Generalised reward generator $\mathbf{Z}(\underline{\mathbf{s}})$ for stochastic fluid models.

Aviva Samuelson[1,3]
with Małgorzata M. O'Reilly[1,3] and Nigel G. Bean[2,3]

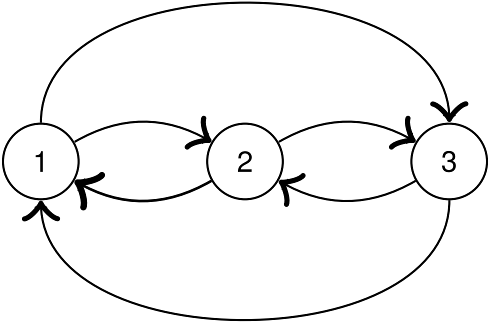
June 28-30, 2016

Matrix Analytic Methods

[1]School of Physical Sciences, University of Tasmania.
[2]School of Mathematical Sciences, University of Adelaide.
[3]ARC Centre of Excellence for Mathematical and Statistical Frontiers.

Outline

Motivating example



State $i = 1, 2, 3$ is a customer class.

SFM definition

Continuous-time process $\{(\varphi(t), X(t)) : t \geq 0\}$ with

phase variable $\varphi(t)$ and unbounded level variable $X(t) \in \mathbb{R}$ such that

- phase process $\{\varphi(t) : t \geq 0\}$ is an irreducible CTMC with generator $\mathbf{T}$ and some finite-state space $\mathcal{S}$
- the rate of change of $X(t)$ at time t is $c_i = dX(t)/dt$ whenever $\varphi(t) = i$.

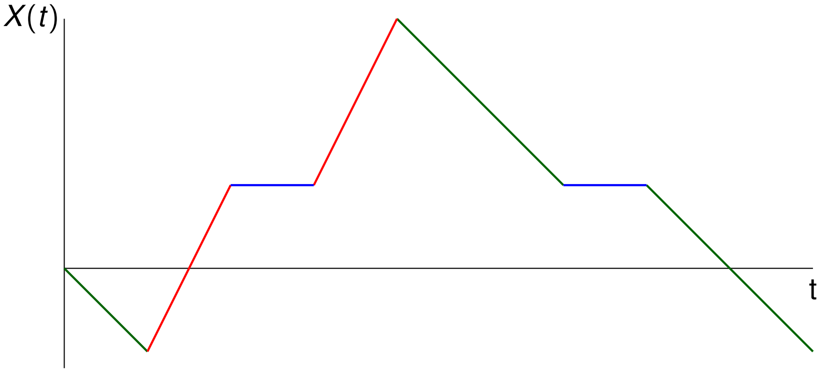
Motivating example - continued

State space: $\mathcal{S} = \{1, 2, 3\}$ where 1, 2, 3 are customer classes.

Fluid $X(t)$: Amount of energy in the grid.

Rates c_i: $c_1 > 0$, $c_2 < 0$, and $c_3 = 0$.

Sample path



Some notation

$$\mathcal{S}_+ = \{i \in \mathcal{S} : c_i > 0\}$$
$$\mathcal{S}_- = \{i \in \mathcal{S} : c_i < 0\}$$
$$\mathcal{S}_0 = \{i \in \mathcal{S} : c_i = 0\}$$

$\mathbf{T}_{+-} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_-$
$\mathbf{T}_{-+} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_-, j \in \mathcal{S}_+$
$\mathbf{T}_{+0} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_0$
etc.

$\mathbf{C}_+ = diag(c_i)$ for all $i \in \mathcal{S}_+$
$\mathbf{C}_- = diag(|c_i|)$ for all $i \in \mathcal{S}_-$

Motivation

In the literature so far, we have

- Laplace-Stieltjes Transforms (LSTs) of the time taken to complete a sample path in $\{(\varphi(t), X(t)) : t \geq 0\}$ using fluid generator $\mathbf{Q}(s)$

- LSTs of the shift in X accumulated during a sample path in Y in $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$ using fluid generator $\mathbf{W}(s)$, where $X(t) \in \mathbb{R}$ is unbounded.

Here, we wish to model, individually,

- i-type rewards accumulated at rates r_i per unit time spent in i during a sample path in $\{(\varphi(t), X(t)) : t \geq 0\}$

  where $X(t)$ may be bounded/unbounded.

In-out fluid $h(t)$

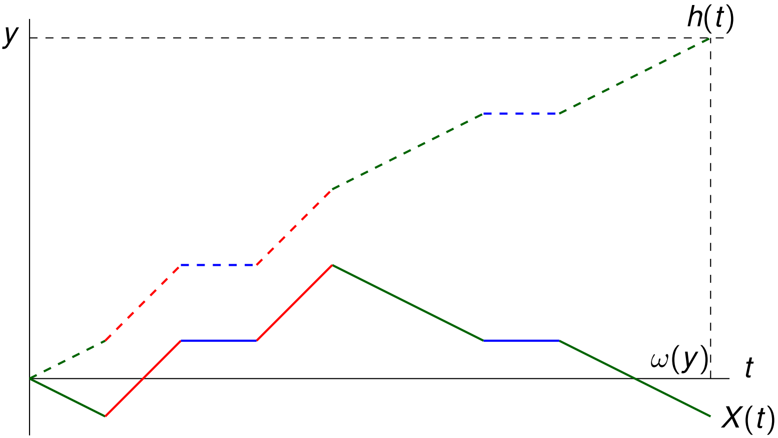
In-out fluid:

$$h(t) = \int_{u=0}^{t} |c_{\varphi(u)}| du.$$

Time at which $h(t)$ hits level y:

$$\omega(y) = \inf\{t > 0 : h(t) = y\}.$$

Evolution of in-out fluid $h(t)$



Total i-type reward

Assume i-type reward is generated at r_i per unit time spent in i.

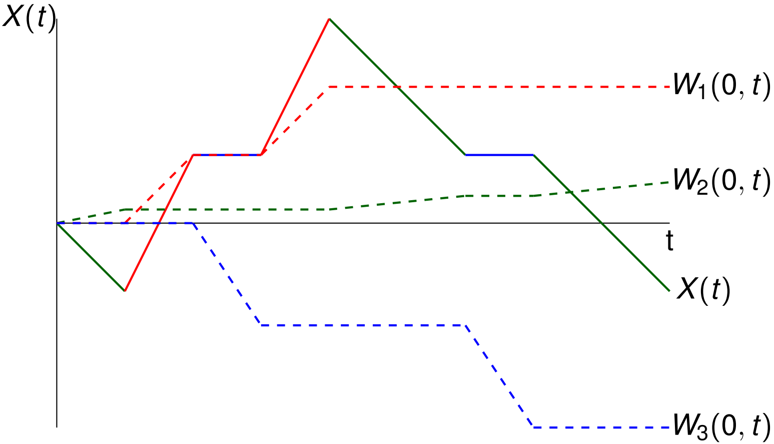
Total i-type reward accumulated during $[z, t]$ is defined as

$$W_i(z, t) = \int_{u=z}^{t} r_i I(\varphi(u) = i) du$$

where $I(\cdot)$ is an indicator function.

A set of total i-type rewards can be expressed as a vector $(W_1(z, t), \ldots, W_n(z, t))$.

Phase i reward $W_i(0, t)$, for $i = 1, 2, 3$



Laplace-Stieltjes transforms (LSTs) of interest

Let $\tilde{\mathbf{\Delta}}^{y}(\underline{\mathbf{s}})$ be a multi-dimensional LST matrix such that,

for any $y > 0$, any vector $\underline{\mathbf{s}} = [s_i]$, and any $i, j \in \mathcal{S}_+ \cup \mathcal{S}_-$,

$$[\tilde{\mathbf{\Delta}}^{y}(\underline{\mathbf{s}})]_{ij} \quad = \quad E\left(e^{-(s_1 W_1(0,\omega(y))+\ldots+s_n W_n(0,\omega(y)))} I(\varphi(\omega(y)) = j) \mid \varphi(0) = i\right)$$

is the LST of the distribution of

$(W_1(0, \omega(y)), \ldots, W_n(0, \omega(y)))$ and $\varphi(\omega(y)) = j$,

given $\varphi(0) = i$.

Key result

Theorem (2)

For any $y > 0$, $\tilde{\boldsymbol{\Delta}}^y(\underline{\mathbf{s}})$ exists and

$$\tilde{\boldsymbol{\Delta}}^y(\underline{\mathbf{s}}) = e^{\mathbf{Z}(\underline{\mathbf{s}})y}.$$

Generalised reward generator $\mathbf{Z}(\mathbf{s})$

Assuming $\chi(\mathbf{T}_{00} - \mathbf{D}_0\mathbf{R}_0) < 0$, define

$$\mathbf{Z}(\underline{\mathbf{s}}) = \begin{bmatrix} \mathbf{Z}_{++}(\underline{\mathbf{s}}) & \mathbf{Z}_{+-}(\underline{\mathbf{s}}) \\ \mathbf{Z}_{-+}(\underline{\mathbf{s}}) & \mathbf{Z}_{--}(\underline{\mathbf{s}}) \end{bmatrix}$$

where $\underline{\mathbf{s}} = [s_i]$ and

$$\begin{aligned}
\mathbf{Z}_{++}(\underline{\mathbf{s}}) &= \mathbf{C}_+^{-1}[\mathbf{T}_{++} - \mathbf{D}_+\mathbf{R}_+ - \mathbf{T}_{+0}(\mathbf{T}_{00} - \mathbf{D}_0\mathbf{R}_0)^{-1}\mathbf{T}_{0+}] \\
\mathbf{Z}_{--}(\underline{\mathbf{s}}) &= \mathbf{C}_-^{-1}[\mathbf{T}_{--} - \mathbf{D}_-\mathbf{R}_- - \mathbf{T}_{-0}(\mathbf{T}_{00} - \mathbf{D}_0\mathbf{R}_0)^{-1}\mathbf{T}_{0-}] \\
\mathbf{Z}_{+-}(\underline{\mathbf{s}}) &= \mathbf{C}_+^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - \mathbf{D}_0\mathbf{R}_0)^{-1}\mathbf{T}_{0-}] \\
\mathbf{Z}_{-+}(\underline{\mathbf{s}}) &= \mathbf{C}_-^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - \mathbf{D}_0\mathbf{R}_0)^{-1}\mathbf{T}_{0+}].
\end{aligned}$$

Some more notation

Recall

$$\mathcal{S}_+ = \{i \in \mathcal{S} : c_i > 0\}$$
$$\mathcal{S}_- = \{i \in \mathcal{S} : c_i < 0\}$$
$$\mathcal{S}_0 = \{i \in \mathcal{S} : c_i = 0\}.$$

Let, for complex s_i,

$$\mathbf{D}_+ = diag(s_i) \text{ for all } i \in \mathcal{S}_+$$
$$\mathbf{D}_- = diag(s_i) \text{ for all } i \in \mathcal{S}_-$$
$$\mathbf{D}_0 = diag(s_i) \text{ for all } i \in \mathcal{S}_0$$

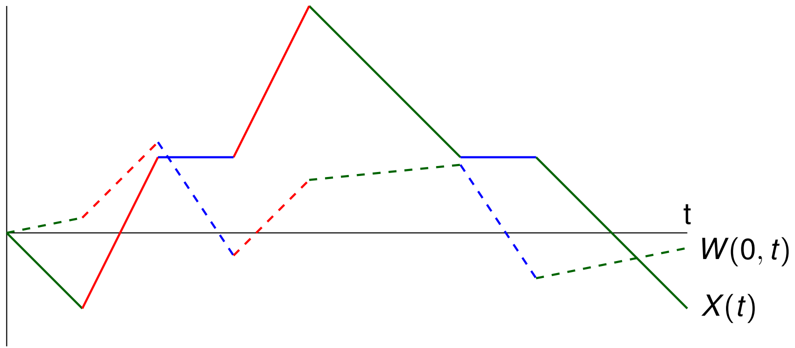
$$\mathbf{R}_+ = diag(r_i) \text{ for all } i \in \mathcal{S}_+$$
$$\mathbf{R}_- = diag(r_i) \text{ for all } i \in \mathcal{S}_-$$
$$\mathbf{R}_0 = diag(r_i) \text{ for all } i \in \mathcal{S}_0.$$

Total reward $W(z, t)$

Total reward accumulated during $[z, t]$ is defined as

$$W(z,t) = \sum_{i \in \mathcal{S}} W_i(z,t).$$

Total reward $W(0, t)$



Projections of $\mathbf{Z}(\underline{\mathbf{s}})$

- Replace rewards $W_i(z,t)$ with total reward $W(z,t)$.
- Replace vector $\underline{\mathbf{s}}$ with scalar $s \in \mathbb{C}$.
- Derive corresponding fluid generators.

Interpretation of $\mathbf{Q}(s)$

We want to know

$$E\left(e^{-(s\omega(y))} I(\varphi(\omega(y)) = j) \mid \varphi(0) = i\right)$$

which is the LST of the distribution of

time $\omega(y)$,

and $\varphi(\omega(y)) = j$,

given $\varphi(0) = i$.

Projection from $\mathbf{Z}(\underline{\mathbf{s}})$ to $\mathbf{Q}(s)$

Let

$$\begin{aligned} \mathbf{R}_+ &= \mathbf{I} \\ \mathbf{R}_- &= \mathbf{I} \\ \mathbf{R}_0 &= \mathbf{I} \end{aligned}$$

so that reward is $r_i = 1$ per unit of time spent in phase i.

Then the total reward is the total elapsed time,

$$W(z,t) = t - z.$$

Fluid generator $\mathbf{Q}(s)$

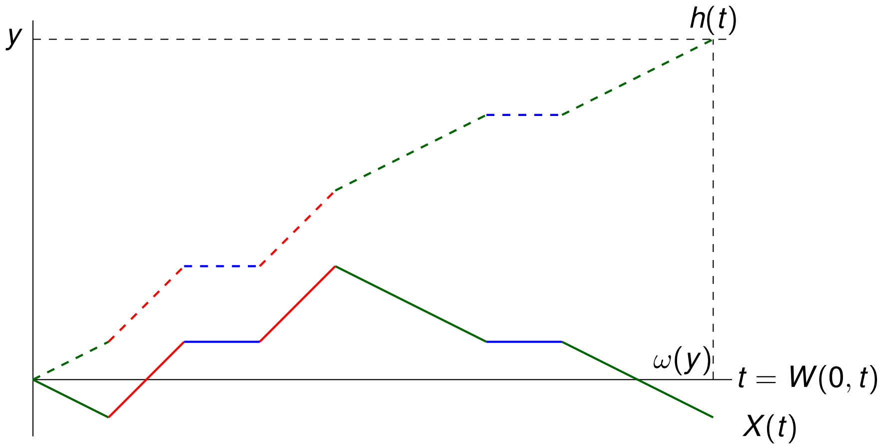
The i, j th entry of the LST of the distribution is $[e^{\mathbf{Q}(s)y}]_{ij}$, where

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{++}(s) & \mathbf{Q}_{+-}(s) \\ \mathbf{Q}_{-+}(s) & \mathbf{Q}_{--}(s) \end{bmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{++}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{++} - s\mathbf{I} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}] \\
\mathbf{Q}_{--}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{--} - s\mathbf{I} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Q}_{+-}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Q}_{-+}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}].
\end{aligned}$$

Interpretation of $\mathbf{Q}(s)$



Interpretation of $\mathbf{W}(s)$

We want to know

$$E\left(e^{-(sW(0,\omega(y)))} I(\varphi(\omega(y)) = j) \mid \varphi(0) = i\right)$$

which is the LST of the distribution of the total shift in $Y(\cdot)$,

accumulated at the time $\omega(y)$

when the *in-out fluid of the process* $X(\cdot)$ first reaches level y,

and $\varphi(\omega(y)) = j$ given $\varphi(0) = i$.

Projection from $\mathbf{Z}(\underline{\mathbf{s}})$ to $\mathbf{W}(s)$

Let

$$\begin{aligned} \mathbf{R}_{+} &= \mathbf{R}_{+} \\ \mathbf{R}_{-} &= \mathbf{R}_{-} \\ \mathbf{R}_{0} &= \mathbf{R}_{0}. \end{aligned}$$

Then the total reward is the total shift in the second (reward) fluid,

$$W(z,t) = Y(t) - Y(z).$$

Fluid generator $W(s)$ (most general projection)

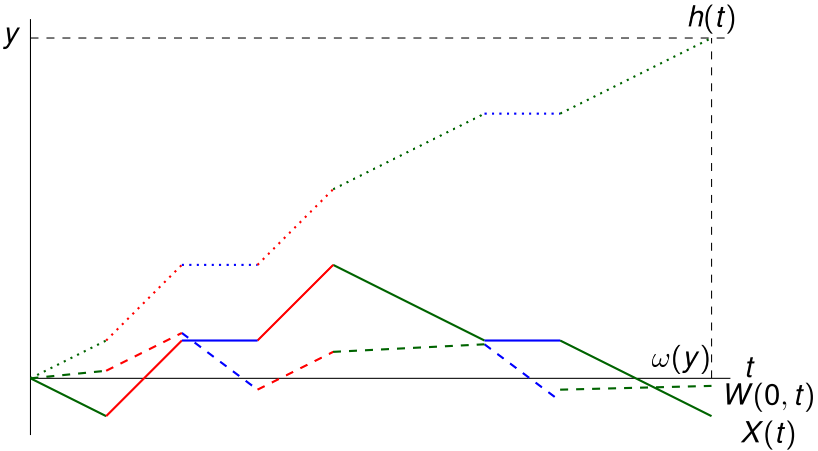
The i, j th entry of the LST of the distribution is $[e^{\mathbf{W}(s)y}]_{ij}$, where

$$\mathbf{W}(s) = \begin{bmatrix} \mathbf{W}_{++}(s) & \mathbf{W}_{+-}(s) \\ \mathbf{W}_{-+}(s) & \mathbf{W}_{--}(s) \end{bmatrix},$$

with

$$\begin{aligned}
\mathbf{W}_{++}(s) &= \mathbf{C}_{+}^{-1}[(\mathbf{T}_{++} - s\mathbf{R}_{+}) - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{R}_{0})^{-1}\mathbf{T}_{0+}] \\
\mathbf{W}_{--}(s) &= \mathbf{C}_{-}^{-1}[(\mathbf{T}_{--} - s\mathbf{R}_{-}) - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{R}_{0})^{-1}\mathbf{T}_{0-}] \\
\mathbf{W}_{+-}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{R}_{0})^{-1}\mathbf{T}_{0-}] \\
\mathbf{W}_{-+}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{R}_{0})^{-1}\mathbf{T}_{0+}].
\end{aligned}$$

Interpretation of $\mathbf{W}(s)$



Interpretation of $\mathbf{Z}^{+}(s)$

Define total upward shift

$$h_{+}(t) = \int_{u=0}^{t} |c_{\varphi(u)}| I(\varphi(u) \in \mathcal{S}_{+}) du.$$

We want to know

$$E\left(e^{-(sh_{+}(t))} I(\varphi(\omega(y)) = j) \mid \varphi(0) = i\right)$$

is the LST of the distribution of

the total upward shift in $X(t)$

accumulated at time $\omega(y)$, and $\varphi(\omega(y)) = j$,

given $\varphi(0) = i$.

Projection to track the total upward shift

Let

$$\begin{aligned} \mathbf{R}_{+} &= \mathbf{C}_{+} \\ \mathbf{R}_{-} &= \mathbf{0} \\ \mathbf{R}_{0} &= \mathbf{0}, \end{aligned}$$

so that the total reward is the total upward shift $h_{+}(t)$ in $X(t)$

$$W(0,t) = h_{+}(t).$$

$\mathbf{Z}^{+}(s)$

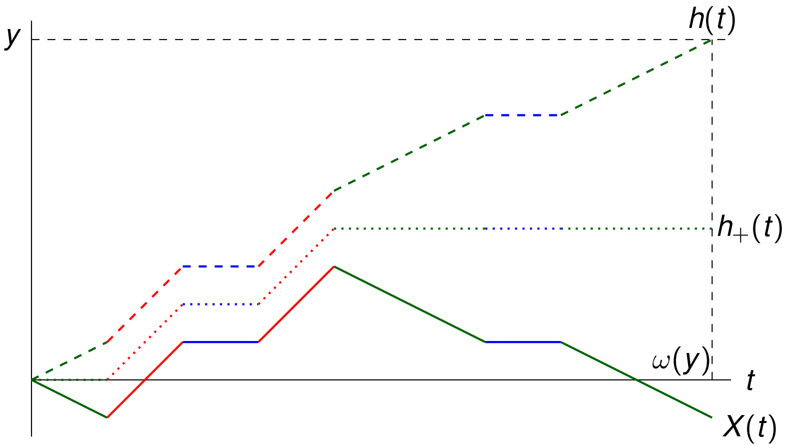
The i, j th entry of the LST of the distribution is $[e^{\mathbf{Z}^{+}(s)y}]_{ij}$ where,

$$\mathbf{Z}^{+}(s) = \begin{bmatrix} \mathbf{Z}^{+}_{++}(s) & \mathbf{Z}^{+}_{+-}(s) \\ \mathbf{Z}^{+}_{-+}(s) & \mathbf{Z}^{+}_{--}(s) \end{bmatrix}$$

where

$$\begin{aligned}
\mathbf{Z}^{+}_{++}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{++} - s\mathbf{C}_{+} - \mathbf{T}_{+0}(\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}] \\
\mathbf{Z}^{+}_{--}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{--} - \mathbf{T}_{-0}(\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Z}^{+}_{+-}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Z}^{+}_{-+}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}].
\end{aligned}$$

Interpretation of $\mathbf{Z}^{+}(s)$



Density $\mathbf{f}_y(x)$

For $0 \leq x \leq y$, let $f_y(x)_{ij}$ be the inverse of the LST $[e^{\mathbf{Z}^+(s)y}]_{ij}$

so that

$$f_y(x)_{ij} \quad = \quad \frac{d}{dx} P(h_+(\omega(y)) \leq x, \varphi(\omega(y)) = j |\, X(0) = 0, \varphi(0) = i)$$

is the probability density that the **total upward shift** in $X(\cdot)$

accumulated at time $\omega(y)$ is $h_+(\omega(y)) = x$

and $\varphi(\omega(y)) = j$, given $\varphi(0) = i$.

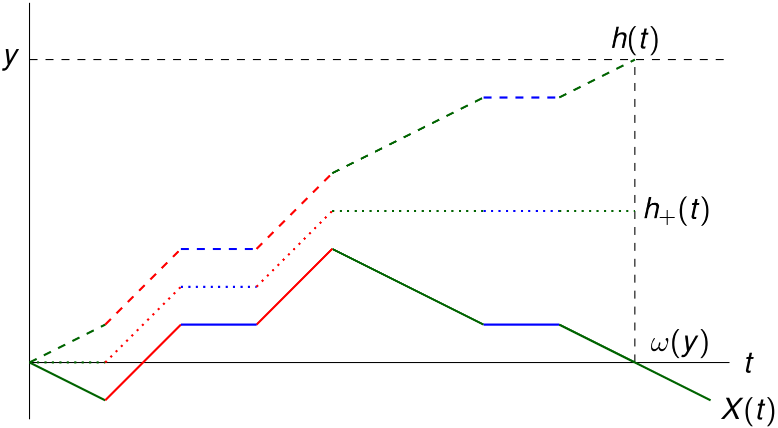
Consider $\mathbf{f}_y(y/2)$

$$f_y(y/2)_{ij}$$

is the probability density that the total upward shift in $X(\cdot)$ accumulated at time $\omega(y)$ is $h_+(\omega(y)) = y/2$ and $\varphi(\omega(y)) = j$, given $\varphi(0) = i$.

Total upward shift = Total downward shift.

Consider $\mathbf{f}_y(y/2)$



Define $\mathbf{M}$

Integrate $f_y(y/2)_{ij}$ over all possible y,

$$M_{ij} = \int_{y=0}^{\infty} f_y(y/2)_{ij} dy$$

which we interpret as

the expected number of visits to state $(j, 0)$,

given that the process starts in state $(i, 0)$,

for all $i, j \in \mathcal{S}_+ \cup \mathcal{S}_-$.

Matrix $\mathbf{M}$

Theorem (3)

We have

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{++} & \mathbf{M}_{+-} \\ \mathbf{M}_{-+} & \mathbf{M}_{--} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}\mathbf{M}_{-+} & (\mathbf{I}-\mathbf{\Psi}\mathbf{\Xi})^{-1}\mathbf{\Psi} \\ \mathbf{\Xi}(\mathbf{I}-\mathbf{\Psi}\mathbf{\Xi})^{-1} & \mathbf{\Xi}\mathbf{M}_{+-} \end{bmatrix}.$$

where $\mathbf{\Psi}, \mathbf{\Xi}$ are respective minimum nonnegative solutions to

$$\mathbf{Q}_{+-} + \mathbf{Q}_{++}\mathbf{\Psi} + \mathbf{\Psi}\mathbf{Q}_{--} + \mathbf{\Psi}\mathbf{Q}_{-+}\mathbf{\Psi} = \mathbf{0},$$

$$\mathbf{Q}_{-+} + \mathbf{Q}_{--}\mathbf{\Xi} + \mathbf{\Xi}\mathbf{Q}_{++} + \mathbf{\Xi}\mathbf{Q}_{+-}\mathbf{\Xi} = \mathbf{0}.$$

Corollary (1)

$\mathbf{\Psi}$ *is a solution to the Riccati equation*

$$\mathbf{M}_{+-} = \mathbf{\Psi} + \mathbf{\Psi}\mathbf{M}_{-+}\mathbf{\Psi}.$$

Corollary (2)

$\mathbf{\Psi}$ *can be explicitly written as*

$$\mathbf{\Psi} = \mathbf{M}_{+-}(\mathbf{I} + \mathbf{M}_{--})^{-1}.$$

Work in progress

- Numerical examples.

- Algorithm for computing $\mathbf{M}$ efficiently.

Conclusion

- Constructed generalised reward generator $\mathbf{Z}(\underline{\mathbf{s}})$
- considered various projections
- concentrated on $\mathbf{Z}^{+}(s)$
  - inverted its corresponding LST
  - integrated over all y to get $\mathbf{M}$
  - and created an explicit equation for $\boldsymbol{\Psi}$.

Thank you for listening

[1] J. Abate and W. Whitt.
Numerical inversion of Laplace transforms of probability distributions.
ORSA Journal on Computing, 7(1):36–36, 1995.

[2] S. Ahn and V. Ramaswami.
Fluid flow models and queues — a connection by stochastic coupling.
Stochastic Models, 19(3):325–348, 2003.

[3] S. Ahn and V. Ramaswami.
Transient analysis of fluid flow models via stochastic coupling to a queue.
Stochastic Models, 20(1):71–101, 2004.

[4] S. Ahn and V. Ramaswami.
Efficient algorithms for transient analysis of stochastic fluid flow models.
Journal of Applied Probability, 42(2):531–549, 2005.

[5] S. Asmussen.
Stationary distributions for fluid flow models with or without Brownian noise.
Stochastic Models, 11(1):21–49, 1995.

[6] A. Badescu, L. Breuer, A. da Silva Soares, G. Latouche, M. A. Remiche, and D. Stanford.
Risk processes analyzed as fluid queues.
Scandinavian Actuarial Journal, 2005(2):127–141, 2005.

[7] N. Bean, M. O'Reilly, and P. Taylor.
Hitting probabilities and hitting times for stochastic fluid flows: The bounded model.
Probability in the Engineering and Informational Sciences, 23(01):121–147, 2009.

[8] N. G. Bean and M. M. O'Reilly.
Stochastic two-dimensional fluid model.
Stochastic Models, 29(1):31–63, 2013.

[9] N. G. Bean, M. M. O'Reilly, and J. E. Sargison.
A stochastic fluid flow model of the operation and maintenance of power generation systems.
IEEE Transactions on Power Systems, 25(3):1361–1374, 2010.

[10] N. G. Bean, M. M. O'Reilly, and P. G. Taylor.
Algorithms for return probabilities for stochastic fluid flows.
Stochastic Models, 21(1):149–184, 2005.

[11] N. G. Bean, M. M. O'Reilly, and P. G. Taylor.
Hitting probabilities and hitting times for stochastic fluid flows.
Stochastic processes and their applications, 115(9):1530–1556, 2005.

[12] N. G. Bean, M. M. O'Reilly, and P. G. Taylor.
Algorithms for the Laplace–Stieltjes transforms of first return times for stochastic fluid flows.
Methodology and Computing in Applied Probability, 10(3):381–408, 2008.

[13] A. da Silva Soares.
Fluid Queues: Building Upon the Analogy with QBD Processes.
PhD thesis, Université Libre de Bruxelles, 2005.

[14] F. Guillemin and B. Sericola.
Volume and duration of losses in finite buffer fluid queues.
Journal of Applied Probability, 52(3):826–840, 2015.

[15] M. M. O'Reilly.
Multi-stage stochastic fluid models for congestion control.
European Journal of Operations Research, 238:514–526, 2014.

[16] M. M. O'Reilly and Z. Palmowski.
Loss rates for stochastic fluid models.
Performance Evaluation, 70(9):593–606, 2013.

[17] V. Ramaswami.
Matrix analytic methods for stochastic fluid flows.
In *ITC16: International Teletraffic Congress*, pages 1019–1030, 1999.

[18] V. Ramaswami.
Passage times in fluid models with application to risk processes.
Methodology and Computing in Applied Probability, 8(4):497–515, 2006.

Generators derived from $\mathbf{Z}(\underline{\mathbf{s}})$

$$\begin{aligned}
\mathbf{J}_1(\underline{\mathbf{s}}) &= \mathbf{Z}_{--}(\underline{\mathbf{s}}) + \mathbf{Z}_{-+}(\underline{\mathbf{s}})\boldsymbol{\Psi}(\underline{\mathbf{s}}) \\
\mathbf{J}_2(\underline{\mathbf{s}}) &= \mathbf{Z}_{++}(\underline{\mathbf{s}}) + \mathbf{Z}_{+-}(\underline{\mathbf{s}})\boldsymbol{\Xi}(\underline{\mathbf{s}})
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}_3(\underline{\mathbf{s}}) &= \mathbf{Z}_{++}(\underline{\mathbf{s}}) + \boldsymbol{\Psi}(\underline{\mathbf{s}})\mathbf{Z}_{-+}(\underline{\mathbf{s}}) \\
\mathbf{J}_4(\underline{\mathbf{s}}) &= \mathbf{Z}_{--}(\underline{\mathbf{s}}) + \boldsymbol{\Xi}(\underline{\mathbf{s}})\mathbf{Z}_{+-}(\underline{\mathbf{s}})
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}_5(\underline{\mathbf{s}}) &= \mathbf{Z}_{++}(\underline{\mathbf{s}}) + \mathbf{Z}_{+-}(\underline{\mathbf{s}})\mathbf{H}^{(b,b)}(\underline{\mathbf{0}}) \\
\mathbf{J}_6(\underline{\mathbf{s}}) &= \mathbf{Z}_{--}(\underline{\mathbf{s}}) + \mathbf{Z}_{-+}(\underline{\mathbf{s}})\mathbf{G}^{(0,b)}(\underline{\mathbf{0}})
\end{aligned}$$