{P}_1 = \mathbf{E} \otimes \vec{g}$,
- service of a customer after which the system remains busy: $\mathbf{P} = \mathbf{P}^* = \mathbf{P}^\# = \mathbf{E} \otimes \vec{g}\vec{f}$,
- arrival phase change when empty: $\mathbf{Q}_0 = \mathbf{D}_0$,
- arrival phase change when system is in the “normal” mode: $\mathbf{Q} = \mathbf{D}_0 \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{G}$,
- arrival phase change when in the “overload” mode: $\mathbf{Q}^* = (\mathbf{D}_0 + \mathbf{D}_{12}) \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{G}$,
- arrival phase change when in the “blocking” mode: $\mathbf{Q}^\# = (\mathbf{D}_0 + \mathbf{D}_1) \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{G}$,
- arrival of a customer to an empty system: $\mathbf{R}_0 = \mathbf{D}_1 \otimes \vec{f}$,
- arrival of a customer to the system in the “normal” mode: $\mathbf{R} = \mathbf{D}_1 \otimes \mathbf{E}$,
- arrival of a customer to the system in the “overload” mode: $\mathbf{R}^* = \mathbf{D}_{11} \otimes \mathbf{E}$.

$$
\mathbb{A} = \left(\begin{array}{ccccccccccccccccccc}
\mathbb{Q}_0 & \mathbb{R}_0 & 0 & \ldots & 0 & \ldots & 0 & & & & & & & & & & & & \\
\mathbb{P}_1 & \mathbb{Q} & \mathbb{R} & \ldots & 0 & \ldots & 0 & & & & & & & & & & & & \\
0 & \mathbb{P} & \mathbb{Q} & \ldots & 0 & \ldots & 0 & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & & & & & & & & & & & \\
0 & 0 & 0 & \ldots & \mathbb{Q} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & & & & & \\
\vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & & & & \\
0 & 0 & 0 & \ldots & 0 & \ldots & \mathbb{Q} & 0 & 0 & 0 & \ldots & \mathbb{R} & \ldots & 0 & & & & & \\
0 & 0 & 0 & \ldots & \mathbb{P}^* & \ldots & 0 & \mathbb{Q}^* & \mathbb{R}^* & 0 & \ldots & 0 & \ldots & 0 & & & & & \\
 & & 0 & \ldots & 0 & \ldots & 0 & \mathbb{P}^* & \mathbb{Q}^* & \mathbb{R}^* & \ldots & 0 & \ldots & 0 & & & & & \\
 & & 0 & \ldots & 0 & \ldots & 0 & 0 & \mathbb{P}^* & \mathbb{Q}^* & \ldots & 0 & \ldots & 0 & & & & & \\
 & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & & & & \\
 & & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \mathbb{Q}^* & \ldots & 0 & 0 & 0 & & 0 & 0 \\
 & & & & & & & & & \vdots & & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & & & & & & & & & 0 & \ldots & 0 & \ldots & \mathbb{Q}^* & 0 & 0 & \ldots & 0 & \mathbb{R}^* \\
 & & & & & & & & & 0 & \ldots & \mathbb{P}^\# & \ldots & 0 & \mathbb{Q}^\# & 0 & \ldots & 0 & 0 \\
 & & & & & & & & & 0 & \ldots & 0 & \ldots & 0 & \mathbb{P}^\# & \mathbb{Q}^\# & \ldots & 0 & 0 \\
 & & & & & & & & & \vdots & & \vdots & & & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & & & & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & \ldots & \mathbb{Q}^\# & 0 \\
 & & & & & & & & & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & \ldots & \mathbb{P}^\# & \mathbb{Q}^\#
\end{array}\right)
$$

Stationary distribution of the original system:

- $p_{k,m,n}$ stationary probability of the state (k, m, n)
- $\vec{p}_{m,n} = (p_{1,m,n}, \ \ldots, p_{NM,m,n})$

Stationary distribution of the new system without queue: $\vec{q}_{0,0}$, $\vec{q}_{0,1}$



Auxiliary matrix:

- $[\mathbf{A}_2]_{(i,j)} = \mathbf{P}\Big\{\mathbf{X}(\tau) = (j,0,1) | \mathbf{X}(0) = (i,0,2)\Big\}, \ \tau = \inf\{t > 0 : \nu(t) = 1\}.$

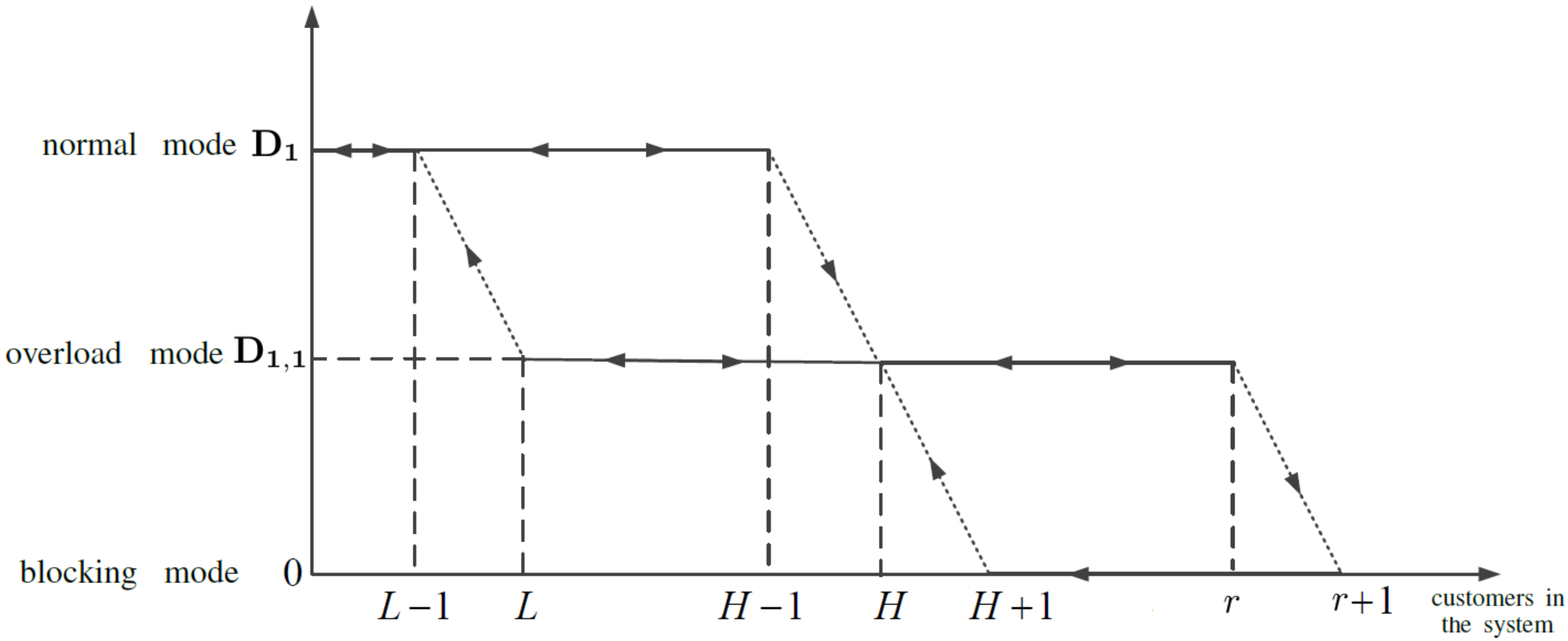
Balance equations for the new system without queue:

$$\begin{aligned} 0 &= \vec{q}_{0,0}\mathbf{Q}_0 + \vec{q}_{0,1}\mathbf{P}_1, \\ 0 &= \vec{q}_{0,0}\mathbf{R}_0 + \vec{q}_{0,1}\mathbf{Q} + \vec{q}_{0,1}\mathbf{R}\mathbf{A}_2. \end{aligned}$$

Due to the restricted Markov chains property these equations are also valid for $\vec{p}_{0,0}$ and $\vec{p}_{0,1}$.

The balance equation for the boundary probabilities in the new system with maximum queue-size n is:

$$0 = \vec{q}_{0,n-1}\mathbf{R} + \vec{q}_{0,n}\mathbf{Q} + \vec{q}_{0,n}\mathbf{R}\mathbf{A}_{n+1}, \ 2 \le n \le L-2.$$


normal mode $\mathbf{D_1}$
overload mode $\mathbf{D_{1,1}}$
blocking mode 0
$L-1$
L
$H-1$
H
$H+1$
r
$r+1$
customers in the system

When queue-size exceeds (H-1) one needs more matrices, which record starting and stopping phases:

- $$[G_{H+1}]_{(i,j)} = \mathbf{P}\left\{\mathbf{X}(\tau) = (j,2,r+1); \mathbf{X}(t) \notin \bigcup_{k=1}^{NM}(k,1,H),\right.$$ $$\left. t \in (0,\tau) | \mathbf{X}(0) = (i,1,H+1)\right\},\ \tau = \inf\{t > 0 : \nu(t) = r+1\},$$
- $$[D_H]_{(i,j)} = \mathbf{P}\left\{\mathbf{X}(\tau) = (j,1,H) | \mathbf{X}(0) = (i,2,r+1)\right\},$$ $\tau = \inf\{t > 0 : \nu(t) = H\}$,
- $$[B_{H-1}]_{(i,j)} = \mathbf{P}\left\{\mathbf{X}(\tau) = (j,0,H); \mathbf{X}(t) \notin \bigcup_{k=1}^{NM}(k,0,L-1),\right.$$ $$\left. t \in (0,\tau) | \mathbf{X}(0) = (i,1,H-1)\right\},\ \tau = \inf\{t > 0 : \nu(t) = H\}.$$

The balance equation for $\vec{q}_{1,H}$ in the new system with queue-size H:

$$0 = \vec{q}_{0,H-1}\mathbf{R} + \vec{q}_{1,H}(\mathbf{Q}^* + \mathbf{P}^*\mathbf{B}_{H-1} + \mathbf{R}^*(\mathbf{A}_{H+1} + \mathbf{G}_{H+1}\mathbf{D}_H)).$$

Final system of balance equations for $\vec{p}_{m,n}$ in the original system:

$$
\begin{aligned}
0 &= \vec{p}_{0,0}\mathbf{Q}_0 + \vec{p}_{0,1}\mathbf{P}_1, \\
0 &= \vec{p}_{0,0}\mathbf{R_0} + \vec{p}_{0,1}\mathbf{Q} + \vec{p}_{0,1}\mathbf{R}\mathbf{A}_2. \\
0 &= \vec{p}_{0,n-1}\mathbf{R} + \vec{p}_{0,n}\mathbf{Q} + \vec{p}_{0,n}\mathbf{R}\mathbf{A}_{n+1},\ 2 \le n \le L-2, \\
0 &= \vec{p}_{0,L-2}R + \vec{p}_{0,L-1}\mathbf{Q} + \vec{p}_{0,L-1}\mathbf{R}\mathbf{A}^*, \\
0 &= \vec{p}_{0,n-1}\mathbf{R} + \vec{p}_{0,n}\mathbf{Q} + \vec{p}_{0,n}\mathbf{R}\mathbf{A}_{n+1},\ L \le n \le H-2, \\
0 &= \vec{p}_{0,H-2}\mathbf{R} + \vec{p}_{0,H-1}\mathbf{Q}, \\
0 &= \vec{p}_{0,H-1}\mathbf{R} + \vec{p}_{1,H}(\mathbf{Q}^* + \mathbf{P}^*\mathbf{B}_{H-1} + \mathbf{R}^*(\mathbf{A}_{H+1} + \mathbf{G}_{H+1}\mathbf{D}_H)), \\
0 &= \vec{p}_{1,n-1}\mathbf{R}^* + \vec{p}_{1,n}\mathbf{R}^*\mathbf{A}_{n+1} + \vec{p}_{1,n}\mathbf{Q}^*,\ \ H+1 \le n \le r-1, \\
0 &= \vec{p}_{1,r-1}\mathbf{R}^* + \vec{p}_{1,r}\mathbf{Q}^*, \\
0 &= \vec{p}_{1,r}\mathbf{R}^* + \vec{p}_{2,r+1}\mathbf{Q}^{\#}, \\
0 &= \vec{p}_{2,n+1}\mathbf{P}^{\#} + \vec{p}_{2,n}\mathbf{Q}^{\#},\ \ H+1 \le n \le r, \\
0 &= \vec{p}_{1,n+1}\mathbf{P}^* + \vec{p}_{1,n}\mathbf{P}^*\mathbf{B}_{n-1} + \vec{p}_{1,n}\mathbf{Q}^*,\ L+1 \le n \le H-1, \\
0 &= \vec{p}_{1,L+1}\mathbf{P}^* + \vec{p}_{1,L}\mathbf{Q}^*.
\end{aligned}
$$

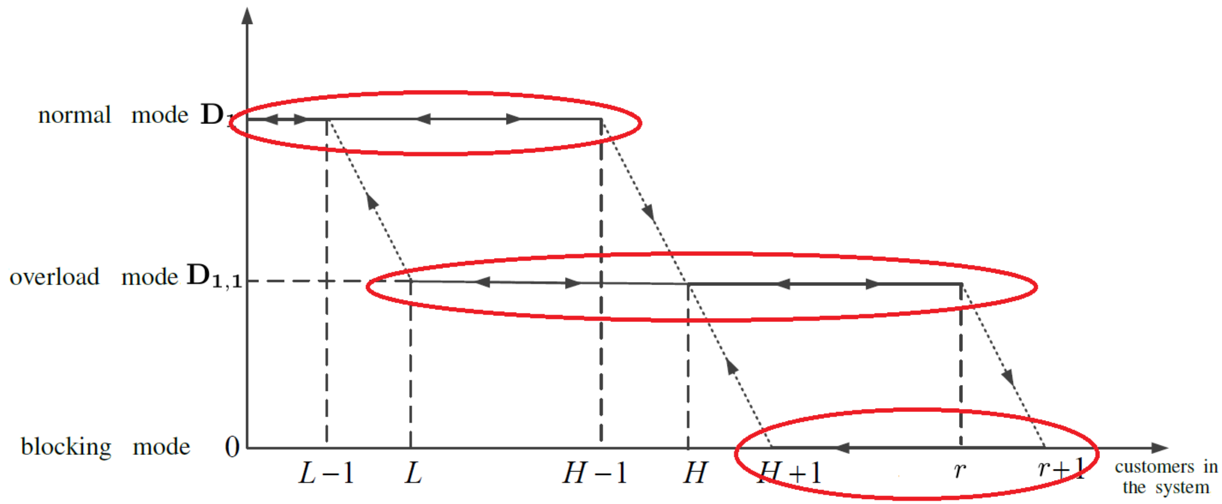
Example of the system of equations for $\mathbf{A}_n,\, H+1 \le n \le r$:

$$
\begin{aligned}
&\mathbf{A}_r = \alpha^* + \beta^*\mathbf{A}_r, \\
&\mathbf{A}_n = \alpha^* + \beta^*\mathbf{A}_n + \gamma^*\mathbf{A}_{n+1}\mathbf{A}_n, \quad H+1 \le n \le r-1.
\end{aligned}
$$

First passage times from overload mode to normal mode:

$$[\mathbf{V}_n(x)]_{(i,j)} = \mathbf{P}\left\{\tau < x, \mathbf{X}(\tau) = (j,0,L-1) | \mathbf{X}(0) = (i,1,n)\right\}, \; L \le n \le r,$$

$$\tau = \inf\{t > 0 : \nu(t) = L-1\}.$$



If n>H, then

time n→(L-1) = time n→H + time H→(L-1).

time n→H = "time n→(n-1) without visiting (r+1)"+"time (n-1)→H"

+ "time n→(r+1) without visiting (n-1)"+"time (r+1)→H".

Time H→(L-1) = time H→(H-1) + time (H-1)→(L-1).

Concluding remarks:

- **Generalization for overlapping hysteretic loops, several incoming flows, multiple servers.**
- **Is it possible to extend the approach for two interconnected systems each with hysteretic policy implemented?**
- **(from application side) behaviour of several interconnected systems: what is the gain of hysteretic control of arrivals with respect to other types of control?**

Thank you for your attention!