Matrix-analytic solution of second order Markov fluid models by using matrix-quadratic equations

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Outline

There are efficient numerical methods for regular (first order) Markov fluid models.
Can we use them for second order Markov fluid models?

Fluid models

Fluid models

- Fluid flows in to and out from an infinite buffer.
- Fluid flow is modulated by a background Markov chain.
  - first order: fluid level changes at a constant rate

$$\mathcal{Z}(t+\Delta) - \mathcal{Z}(t) = r_i \Delta \quad \text{in state } i.$$

  - second order: normal distributed fluid increment

$$\mathcal{Z}(t+\Delta) - \mathcal{Z}(t) = \mathcal{N}(r_i \Delta, \sigma_i^2 \Delta) \quad \text{in state } i.$$

First order fluid models

Nice special cases allow symbolic solutions (Anick,Mitra,Sondhi).

Numerical methods

- *Spectral decomposition based methods (Kulkarni)*
- *Ricatti equation based solutions (Ahn-Ramaswami, Soares-Latouche)*
- *Quadratic matrix equation based solution (Ramaswami)*

Second order fluid models

Numerical methods

- Spectral decomposition based methods (Karandikar-Kulkarni)
- Transformation to first order differential equation with larger state space (Kulkarni)
- Quadratic matrix equation based solution

Second order Markov Fluid model

Main characterization of the stochastic processes:

- infinite buffer with lower boundary at level 0.
- $\mathcal{Z}(t)$: fluid level process.
- fluid increment process is characterized by
  - $\mathcal{X}(t)$: modulating CTMC with state space $\mathcal{S} = \{1, \ldots, L\}$ and generator $\boldsymbol{Q}$,
  - fluid rates, r_j, $j \in \{1, \ldots, L\}$ described by diagonal matrix $\mathbf{R} = diag(r_1, \ldots, r_L)$.
  - variance parameter, σ_j, $j \in \{1, \ldots, L\}$ described by diagonal matrix $\mathbf{S} = diag(\sigma_1^2/2, \ldots, \sigma_L^2/2)$

Second order Markov Fluid model

Performance measures of interest:

- stationary fluid level distribution

$$f_i(x) = \lim_{t\to\infty} \frac{d}{dx} P(\mathcal{X}(t) < x, \mathcal{Z}(t) = i),$$

- stationary buffer empty probability

$$p_i = \lim_{t\to\infty} P(\mathcal{X}(t) = 0, \mathcal{Z}(t) = i).$$

Analytical description

$f(x)$ satisfies the following differential and boundary equations

$$\frac{d}{dx}f(x)\boldsymbol{R} - \frac{d^2}{dx^2}f(x)\boldsymbol{S} = f(x)\boldsymbol{Q}, \tag{1}$$

$$f(0)\boldsymbol{R} - f'(0)\boldsymbol{S} = p\boldsymbol{Q}, \tag{2}$$

where $f'(0) = \frac{d}{dx}f(x)|_{x=0}$.

State classification

The states in $\mathcal{S}$ are divided into

- first order states with $\sigma_i^2 = 0$ and
- second order states ($\sigma_i^2 > 0$)
  - with *reflecting* boundary.
  - with *absorbing* boundary.

Properties

- In first order states:

$$p_i > 0,\ \forall i : r_i < 0 \ \text{ and } \ p_i = 0,\ \forall i : r_i > 0.$$

- In second order
  - *reflecting* states:

$$p_i = 0,\ \forall i : \sigma_i^2 > 0,$$

  - *absorbing* states:

$$f_i(0) = 0,\ \forall i : \sigma_i^2 > 0.$$

State classification

The state space $\mathcal{S}$ is partitioned according to the sign of the rates and variances as follows:

- $\mathcal{S}^{+} = \{i : r_i > 0, \sigma_i^2 = 0\}$, $\mathcal{S}^{-} = \{i : r_i < 0, \sigma_i^2 = 0\}$,
- $\mathcal{S}^{\sigma+} = \{i : r_i > 0, \sigma_i^2 > 0\}$, $\mathcal{S}^{\sigma-} = \{i : r_i < 0, \sigma_i^2 > 0\}$.

The set of states is decomposed as
$\mathcal{S} = \mathcal{S}^{+} \cup \mathcal{S}^{\sigma+} \cup \mathcal{S}^{\sigma-} \cup \mathcal{S}^{-} = \mathcal{S}^{\bullet} \cup \mathcal{S}^{-}$, where
$\mathcal{S}^{\bullet} = \mathcal{S}^{+} \cup \mathcal{S}^{\sigma+} \cup \mathcal{S}^{\sigma-}$.
We assume that the states are numbered according to the $\mathcal{S}^{+}, \mathcal{S}^{\sigma+}, \mathcal{S}^{\sigma-}, \mathcal{S}^{-}$ *order of subsets.*

We exclude $r_i = 0$ *!!!*

Solution of the differential equation

Similar to first order models, $f(x)$ can be expressed in a matrix-exponential form (Karandikar-Kulkarni)

$$f(x) = \pi e^{\mathbf{K}x} \begin{bmatrix} \mathbf{I} & \mathbf{\Psi} \end{bmatrix}, \tag{3}$$

where

- π is a row vector of size $|\mathcal{S}^{\bullet}|$,
- the size of $\mathbf{K}$ is $|\mathcal{S}^{\bullet}| \times |\mathcal{S}^{\bullet}|$
- and the size of $\mathbf{\Psi}$ is $|\mathcal{S}^{\bullet}| \times |\mathcal{S}^{-}|$.

It remains to solve

- matrices $\mathbf{K}$ and $\mathbf{\Psi}$,
- vector π,
- and the vector of probability masses at level 0 p.

Quadratic equations

Substituting (3) into the differential equation (1) gives

$$\boldsymbol{K}\boldsymbol{R}_{\bullet} - \boldsymbol{K}^2\boldsymbol{S}_{\bullet} = \boldsymbol{Q}_{\bullet\bullet} + \boldsymbol{\Psi}\boldsymbol{Q}_{-\bullet},$$
$$\boldsymbol{K}\boldsymbol{\Psi}\boldsymbol{R}_{-} - \underbrace{\boldsymbol{K}^2\boldsymbol{\Psi}\boldsymbol{S}_{-}}_{\mathbf{0}} = \boldsymbol{Q}_{\bullet-} + \boldsymbol{\Psi}\boldsymbol{Q}_{--},$$

where $\boldsymbol{S}_{-} = \mathbf{0}$ has been exploited.

Our goal is to transform this set of quadratic equations into a single one of size $|\mathcal{S}|$ with proper signs of the coefficients.

Main theorem

Theorem

The minimal non-negative solution of the matrix-quadratic equation $\boldsymbol{F} + \mathbb{R}\boldsymbol{L} + \mathbb{R}^2\boldsymbol{B} = \boldsymbol{0}$ *defined by the QBD with forward, local and backward matrix blocks*

$$\boldsymbol{F} = \begin{bmatrix} \hat{\boldsymbol{Q}}_{\bullet\bullet} + \hat{\boldsymbol{I}}_{\bullet} + \hat{\boldsymbol{S}}_{\bullet} & \hat{\boldsymbol{Q}}_{\bullet-} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \boldsymbol{L} = \begin{bmatrix} -\hat{\boldsymbol{I}}_{\bullet} - 2\hat{\boldsymbol{S}}_{\bullet} & \boldsymbol{0} \\ \hat{\boldsymbol{Q}}_{-\bullet} & \hat{\boldsymbol{Q}}_{--} - \boldsymbol{I}_{-} \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} \hat{\boldsymbol{S}}_{\bullet} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{-} \end{bmatrix} \quad \textit{is} \quad \mathbb{R} = \begin{bmatrix} \hat{\boldsymbol{K}} + \boldsymbol{I}_{\bullet} & \hat{\boldsymbol{\Psi}} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Main theorem

Where the notations with $\hat{}$ denotes properly scaled quantities

$$\boldsymbol{C}_\bullet = \begin{bmatrix} \boldsymbol{R}_+ & & \\ & \boldsymbol{R}_{\sigma+} & \\ & & -\boldsymbol{R}_{\sigma-} \end{bmatrix} \text{ and } \boldsymbol{C}_- = -\boldsymbol{R}_-,$$

$$c > \max\left(\max_{i\in\mathcal{S}^+} \frac{-q_{ii}}{r_i}, \max_{i\in\mathcal{S}^{\sigma-}\bigcup\mathcal{S}^{\sigma+}} \frac{-r_i + \sqrt{r_i^2 - 2\sigma_i^2 q_{ii}}}{\sigma_i^2} \right),$$

$$\hat{\boldsymbol{K}} = \frac{1}{c}\boldsymbol{C}_\bullet^{-1}\boldsymbol{K}\boldsymbol{C}_\bullet,\ \hat{\boldsymbol{\Psi}} = \frac{1}{c}\boldsymbol{C}_\bullet^{-1}\boldsymbol{\Psi}\boldsymbol{C}_-,\ \hat{\boldsymbol{S}}_\bullet = c\boldsymbol{C}_\bullet^{-1}\boldsymbol{S}_\bullet,\ \hat{\boldsymbol{Q}} = \frac{1}{c}\boldsymbol{C}^{-1}\boldsymbol{Q},$$

and $$\hat{\boldsymbol{I}}_\bullet = \boldsymbol{C}_\bullet^{-1}\boldsymbol{R}_\bullet = \begin{bmatrix} \boldsymbol{I}_+ & & \\ & \boldsymbol{I}_{\sigma+} & \\ & & -\boldsymbol{I}_{\sigma-} \end{bmatrix}.$$

Main theorem

The elements of the proof:

- Simple substitution provides the identity with the differential equation.
- The scaling ensures that $\boldsymbol{B}$, $\boldsymbol{L}$ and $\boldsymbol{F}$ are proper QBD matrix blocks.
- The QBD solution ensured the (minimal non-negative) solution with the proper eigenvalues.

Reflecting boundary at second order states

In all second order states then $p_\bullet = 0$
Inserting the matrix-exponential solution into (2) leads to

$$\pi \boldsymbol{R}_\bullet - \pi \boldsymbol{K} \boldsymbol{S}_\bullet = p_- \boldsymbol{Q}_{-\bullet}, \tag{4}$$

$$\pi \boldsymbol{\Psi} \boldsymbol{R}_- = p_- \boldsymbol{Q}_{--}, \tag{5}$$

since $\boldsymbol{S}_- = 0$.
From (5) and (4)

$$\pi(\boldsymbol{R}_\bullet - \boldsymbol{K}\boldsymbol{S}_\bullet - \boldsymbol{\Psi}\boldsymbol{R}_-(\boldsymbol{Q}_{--})^{-1}\boldsymbol{Q}_{-\bullet}) = 0,$$
$$p_- = \pi \boldsymbol{\Psi} \boldsymbol{R}_-(\boldsymbol{Q}_{--})^{-1},$$

where $\boldsymbol{R}_-(\boldsymbol{Q}_{--})^{-1}$ is a non-negative matrix.
The normalization condition ($\int_x f(x)\mathbb{1} + p_-\mathbb{1} = 1$), is

$$\pi \left((-\boldsymbol{K})^{-1} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Psi} \end{bmatrix} \mathbb{1} + \boldsymbol{R}_-(\boldsymbol{Q}_{--})^{-1}\mathbb{1} \right) = 1.$$

Absorbing boundary at second order states

$f_{\sigma+}(0) = 0$ and $f_{\sigma-}(0) = 0$.
$f(0) = \pi \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Psi} \end{bmatrix}$ implies that $\pi_{\sigma+} = 0$ and $\pi_{\sigma-} = 0$.
Substituting it into (2) gives

$$f(0)\boldsymbol{R} = \begin{bmatrix} \pi_+\boldsymbol{R}_+ & 0 & 0 & \pi_+\boldsymbol{\Psi}_{+-}\boldsymbol{R}_- \end{bmatrix},$$
$$f'(0)\boldsymbol{S} = \begin{bmatrix} 0 & \pi_+\boldsymbol{K}_{+,\sigma_+}\boldsymbol{S}_{\sigma_+} & \pi_+\boldsymbol{K}_{+,\sigma_-}\boldsymbol{S}_{\sigma_-} & 0 \end{bmatrix},$$

since $\boldsymbol{S}_+ = \boldsymbol{S}_- = 0$.

Absorbing boundary at second order states

For the partitioned vectors and block matrices (2) can be rewritten as

$$\begin{aligned}
\pi_+ \boldsymbol{R}_+ &= p_{\sigma+} \boldsymbol{Q}_{\sigma+,+} + p_{\sigma-} \boldsymbol{Q}_{\sigma-,+} + p_- \boldsymbol{Q}_{-,+},\\
-\pi_+ \boldsymbol{K}_{+,\sigma_+} \boldsymbol{S}_{\sigma_+} &= p_{\sigma+} \boldsymbol{Q}_{\sigma+,\sigma_+} + p_{\sigma-} \boldsymbol{Q}_{\sigma-,\sigma_+} + p_- \boldsymbol{Q}_{-,\sigma_+},\\
-\pi_+ \boldsymbol{K}_{+,\sigma_-} \boldsymbol{S}_{\sigma_-} &= p_{\sigma+} \boldsymbol{Q}_{\sigma+,\sigma_-} + p_{\sigma-} \boldsymbol{Q}_{\sigma-,\sigma_-} + p_- \boldsymbol{Q}_{-,\sigma_-},\\
\pi_+ \boldsymbol{\Psi}_{+-} \boldsymbol{R}_- &= p_{\sigma+} \boldsymbol{Q}_{\sigma+,-} + p_{\sigma-} \boldsymbol{Q}_{\sigma-,-} + p_- \boldsymbol{Q}_{-,-},
\end{aligned}$$

which gives the linear system

$$\begin{bmatrix} \pi_+ & p_{\sigma+} & p_{\sigma-} & p_- \end{bmatrix} \cdot \begin{bmatrix} -\boldsymbol{R}_+ & \boldsymbol{K}_{+,\sigma_+} \boldsymbol{S}_{\sigma_+} & \boldsymbol{K}_{+,\sigma_+} \boldsymbol{S}_{\sigma_-} & -\boldsymbol{\Psi}_{+-} \boldsymbol{R}_- \\ \boldsymbol{Q}_{\sigma+,+} & \boldsymbol{Q}_{\sigma+,\sigma_+} & \boldsymbol{Q}_{\sigma+,\sigma_-} & \boldsymbol{Q}_{\sigma+,-} \\ \boldsymbol{Q}_{\sigma-,+} & \boldsymbol{Q}_{\sigma-,\sigma_+} & \boldsymbol{Q}_{\sigma-,\sigma_-} & \boldsymbol{Q}_{\sigma-,-} \\ \boldsymbol{Q}_{-,+} & \boldsymbol{Q}_{-,\sigma_+} & \boldsymbol{Q}_{\sigma-,-} & \boldsymbol{Q}_{-,-} \end{bmatrix} = 0.$$

Absorbing boundary at second order states

Whose normalization condition is

$$\begin{bmatrix}\pi_+ & p_{\sigma+} & p_{\sigma-} & p_-\end{bmatrix} \cdot \begin{bmatrix} (-\boldsymbol{K})^{-1}_{+\bullet} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Psi} \end{bmatrix} \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \end{bmatrix} = 1.$$

Final remarks

Summary

1. It was possible to recycle the methodology developed for first order fluid models for the analysis of second order fluid models.
2. The proof is based on known properties of QBD quadratic matrix equations.
3. Similar to first order fluid models, based on $\boldsymbol{K}$ and $\boldsymbol{\Psi}$ the boundary conditions are obtained from the solution of a linear system.

Plans

1. analysis of further second order fluid models.