Analysis of tandem fluid queues

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Outline

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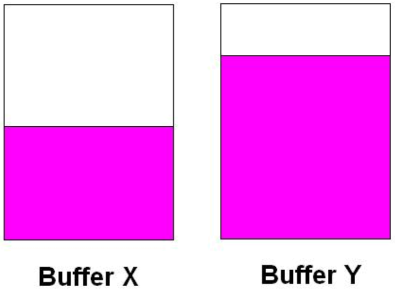
Model: two fluid queues driven by $\varphi(t)$

- CTMC $\varphi(t)$ with finite state space S, generator $\mathbf{T}$
- Two fluid queues, contents $X(t)$ and $Y(t)$, both $\in [0, \infty)$



First queue $X(t)$ driven by $\varphi(t)$

- $(\varphi(t), X(t))$ is standard fluid queue
- Fluid rates in $\mathbf{R} = diag(r_i)_{i\in\mathcal{S}}$

$$\frac{d}{dt}X(t) = r_{\varphi(t)} \qquad \text{when } X(t) > 0,$$
$$\frac{d}{dt}X(t) = \max(0, r_{\varphi(t)}) \qquad \text{when } X(t) = 0.$$

- $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_\bigcirc$, e.g. $\mathcal{S}_+ = \{i \in \mathcal{S} : r_i > 0\}$ (upstates, downstates, zero-states)
- also: $\mathcal{S}_\ominus = \mathcal{S}_- \cup \mathcal{S}_\bigcirc$ ("zero-states at $X(t) = 0$")
- after ordering,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{++} & \mathbf{T}_{+-} & \mathbf{T}_{+\bigcirc} \\ \mathbf{T}_{-+} & \mathbf{T}_{--} & \mathbf{T}_{-\bigcirc} \\ \mathbf{T}_{\bigcirc+} & \mathbf{T}_{\bigcirc-} & \mathbf{T}_{\bigcirc\bigcirc} \end{bmatrix}.$$

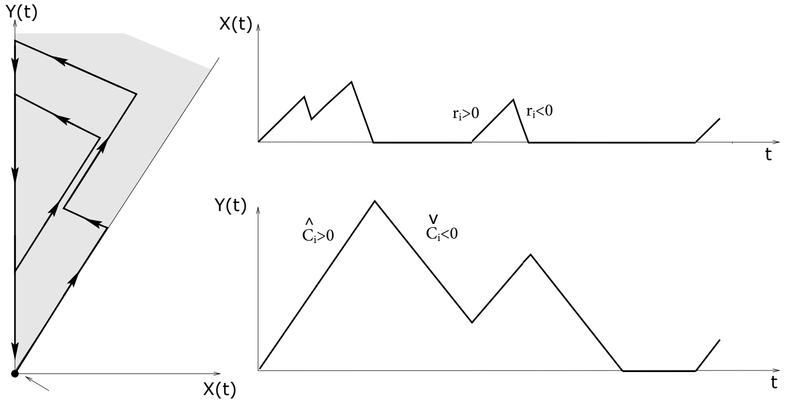
Second queue $Y(t)$ driven by $(\varphi(t), X(t))$

- $Y(t)$ increases when $X(t) > 0$, at rate $\widehat{c}_i > 0$
- $Y(t)$ decreases when $X(t) = 0$, at rate $\breve{c}_i < 0$ (unless $Y(t) = 0$)
- So

$$\begin{aligned}
\frac{d}{dt}Y(t) &= \widehat{c}_{\varphi(t)} > 0 && \text{when } X(t) > 0, \\
\frac{d}{dt}Y(t) &= \breve{c}_{\varphi(t)} < 0 && \text{when } X(t) = 0,\ Y(t) > 0, \\
\frac{d}{dt}Y(t) &= \widehat{c}_{\varphi(t)} \cdot 1\{\varphi(t) \in \mathcal{S}_+\} && \text{when } X(t) = 0,\ Y(t) = 0.
\end{aligned}$$

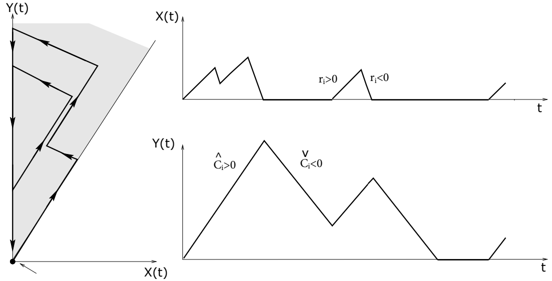
- $\widehat{\mathbf{C}} = diag(\widehat{c}_i)_{i \in \mathcal{S}}$ and $\breve{\mathbf{C}} = diag(\breve{c}_i)_{i \in \mathcal{S}}$

Special case: $\mathcal{S}_\circ = \emptyset$, $|\mathcal{S}_+| = |\mathcal{S}_-| = 1$, $\quad \widehat{\mathbf{C}} = -\check{\mathbf{C}} = \mathbf{I}$



[Kroese and Scheinhardt. Joint Distributions for Interacting Fluid Queues, *Queueing Systems*, 2001]

Qualitative behaviour

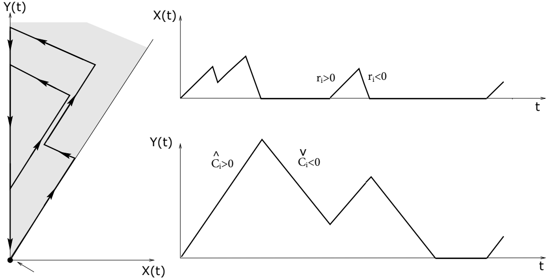


Assuming stability (see paper) process $(\varphi(t), X(t), Y(t))$ alternates between:

(i) periods on $x = 0$

(ii) periods on $x > 0$

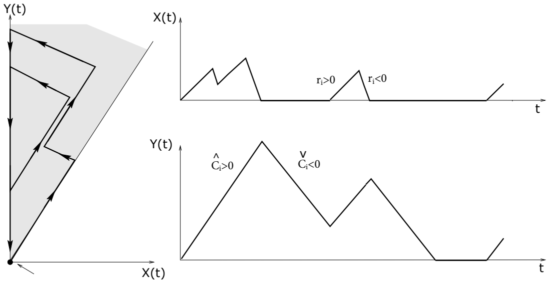
Qualitative behaviour (i) on $x = 0$



(i) periods on $x = 0$

- $Y(t)$ decreasing, unless at $x = 0, y = 0$
- $\varphi(t)$ in $\mathcal{S}_\ominus$
- starts at $x = 0, y > 0$, with $\varphi(t)$ in $\mathcal{S}_-$
- ends at $x = 0, y \geq 0$, with $\varphi(t)$ jumping from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$

Qualitative behaviour (ii) on $x > 0$



(ii) periods on $x > 0$

- $Y(t)$ increasing (while $X(t)$ can either increase or decrease)
- $\varphi(t)$ in $\mathcal{S}$ (any phase)
- starts at $x = 0$, $y \geq 0$, with $\varphi(t) \in \mathcal{S}_+$
- ends at $x = 0$, $y > 0$, with $\varphi(t) \in \mathcal{S}_-$

Stationary distribution

has following form (all *vectors* with $|\mathcal{S}|$ components):

(i)
- 1-dimensional densities $\boldsymbol{\pi}(0, y)$
  at $x = 0, y > 0$
- point masses $\mathbf{p}(0, 0)$
  at $(0, 0)$

(ii)
- 2-dimensional densities $\boldsymbol{\pi}(x, y)$
  on $\{(x, y) : x > 0, y > x \cdot \min_{i \in \mathcal{S}_+} \{\widehat{c}_i / r_i\}\}$
- 1-dimensional density $\pi^i(x, x\widehat{c}_i / r_i)$
  on line $y = x\widehat{c}_i / r_i$, $i \in \mathcal{S}_+$

Outline

Approach

Several steps:

- Introduce embedded discrete-time process J_k
- Find its stationary distribution ξ_y
- Take a deep breath...
- Express $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y, using down-shift in Y
- Normalise based on knowledge of $(\varphi(t), X(t))$
- Express $\pi(x, y)$ in $\pi(0, y)$ and $\mathbf{p}(0, 0)$, using up-shift in Y
- Express $\pi^i(x, x\widehat{c}_i/r_i)$ in $\mathbf{p}(0, 0)$

Mostly as LST's (but not always)

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Mostly as LST's (but not always)

Intermezzo (i) on down-shift: $\check{\mathbf{Q}}_{\ominus\ominus}$ and $\check{\mathbf{Q}}_{\ominus+}$

Define generator matrix

$$\check{\mathbf{Q}}_{\ominus\ominus} = (|\check{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus\ominus},$$

then for $i, j \in \mathcal{S}_{\ominus}$, and $z > 0$,

$$[e^{\check{\mathbf{Q}}_{\ominus\ominus}z}]_{ij} \;=\; P(\varphi(t_z) = j, \varphi(u) \in \mathcal{S}_{\ominus}, 0 \le u \le t_z \mid \varphi(0) = i, X(0) = 0)$$

Also,

$$\check{\mathbf{Q}}_{\ominus+} = (|\check{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus+},$$

is a matrix of transition rates (w.r.t. level) to phases in $\mathcal{S}_+$
(for times at which X and Y start increasing)

[Bean, O'Reilly and Taylor. Hitting probabilities and hitting times for stochastic fluid flows, *SPA* 2005]

Intermezzo (ii) on up-shift: $\widehat{\mathbf{Q}}(s)$ and $\widehat{\mathbf{\Psi}}(s)$

Let $\theta = \inf\{t > 0 : X(t) = 0\}$ and $U(t) = \int_{u=0}^{t} \widehat{c}_{\varphi(u)} du$, then $U(\theta)$ is total up-shift in Y during Busy Period of X

Its $|\mathcal{S}_+| \times |\mathcal{S}_-|$ density matrix $\widehat{\psi}(z)$ is given via LST

$$\widehat{\mathbf{\Psi}}(s) = \int_{z=0}^{\infty} e^{-sz} \widehat{\psi}(z) dz,$$

as

$$[\widehat{\mathbf{\Psi}}(s)]_{ij} = E(e^{-sU(\theta)} 1\{\varphi(\theta) = j\} \mid \varphi(0) = i, X(0) = 0),$$

[Bean and O'Reilly. A stochastic two-dimensional fluid model, *Stochastic Models*, 2013]

Intermezzo (ii) on up-shift: $\widehat{\mathbf{Q}}(s)$ and $\widehat{\mathbf{\Psi}}(s)$

To find $\widehat{\mathbf{\Psi}}(s)$ define Key generator matrix $\widehat{\mathbf{Q}}(s)$, as

$$\widehat{\mathbf{Q}}(s) = \begin{bmatrix} \widehat{\mathbf{Q}}(s)_{++} & \widehat{\mathbf{Q}}(s)_{+-} \\ \widehat{\mathbf{Q}}(s)_{-+} & \widehat{\mathbf{Q}}(s)_{--} \end{bmatrix}$$

$$\widehat{\mathbf{Q}}(s)_{++} = (\mathbf{R}_+)^{-1} \left(\mathbf{T}_{++} - s\widehat{\mathbf{C}}_+ - \mathbf{T}_{+\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc+} \right)$$

$$\widehat{\mathbf{Q}}(s)_{+-} = (\mathbf{R}_+)^{-1} \left(\mathbf{T}_{+-} - \mathbf{T}_{+\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc-} \right)$$

$$\widehat{\mathbf{Q}}(s)_{-+} = (|\mathbf{R}_-|)^{-1} \left(\mathbf{T}_{-+} - \mathbf{T}_{-\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc+} \right)$$

$$\widehat{\mathbf{Q}}(s)_{--} = (|\mathbf{R}_-|)^{-1} \left(\mathbf{T}_{--} - s\widehat{\mathbf{C}}_- - \mathbf{T}_{-\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc-} \right)$$

Then $\widehat{\mathbf{\Psi}}(s)$ is minimum nonnegative solution of Riccati eq.

$$\widehat{\mathbf{Q}}(s)_{+-} + \widehat{\mathbf{Q}}(s)_{++}\widehat{\mathbf{\Psi}}(s) + \widehat{\mathbf{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{--} + \widehat{\mathbf{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}\widehat{\mathbf{\Psi}}(s) = \mathbf{O},$$

[Bean and O'Reilly. A stochastic two-dimensional fluid model, *Stochastic Models*, 2013]

Back on track... Embedded process J_k

Let $J_k = (\varphi(\theta_k), Y(\theta_k))$, with state space $\mathcal{S}_- \times (0, \infty)$, where θ_k is k-th time that $(\varphi(t), X(t), Y(t))$ hits $x = 0$

Lemma

The transition kernel of J_k is given by

$$\begin{aligned}\mathbf{P}_{z,y} &= \int_{u=[z-y]^+}^{z} \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} u} \check{\mathbf{Q}}_{\ominus +} \widehat{\psi}(y - z + u) du \\ &+ \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (-\check{\mathbf{Q}}_{\ominus\ominus})^{-1} \check{\mathbf{Q}}_{\ominus +} \widehat{\psi}(y).\end{aligned}$$

where $[x]^+$ denotes $\max(0, x)$, and $\begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix}$ is a $|\mathcal{S}_-| \times |\mathcal{S}_\ominus|$ matrix.

Embedded process J_k

Proof. Based on Lindley-type recursion,

$$Y(\theta_{k+1}) = [Y(\theta_k) - D_k]^+ + U_k, \tag{1}$$

where

$$D_k = \int_{u=\theta_k}^{\tau_k} |\check{c}_{\varphi(u)}| du, \quad U_k = \int_{u=\tau_k}^{\theta_{k+1}} \widehat{c}_{\varphi(u)} du$$

So (i) $Y(t)$ first has down-shift $-D$, as long as $\varphi(t) \in \mathcal{S}_\ominus$
(ii) after jump $\mathcal{S}_\ominus \to \mathcal{S}_+$, $Y(t)$ has up-shift U, during busy period of X.
Then use previous knowledge; note that J_k moves from (i, z) to (j, y) without or with returning to 0 during (θ_k, θ_{k+1}). □

Embedded process J_k

Corollary

The Laplace-Stieltjes transform of $\mathbf{P}_{z,y}$ *w.r.t.* y *is given by*

$$\begin{aligned}\mathbf{P}_{z,\cdot}(s) &= \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{-sz} \left(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}\right)^{-1} \left(e^{\left(\check{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I}\right)z} - \mathbf{I}\right) \\ &\quad \times \check{\mathbf{Q}}_{\ominus+} \widehat{\mathbf{\Psi}}(s) \\ &\quad + \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (-\check{\mathbf{Q}}_{\ominus\ominus})^{-1} \check{\mathbf{Q}}_{\ominus+} \widehat{\mathbf{\Psi}}(s).\end{aligned}$$

Proof. Using lemma, or based on (1) directly □

Embedded process J_k – stationary distribution ξ_y

Stationary distribution of J_k is given by row vector $\boldsymbol{\xi}_z = [\xi_{i,z}]_{i \in \mathcal{S}_-}$ of densities, satisfying

$$\begin{cases} \int_{z=0}^{\infty} \boldsymbol{\xi}_z \mathbf{P}_{z,y} dz & = & \boldsymbol{\xi}_y \\ \int_{y=0}^{\infty} \boldsymbol{\xi}_y dy \mathbf{1} & = & 1, \end{cases}$$

Will be solved numerically.

Next step (after deep breath):

Express stationary distribution of $(\varphi(t), X(t), Y(t))$ at level $x = 0$ in terms of ξ_z.

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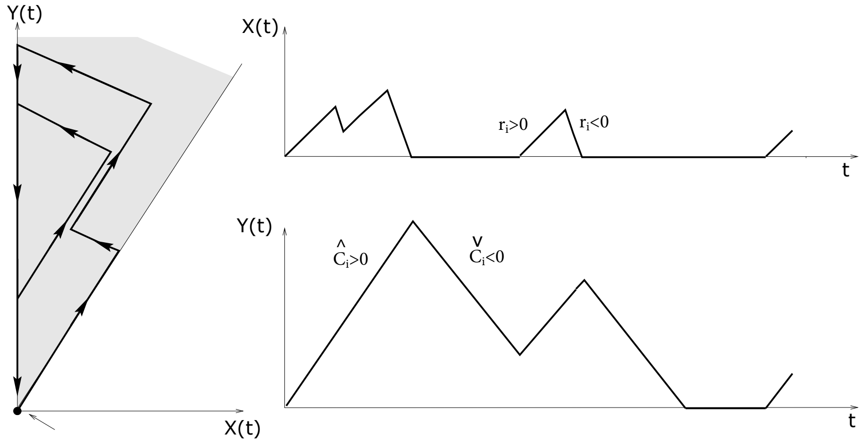
Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y



Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y

Lemma

We have $\pi(0, y) = \begin{bmatrix} \mathbf{0} & \pi(0, y)_\ominus \end{bmatrix}$, *where*

$$\pi(0, y)_\ominus = \alpha \int_{z=y}^{\infty} \begin{bmatrix} \xi_z & \mathbf{0} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus}(z-y)} (|\check{\mathbf{C}}_\ominus|)^{-1} dz,$$

and $\mathbf{p}(0, 0) = \begin{bmatrix} \mathbf{0} & \mathbf{p}(0, 0)_\ominus \end{bmatrix}$, *where*

$$\mathbf{p}(0, 0)_\ominus = \alpha \int_{z=0}^{\infty} \begin{bmatrix} \xi_z & \mathbf{0} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1}.$$

Here, α is a normalization constant
In fact α is the total rate of hitting $x = 0$.

Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y

Proof. Consider "cycles" defined by hitting times of $x = 0$, and condition on where previous hit took place.

[Latouche and Taylor. A stochastic fluid model for an ad hoc mobile network, *Queueing Systems*, 2009]

□

Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y

LST of density part: let

$$\pi(0,\cdot)(s) = \int_{z=0}^{\infty} e^{-sy}\pi(0,y)dy$$

Corollary

We have $\pi(0,\cdot)(s) = \left[\begin{array}{cc} \mathbf{0} & \pi(0,\cdot)(s)_{\ominus} \end{array}\right]$, *where*

$$\begin{aligned}\pi(0,\cdot)(s)_{\ominus} &= \alpha \int_{z=0}^{\infty} \left[\begin{array}{cc} \xi_z & \mathbf{0} \end{array}\right] e^{\check{\mathbf{Q}}_{\ominus\ominus}z} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\ &\quad \times \left(\mathbf{I} - e^{-(\check{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})z}\right) (|\check{\mathbf{C}}_{\ominus}|)^{-1} dz.\end{aligned}$$

Proof. Straightforward. □

Normalise, based on 1-dim fluid queue $(\varphi(t), X(t))$

Lemma

The normalisation constant α *is given by*

$$\alpha = \Big\{ \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} \Big(\mathbf{1} + \mathbf{T}_{\ominus+}\mathbf{K}^{-1} \begin{bmatrix} (\mathbf{R}_{+})^{-1} & \boldsymbol{\Psi}(|\mathbf{R}_{-}|)^{-1} \end{bmatrix} \times \Big(\mathbf{1} + \mathbf{T}_{\pm\bigcirc}(-\mathbf{T}_{\bigcirc\bigcirc})^{-1}\mathbf{1} \Big) \Big) \Big\}^{-1},$$

where, $\boldsymbol{\xi} = \int_{z=0}^{\infty} \boldsymbol{\xi}_z dz$, $\boldsymbol{\Psi} = \widehat{\boldsymbol{\Psi}}(s)|_{s=0}$ *and* $\mathbf{K} = \widehat{\mathbf{K}}(s)|_{s=0}$ *with*

$$\widehat{\mathbf{K}}(s) = \widehat{\mathbf{Q}}(s)_{++} + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}.$$

Normalise, based on 1-dim fluid queue $(\varphi(t), X(t))$

Proof. Integrating $\boldsymbol{\pi}(0, y)$ and adding $\mathbf{p}(0, 0)$ yields the probability mass vector of $\varphi(t)$ at $x = 0$, which is also known from 1-dim fluid queue:

$$\left[\begin{array}{cc} \mathbf{p}_{-} & \mathbf{p}_{\circ} \end{array}\right] = \alpha \left[\begin{array}{cc} \boldsymbol{\xi} & \mathbf{0} \end{array}\right] \left(-\mathbf{T}_{\ominus\ominus}\right)^{-1}$$

Similarly, we have expression for density $\boldsymbol{\pi}(x)$ at $x > 0$.

Now solve α from

$$\mathbf{p1} + \int_{x=0}^{\infty} \boldsymbol{\pi}(x) dx \mathbf{1} = 1$$

□

Expressing $\pi(x,y)$ in $\pi(0,y)$ and $\mathbf{p}(0,0)$

Lemma

We have

$$\pi(x,\cdot)(s) = \begin{bmatrix} \pi(x,\cdot)(s)_+ & \pi(x,\cdot)(s)_- & \pi(x,\cdot)(s)_\bigcirc \end{bmatrix}$$

with

$$\begin{bmatrix} \pi(x,\cdot)(s)_+ & \pi(x,\cdot)(s)_- \end{bmatrix} = (\pi(0,\cdot)(s)_\ominus + \mathbf{p}(0,0)_\ominus) \\ \times \mathbf{T}_{\ominus +} e^{\widehat{\mathbf{K}}(s)x} \times \begin{bmatrix} (\mathbf{R}_+)^{-1} & \widehat{\mathbf{\Psi}}(s)(|\mathbf{R}_-|)^{-1} \end{bmatrix},$$

and

$$\pi(x,\cdot)(s)_\bigcirc = \begin{bmatrix} \pi(x,\cdot)(s)_+ & \pi(x,\cdot)(s)_- \end{bmatrix} \\ \times \mathbf{T}_{\pm\bigcirc}(s\widehat{\mathbf{C}}_\bigcirc - \mathbf{T}_{\bigcirc\bigcirc})^{-1}.$$

Expressing $\pi(x, y)$ in $\pi(0, y)$ and $\mathbf{p}(0, 0)$

Let $\pi(\cdot,\cdot)(v,s) = \int_{x=0}^{\infty} e^{-vx}\pi(x,\cdot)(s)dx$.

Corollary

We have

$$\pi(\cdot,\cdot)(v,s) = \left[\begin{array}{ccc} \pi(\cdot,\cdot)(v,s)_{+} & \pi(\cdot,\cdot)(v,s)_{-} & \pi(\cdot,\cdot)(s)_{\bigcirc} \end{array}\right]$$

with

$$\left[\begin{array}{cc} \pi(\cdot,\cdot)(v,s)_{+} & \pi(\cdot,\cdot)(v,s)_{-} \end{array}\right] = (\pi(0,\cdot)(s)_{\ominus} + \mathbf{p}(0,0)_{\ominus}) \times \left[\begin{array}{c} \mathbf{T}_{-+} \\ \mathbf{T}_{\bigcirc+} \end{array}\right] (-\widehat{\mathbf{K}}(s) + v\mathbf{I})^{-1} \left[\begin{array}{cc} (\mathbf{R}_{+})^{-1} & \widehat{\mathbf{\Psi}}(s)(|\mathbf{R}_{-}|)^{-1} \end{array}\right]$$

and

$$\pi(\cdot,\cdot)(s)_{\bigcirc} = \left[\begin{array}{cc} \pi(\cdot,\cdot)(s)_{+} & \pi(\cdot,\cdot)(s)_{-} \end{array}\right] \mathbf{T}_{\pm\bigcirc} \times (s\widehat{\mathbf{C}}_{\bigcirc} - \mathbf{T}_{\bigcirc\bigcirc})^{-1}.$$

Expressing $\pi^i(x, x\widehat{c}_i/r_i)$ in $\mathbf{p}(0,0)$

Lemma

For all $i \in \mathcal{S}_+$,

$$\pi^i(x, x\widehat{c}_i/r_i) = \sum_{j \in \mathcal{S}_\ominus} \mathbf{p}_j(0,0) T_{ji} \exp(-(T_{ii}/r_i)x)/r_i$$

Proof. Consider “cycle” starting when $(0,0)$ is left, and consider expected number of visits to $(i, x, x\widehat{c}_i/r_i)$ before return to $(0,0)$. □

Main result

Theorem

Stationary distribution of $(\varphi(t), X(t), Y(t))$ *is found, as mixture of densities and LSTs.*

Numerical scheme

- Discretize the DTMC J_k and truncate its state space:

$$\tilde{\mathbf{P}}_{\ell m} = \int_{y=m\Delta u}^{(m+1)\Delta u} \mathbf{P}_{\ell \Delta u, y} dy, \quad \ell, m = 0, 1, 2, \ldots L$$

- Normalize this to obtain $\mathbf{P}_{\ell m}$ with $\sum_{m=0}^{L} \mathbf{P}_{\ell m} \mathbf{1} = \mathbf{1}$.
- Find $\bar{\xi}_\ell = [\bar{\xi}_{j;\ell}]_{j \in \mathcal{S}_-}$ by solving $\bar{\xi}\mathbf{P} = \bar{\xi}, \quad \bar{\xi}\mathbf{1} = \mathbf{1}$.
- Use this to approximate e.g.

$$\begin{aligned}\mathbf{p}(0,0)_{\ominus} &= \alpha \int_{z=0}^{\infty} \xi_z e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\ &\approx \alpha \sum_{\ell=0}^{L} \bar{\xi}_\ell e^{\check{\mathbf{Q}}_{\ominus\ominus} \ell \Delta u} (-\mathbf{T}_{\ominus\ominus})^{-1}.\end{aligned}$$

- Similar for $\pi(0, y)$ etc; invert using Abate and Whitt
- Work in progress

Numerical scheme

$$
\begin{array}{ccccccc}
\widehat{\boldsymbol{\Psi}}(s) & & & & & & \\
\downarrow & & & & & & \\
\mathbf{P}_{z,\cdot}(s) & & \xi(s) & & \pi(0,\cdot)(s) & \rightarrow & \pi(x,\cdot)(s) \\
\downarrow & & & & \uparrow & & \downarrow \\
\mathbf{P}_{z,y} & \rightarrow & \xi_z & \rightarrow & \pi(0,y) & & \pi(x,y)
\end{array}
$$

Conclusions and future work

- Stationary distribution found, as mixture of densities and LSTs (as opposed to closed form LST in special case)
- Finish numerical scheme
- Consider dual model