Optimal Control of Service Rates in a MAP/M/1 Queue *

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Background

- Service rate control of queuing systems
  - M/M/1, tandem queue, cyclic queue, Jackson network, ...
- Poisson arrival → general arrival
  - Markovian arrival process (MAP)
    - approximate almost all arrival process at the cost of increasing model complexity
    - Coefficient of variation of MAP could be any positive #
- In this paper, we study service rate control of MAP/M/1 queue

Background

- Main difficulty
  - Increase the model complexity
    - State: (customer#, phase#)
    - Complicated Bellman equation
- Our idea
  - MAM (matrix analytic method)
    - Numerical algorithm to study QBD structure
  - SBO (sensitivity based optimization)
    - Difference formula provides a new perspective for optimization, utilize the problem structure
  - MAM + SBO: efficient way to compute value function; derive optimality property; algorithm
    - Promote: MAM community → optimization

Problem formulation

- MAP/M/1
  - MAP with m phases: $D_0,\ D_1$
  - Equivalent arrival rate: $\lambda = \omega D_1 e$
  - System state: $(N(t), J(t))$
  - Service rate: $\mu_{n,j}$, state-dependent
  - Cost function: $f(n,j) = \phi(n,j) + b\ \mu_{n,j}$
  - long-run average cost: η

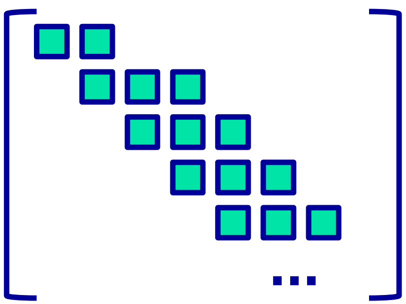
$$\boldsymbol{\mu}^* = \underset{\mu_{n,j}^{\min} \le \mu_{n,j} \le \mu_{n,j}^{\max}}{\arg\min} \left\{ \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T f(N(t), J(t)) dt \right] \right\}$$ for all n,j

Problem analysis

$$\boldsymbol{\mu}^{*} = \underset{\mu_{n,j}^{\min} \le \mu_{n,j} \le \mu_{n,j}^{\max}}{\arg\min} \left\{ \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T f(N(t), J(t))dt \right] \right\}$$

- Infinite dimension MDP
  - Optimization variables: $\mu^{*}_{n,j}$ at every (n, j)
  - Bellman optimality equation: difficult to use
- Special structure
  - QBD structure
- SBO + MAM

B=



SBO

- Sensitivity-based optimization (SBO)
  - Much more beyond perturbation analysis (PA)
  - Difference formula to study Markov systems
- Key formulas
  - Performance potential (relative value function)

$$g(n,j) = \lim_{T\to\infty} E\left\{ \int_{t=0}^{T} [f(N(t),J(t)) - \eta] dt \,|_{(N(0),J(0))=(n,j)} \right\}$$

  - Difference formula (change to a new policy **B'**, **f'**)

$$\eta' - \eta = \boldsymbol{\pi}'[(\boldsymbol{B}' - \boldsymbol{B})\boldsymbol{g} + (\boldsymbol{f}' - \boldsymbol{f})]$$

Difference of average cost

unknown, but always positive

Choose proper B', f' to make column vector negative

SBO

- With the QBD structure of B, we have

$$\eta' - \eta = \sum_{n=1}^{\infty} \sum_{j=1}^{m} \left(\mu'_{n,j} - \mu_{n,j} \right) \pi'(n,j) G(n,j)$$

$$G(n,j) := g(n-1,j) - g(n,j) + b$$

Physical explanation: the change of **g** at two adjacent states, which indicates an event of customer departure

very simple rule for the optimization:

If G(n,j)>0, we choose a smaller $\mu'_{n,j}$;
If G(n,j)<0, we choose a larger $\mu'_{n,j}$;

How to compute G?

Optimality property and theorems

Monotone property

Theorem 1. The average cost is monotone w.r.t the service rate $\mu_{n,j}$, for all $n = 1,2,\cdots;\ j = 1,2,\cdots,m$.

Bang-bang control

Theorem 2. The optimal service rate can be either minimum or maximum. More specifically,
if $G^*(n,j) \geq 0,\ \ \mu^*_{n,j}=\mu^{min}_{n,j}$; if $G^*(n,j) < 0,\ \ \mu^*_{n,j}=\mu^{max}_{n,j}$.

Optimality property and theorems

Quasi-threshold optimality

Theorem 3. Assume $\mu_{N+k,j}^{min} = 0$ and other mild conditions, if $\mu_{N,i}^{*} = \mu_{N,i}^{max}$, for some N and i, then we have $\mu_{N+k,j}^{*} = \mu_{N+k,j}^{max}$, for any k and $j = 1,2,\cdots,m$.

Remark. Quasi-threshold type policy: there exists a threshold N such that, if $n > N$, then $\mu_{n,j}^{*} = \mu_{n,j}^{max}$ for any j .

Strong quasi-threshold optimality

Corollary 1. if $\mu_{n,j}^{min} = 0$ and $\mu_{n,j}^{max}$ non-decreasing in n, then there exists a threshold N such that, if $n > N$, $\mu_{n,j}^{*} = \mu_{n,j}^{max}$; if $n < N$, $\mu_{n,j}^{*} = 0$; if $n = N$, $\mu_{n,j}^{*}$ can be either 0 or $\mu_{n,j}^{max}$ at different phase j .

Optimality property and theorems

Service rate control for M/M/1

Corollary 2. Since M/M/1 is a special case of MAP/M/1, all the previous results hold for M/M/1. That is, the monotonicity and the optimality of bang-bang control hold.

Threshold optimality for M/M/1

Theorem 4. If we consider the service rate control of M/M/1, then we have $\mu_n^* = \mu_n^{max}$ for $n > \theta$; $\mu_n^* = 0$ for $n \leq \theta$, where θ is the optimal threshold.

Computation and algorithms

- G is the key, how to compute it?
  - with QBD structure, use MAM to compute G
    - recursive, numerical algorithm

Algorithm 1. recursive numerical computation for $\tilde{A}_\infty$

- Initialize $\tilde{\boldsymbol{A}}_\infty$ arbitrarily, e.g., set $\tilde{\boldsymbol{A}}_\infty^{(0)} = \boldsymbol{0}$, $k = 0$, and $\epsilon > 0$.
- Repeat:

$$\tilde{\boldsymbol{A}}_\infty^{(k+1)} = (\mathrm{diag}(\boldsymbol{\mu}_\infty^{\max}) - \boldsymbol{D}_0)^{-1}\mathrm{diag}(\boldsymbol{\mu}_\infty^{\max}) + (\mathrm{diag}(\boldsymbol{\mu}_\infty^{\max}) - \boldsymbol{D}_0)^{-1}\boldsymbol{D}_1(\tilde{\boldsymbol{A}}_\infty^{(k)})^2;$$

$$k \leftarrow k + 1;$$

  Until the stopping criterion $||\tilde{\boldsymbol{A}}_\infty^{(k)} - \tilde{\boldsymbol{A}}_\infty^{(k-1)}|| < \epsilon$ is satisfied.
- Output $\tilde{\boldsymbol{A}}_\infty^{(k)}$ as the value of $\tilde{\boldsymbol{A}}_\infty$.

$\tilde{A}_\infty$ is used to further compute G iteratively

Optimization algorithms

Algorithm 2. iterative algorithm to find $\boldsymbol{\mu}^*$ of MAP/M/1

- Initialize system parameters $(m, \boldsymbol{D}_0, \boldsymbol{D}_1)$, $\boldsymbol{\mu}^{\max}$, ϕ, b, and the initial policy $\boldsymbol{\mu}'$. Determine N, $\boldsymbol{\mu}_\infty^{\max}$, and $\mathrm{d}\phi_\infty$.
- Compute $\tilde{\boldsymbol{A}}_\infty$ and $\boldsymbol{A}_\infty$ by using Algorithm 1. Compute $\boldsymbol{\xi}_\infty$ by using (55).
- Repeat:

  Let $\boldsymbol{\mu} \leftarrow \boldsymbol{\mu}'$;

  Compute $\{\boldsymbol{A}_n,\ n = N, N-1, \cdots, 1\}$ and $\{\boldsymbol{\xi}(n),\ n = N, N-1, \cdots, 1\}$ with (44);

  Set $\boldsymbol{G}(1) = \boldsymbol{\xi}(1)$, compute $\{\boldsymbol{G}_n,\ n = 2, 3, \cdots, N\}$ by using (43);

  Use (63) to generate a new policy $\boldsymbol{\mu}'$;

  Until the stopping criterion $\boldsymbol{\mu} = \boldsymbol{\mu}'$ is satisfied.
- Output $\boldsymbol{\mu}$ as the optimal policy $\boldsymbol{\mu}^*$.

- A policy iteration type algorithm
- MAM algorithm to compute G, difference of value functions
  - How to compute value function is key for MDP
    - Deep learning, use deep neural network to compute it

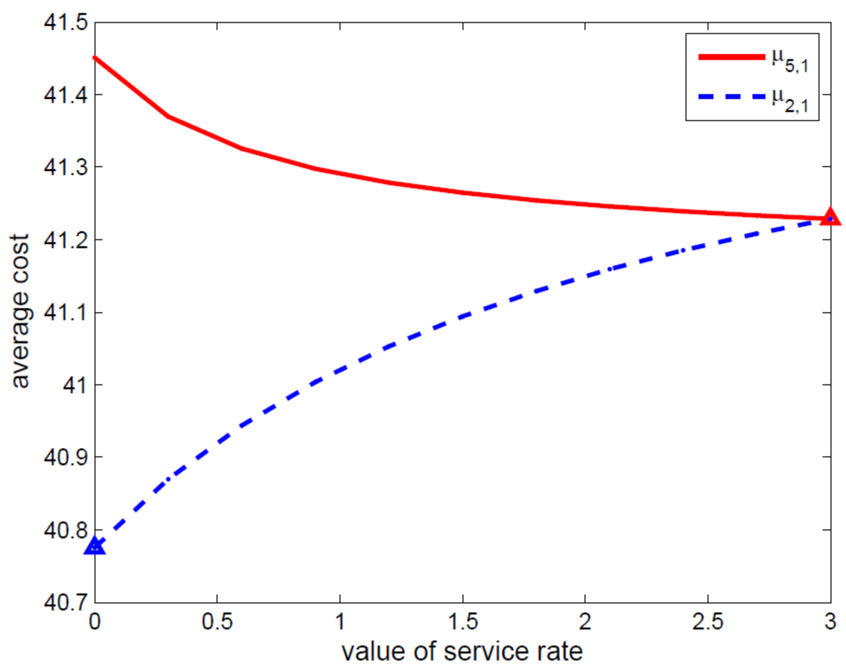
Numerical experiments

- Consider a MAP/M/1 with parameters

$$m = 2,\ \boldsymbol{D}_0 = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix},\ \boldsymbol{D}_1 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix};$$

$$\boldsymbol{\mu}^{\max} = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots \end{pmatrix}; \quad \boldsymbol{\mu}^{\min} = \mathbf{0};$$

$$\phi(n,j) = \frac{10}{n+1} + 3\sqrt{n} + 10j, \quad \text{for } n \geq 0,\ j = 1, 2.$$



Monotonicity is validated.

Numerical experiments

- Consider another example

$$m = 2,\ \boldsymbol{D}_0 = \begin{pmatrix} -0.1 & 0 \\ 0.2 & -3 \end{pmatrix},\ \boldsymbol{D}_1 = \begin{pmatrix} 0.09 & 0.01 \\ 0 & 2.8 \end{pmatrix};$$

$$\boldsymbol{\mu}^{\max} = \begin{pmatrix} 3 & 5 & 4 & 4 & 2 & 3 & 3 & 4 & 2 & 2 & 2 & \cdots \\ 8 & 3 & 2 & 5 & 2 & 2 & 5 & 5 & 7 & 7 & 7 & \cdots \end{pmatrix};$$

$$\boldsymbol{\mu}^{\min} = \boldsymbol{0};\ \phi(n,j) = \frac{15}{n+1} + 2\sqrt{n} + 30j,\ \text{for } n \geq 0,\ j = 1, 2.$$

- Use MAM+SBO, Algorithm 1&2 to find

$$\boldsymbol{\mu}^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 3 & 4 & 2 & 2 & 2 & \cdots \\ 0 & 0 & 0 & 0 & 2 & 2 & 5 & 5 & 7 & 7 & 7 & \cdots \end{pmatrix}.$$

- Bang-bang control is validated
- Optimality of strong quasi-threshold type policy
  - n>5, $\mu_{n,j}^* = \mu_{n,j}^{max}$; n<5, $\mu_{n,j}^* = 0$; n=5, $\mu_{n,j}^*$ is either 0 or max

Conclusion

- Service rate control of MAP/M/1
- Optimality properties
  - Monotone, bang-bang, quasi-threshold
- MAM+SBO: optimization algorithm
  - MAM to recursively compute G
  - SBO to iteratively compute $\boldsymbol{\mu}^*$
- Computation of value function
  - Very important topic in MDP and AI
    - Deep learning, AlphaGo, reinforcement learning, ADP
  - SBO provides a powerful method to do optimization
  - MAM provides a promising way, recursive algo.,
  - recursive numerical approach is important
    - Google PageRank to compute $\boldsymbol{\pi}$: $\boldsymbol{P}^n \rightarrow \boldsymbol{\pi}$

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Thank You!