Lattice and Non-Lattice Markov Additive Models

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Markov Additive Processes

A *Markov additive process (MAP)* is a bivariate Markov process $(X(t), J(t))$ living on $\mathbb{R} \times \{1, \ldots, N\}$. The components X and J are referred to as the *level* and *phase* respectively.

For any time T and any phase i, conditional on $\{J(T) = i\}$, the process $(X(T + t) - X(T), J(T + t))$ is independent of $\mathcal{F}_T$ and has the law of $(X(t) - X(0), J(t))$ given $\{J(0) = i\}$.

So the increments of the level process are governed by the phase process $J(t)$, which evolves as a finite-state continuous-time Markov chain with some irreducible transition rate matrix Q.

Markov Additive Processes

A general MAP can be thought of as a *Markov modulated Lévy process*. We shall deal with the *spectrally-positive* case where there are no negative jumps.

In this case, as long as $J(t) = i$, $X(t)$ evolves as a Lévy process $X^{(i)}(t)$ with Lévy exponent,

$$
\begin{aligned}
\psi_i(\alpha) &\equiv \frac{1}{t}\log\left(\mathbb{E}[\exp(\alpha X^{(i)}(t))]\right) \\
&= \frac{1}{2}\sigma_i^2\alpha^2 + a_i\alpha + \int_0^\infty (e^{\alpha x} - 1 - \alpha x 1x < 1)\nu_i(\mathrm{d}x),
\end{aligned}
$$

defined at least for $\Re(\alpha) \leq 0$, and there are additional jumps distributed as U_{ij} when $J(t)$ switches from i to j.

Markov Additive Processes

The data that defines such a MAP is a matrix-valued function

$$F(\alpha) := Q \circ (\mathbb{E}[\exp(\alpha U_{ij})] + \mathrm{diag}(\psi_1(\alpha), \ldots, \psi_N(\alpha)),$$

where

- $\circ$ stands for entry-wise matrix multiplication,
- $\psi_i(\alpha) = \log \mathbb{E}[\exp(\alpha X^{(i)}(1))]$ is the Lévy exponent of $X^{(i)}(t)$, and
- $\alpha \leq 0$.

Then we have

$$\mathbb{E}[\exp^{\alpha X(t)}; J(t)] = \exp(tF(\alpha)).$$

Markov Additive Processes

An $M/G/1$ type matrix analytic model has $X(t) \in \mathbb{Z}$ and all the Lévy processes $X_i(t)$ are compound Poisson processes with jumps B_i and U_{ij} taking values in $\{-1, 0, 1, \ldots\}$.

In this case, it is more convenient to describe the process with a discrete generating function, defined at least for $|z| \leq 1, z \neq 0$ by

$$F(z) = z \sum_{m=-1}^{\infty} z^m A_m,$$

where the coefficient outside the sum ensures that F does not have a singularity at 0.

Then we have

$$\mathbb{E}[z^{X(t)}; J(t)] = \exp(tF(z)/z).$$

Markov Additive Processes

We can think of a matrix-analytic model as a *lattice* version of a MAP, with the changes in level restricted to be integers. Conversely, a general MAP can be considered to be *non-lattice*.

An M/G/1 type model is one-sided in the sense that it is skip-free to the left. It can be considered to be the lattice analogue of a spectrally-positive non-lattice model.

Our purpose is to look at these one-sided lattice and non-lattice MAPs side by side, interpret results that are standard in one tradition in the other and capture new perspectives.

The Scale Function

An object of interest in the study of one-sided scalar Lévy processes is the *scale function* W.

It relates

- the first hitting time $\tau_x := \inf\{t \geq 0 : X(t) = x\}$ on level x for $x \in \mathbb{R}$,
- the first hitting time $\tau_x^+ := \inf\{t > 0 : X(t) \geq x\}$ above level x for $x \in \mathbb{R}$,
- the local time $L(x,t) := \int_0^t 1(X(s) = x)\mathrm{d}s$ conditional on X starting in level 0: we write $H := \mathbb{E}L(0,\infty)$ which is nonsingular in the non-zero drift case, and
- the probability g_x that the process ever visits level $-x$.

The Scale Function

The scalar scale function W has the properties

- $\int_0^\infty e^{\alpha x} W(x) \mathrm{d}x = F(\alpha)^{-1}$.
- $W(x)$ is non zero for $x > 0$ and, for $a, b \geq 0$ with $a + b > 0$,

$$\mathrm{P}[\tau_{-a} < \tau_b^+] = \frac{W(b)}{W(a+b)},$$

- $W(x) = g_x \Theta(x)$, where

$$\Theta(x) := \mathbb{E} L(0, \tau_{-x}) \text{ for } x > 0,$$

the expected time that the process spends at level zero before it first visits level $-x$.

The Scale Function: Non-Lattice Case

Ivanovs and Palmovski (2012): There exists a continuous matrix-valued function $W(x), x \geq 0$ such that

- $\int_0^\infty e^{\alpha x} W(x) \mathrm{d}x = F(\alpha)^{-1}$,
- $W(x)$ is non-singular for $x > 0$ and

$$\mathrm{P}[\tau_{-a} < \tau_b^+, J(\tau_{-a})] = W(b)W(a+b)^{-1}$$

- $W(x) = e^{-Gx}\Theta(x),$ where $\Theta(x)$ is now a matrix and and G is the non-conservative transition matrix of the *ladder height continuous-time Markov chain* $Y(x) \equiv X(\tau_{-x})$.
- when the drift is nonzero, $\mathrm{P}[J_{\tau_x}] = e^{-Gx} - W(x)H^{-1}$.

The Scale Function: Lattice Case

What is the lattice version of this result?

Theorem

The usual M/G/1 matrix G is nonsingular if and only if A_{-1} is nonsingular. In this case, there exists a nonsingular matrix $W(m)$ with $W(m) = \mathbb{O}$ for $m \leq 0$, such that

- $\sum_{m=1}^{\infty} z^m W(m) = zF(z)^{-1}$,
- $P[\tau_{-l} < \tau_m^+, J_{\tau_{-l}}] = W(m)W(m+l)^{-1}$,
- $W(m) = G^{-m}\Theta(m)$ *where* $\Theta(m) = \mathbb{E}L(0, \tau_{-m})$,
- $P[J_{\tau_m}] = G^{-m} - W(m)H^{-1}, m \in \mathbb{Z}$ *in the nonzero-drift case.*

The Scale Function: Lattice Case

What if A_{-1} is singular?

Theorem

- *The matrix* $\Xi(m) \equiv \mathbb{E}L(0, \tau_m^+)$ *is nonsingular (even in the zero drift case),*
- $P[\tau_{-l} < \tau_m^+, J_{\tau_{-l}}] = \Xi(m)\hat{R}^l\Xi(m+l)^{-1}$ *with* $\hat{R}$ *the usual R-matrix for the level reversed GI/M/1-type Markov chain,*
- *in the QBD case,* $\Xi(m) = \sum_{\nu=0}^{m-1} G^\nu \left(-U\right)^{-1} R^\nu$ *with* R *and* U *the usual matrices.*
- *We are still working on the best way to derive* $\Xi(m)$ *in the general case and what the relationship to* $zF(z)^{-1}$ *might be.*