Acyclic Phase-Type Distributions in Fault Trees

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The 9th International Workshop on Performability Modeling of Computer and Communication Systems

Reliability analysis

What is the likelihood of *system failure*?

given

the likelihood of *component failure*?

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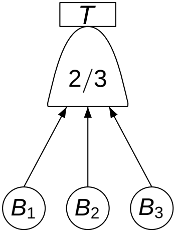
Outline

1. Theory
   - Fault Trees
   - Phase-Type distributions
2. Practice
   - Dynamic Fault Trees
   - Case study
3. Conclusion

Fault Trees - from component failure to system failure



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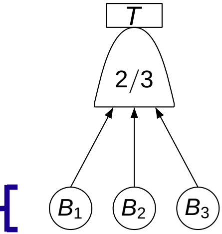

Basic events

'Component Failure'

Failure probability is given

Fault Trees - from component failure to system failure

Gate

'T occurs after 2 of B_1, B_2, B_3 occur'

'T fails after 2 of its 3 subcomponents fail'

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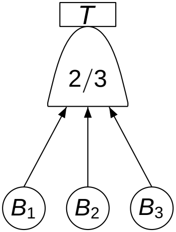

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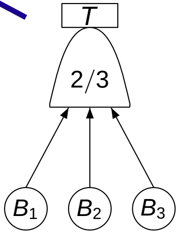

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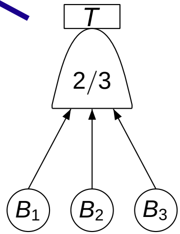

But actually...

A Fault Tree is a Boolean Function

Some terminology

- The state of the basic events is a random boolean vector $\vec{B} = (B_1, \ldots, B_n)$,
- The fault tree is a function f from $\{0,1\}^n$ to $\{0,1\}$. And now,

$$P(T = 1) = P(f(\vec{B}) = 1).$$

- This problem can be solved efficiently with binary decision diagrams.
- We only consider *coherent* fault trees where events are irrevocable.

Truth table of f

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Fault Trees with Time

Adding Time...

- State of BEs at time t is a stochastic process $\vec{B}^{(t)} = (B_1^{(t)}, \ldots, B_n^{(t)})$,
- $P(B_1^{(t)} = 1)$ is the probability that event B_1 has occurred on or before time-point t, and
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How do we represent the distribution of basic events and the top event?

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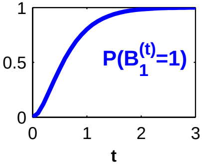

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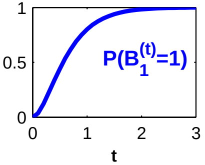

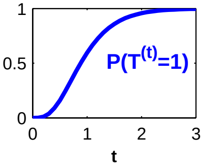

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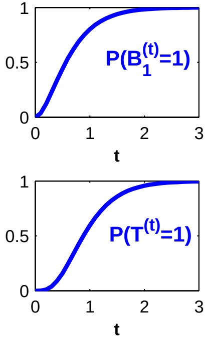

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PH distribution overview

Properties

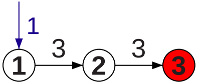
- A PH-distribution is represented by a CTMC with a single absorbing state,
- Matrix characterization,
- For a random variable Z PH-distributed with representation X we have,

$$P(Z \leq t) = P(X^{(t)} = 3) = \vec{\alpha} e^{Qt} \vec{\omega}.$$

- Infinitely many different representations,
- Acyclic PH-distributions (APH),
- For FTs:

$$P(B_1^{(t)} = 1) = P(X^{(t)} = 3).$$

X



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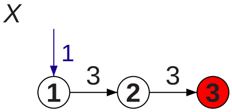

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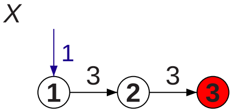

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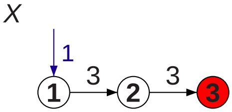

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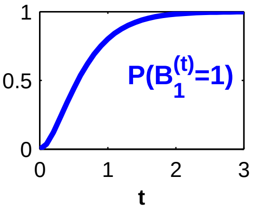


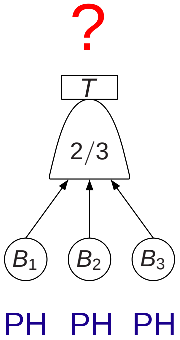
FT and PH

Theorem

The top event of a coherent fault tree with PH-distributed basic events is itself PH-distributed.

Corollary

The top event of a coherent fault tree with APH-distributed basic events is itself APH-distributed.



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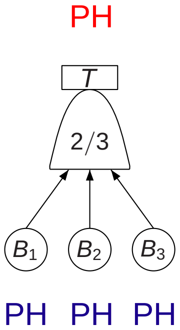

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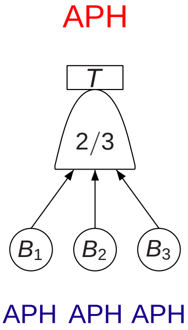
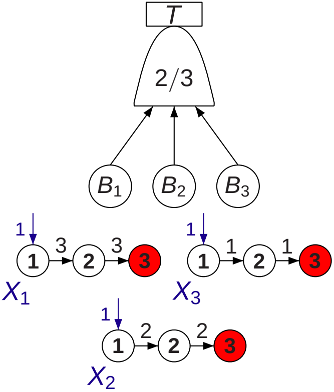


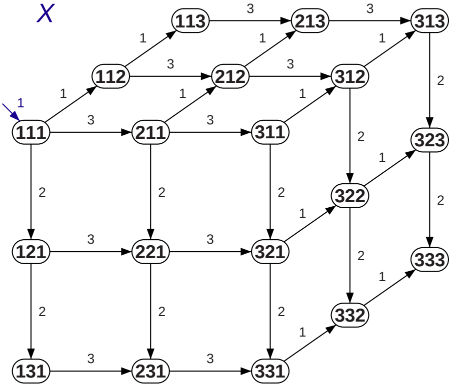
Proof by Construction

Constructing a representation for the top event

- For a coherent FT with n basic events,
- Parallel composition of representations: X
  Initial distribution $\vec{\alpha} = \vec{\alpha}_1 \otimes \ldots \otimes \vec{\alpha}_n$
  Generator matrix $\mathbf{Q} = \mathbf{Q}_1 \oplus \ldots \oplus \mathbf{Q}_n$.
- Mark occurrence of basic events. Per state a boolean vector $\vec{b} \in \{0, 1\}^n$,
- Group states by $f(\vec{b})$: Two sets S_0 and S_1,
- Collapse S_1 to a single state (Note: S_1 is absorbing),
- The resulting CTMC Y represents the PH distribution of the top event of the FT.

Construction example: parallel composition

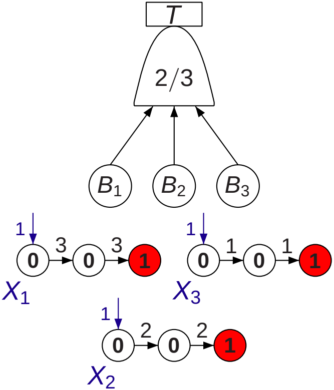


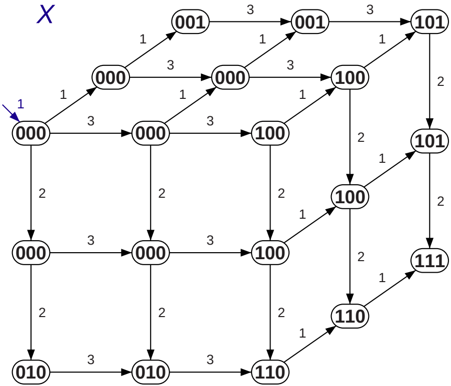


The connection

$$P(X_1^{(t)} = 3 \wedge X_2^{(t)} = 1 \wedge X_3^{(t)} = 2) = P(X^{(t)} = 312)$$

Construction examples: From states to events





The connection

$$P(\vec{B}^{(t)} = (100)) = P(X_1^{(t)} \in S_1 \wedge X_2^{(t)} \in S_0 \wedge X_3^{(t)} \in S_0) = P(X^{(t)} \in S_{100})$$

Construction: Apply function f

B_1	B_2	B_3	T
0	0	0	0
0	0	1	0
0	1	0	0
1	0	0	0
0	1	1	1
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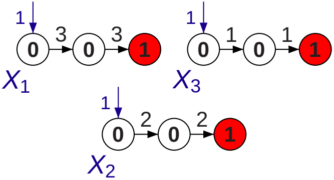

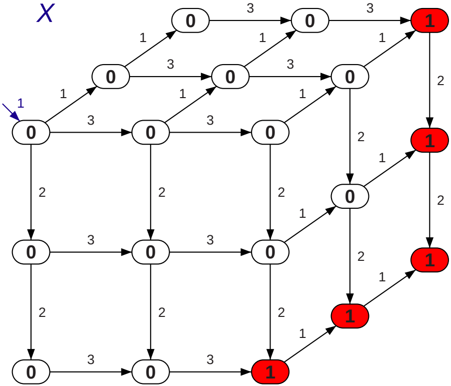

The connection

$$P(T^{(t)} = 1) = P(f(\vec{B}^{(t)}) = 1) = P(X^{(t)} \in S_1)$$

Construction: Collapse set ‘1’

B_1	B_2	B_3	T
0	0	0	0
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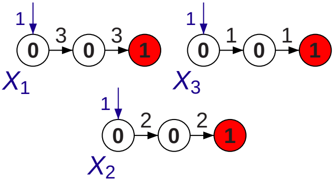
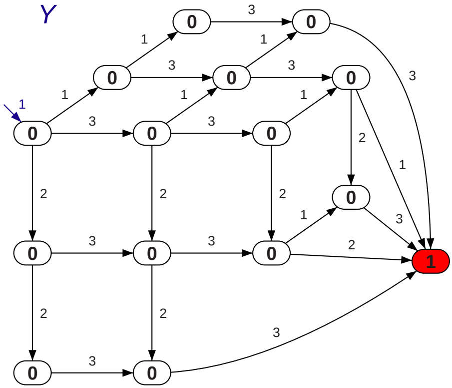




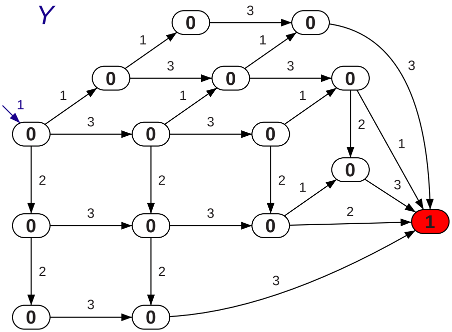
The connection

$$P(T^{(t)} = 1) = P(X^{(t)} \in S_1) = P(Y^{(t)} = 1)$$

Minimal Representation for APH

A better representation

- Y has 21 states and 45 transitions,
- Smallest representation: 14 states and 13 transitions,
- For APH: find smallest representation with APHMIN.



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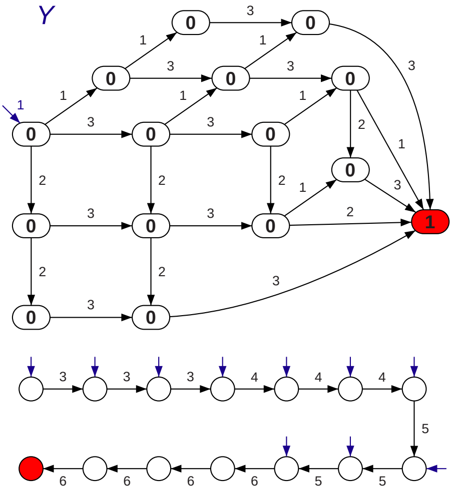

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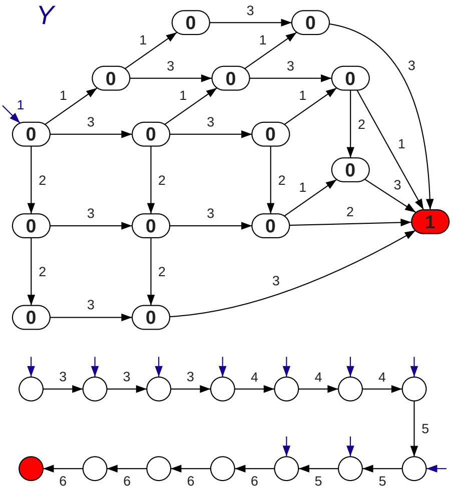


APHMIN Sketch

Computing the 'minimal' representation

1. Convert the APH representation to bidiagonal form,
2. Consider the Laplace-Stieltjens transform,
3. For each state:
    1. Check for states that are linear combinations of other states,
    2. Compute a new initial 'distribution',
    3. Check if the new initial 'distribution' is a distribution.

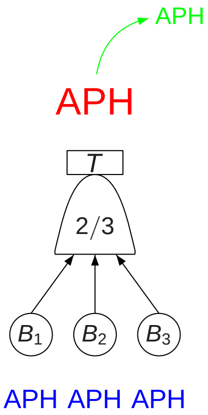
So far

We have seen...

- Describe BEs with APH distributions,
- Top event is also APH distributed, and
- APH representations can be minimized.

But...

- Solving a FT is anyway easy!
- Perhaps FTs are too simple...



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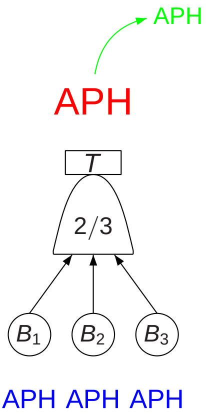

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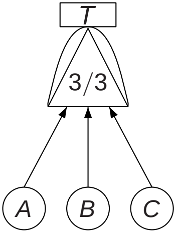
Bringing order to FTs

Dynamic Fault Trees

- Adds three gates to FTs,
- Example: T only occurs if A, B and C happen in the correct order,
- A DFT is not a boolean function!
- We must construct a representation of the distribution of event T.
- FT construction does not work.



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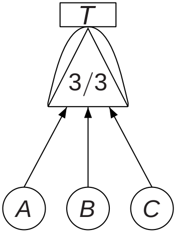
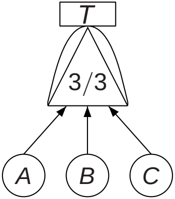


Compositional Aggregation

Generating the state space step by step

1. Translate syntactic elements to interactive models,
2. Select a subset of interactive models,
3. Compose them,
4. Minimize the result,
5. More than one model left: goto 2.

Constructing CTMC with CORAL



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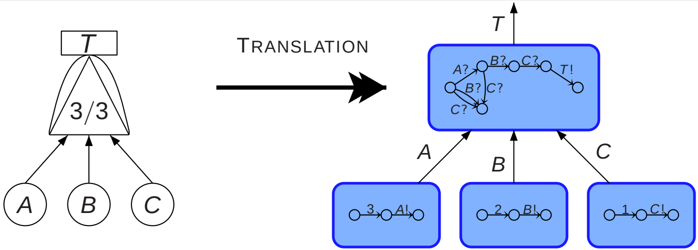

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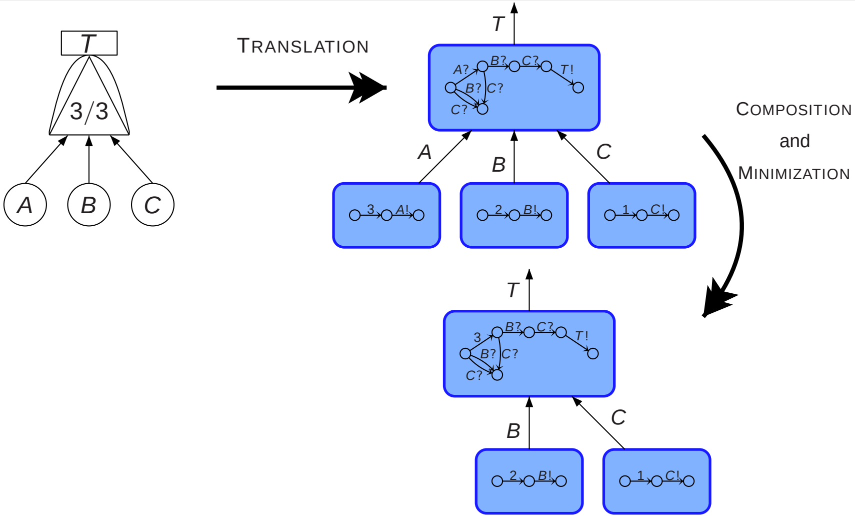

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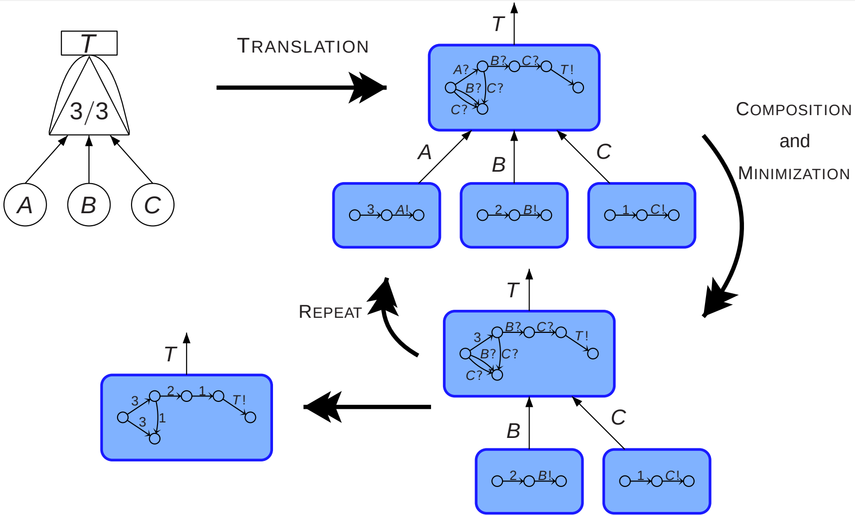
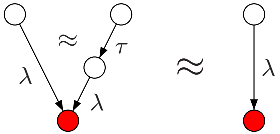


How to minimize?

Weak bisimulation



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- Equality = same observable transitions,
- Eliminate equivalent states,
- For CTMCs and IOIMCs,
- Partition refinement algorithm.

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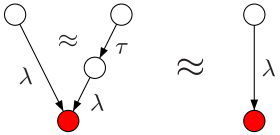

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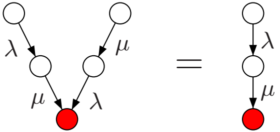


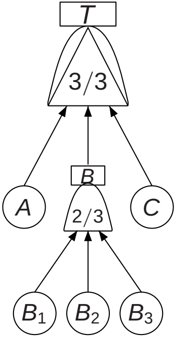
Properties

- Considers Laplace-Stieltjes transform,
- Eliminate states that are linear combinations of other states,
- For APH representations,
- Weaker than weak!

Improving Compositional Aggregation

Which minimization?

- Compositional aggregation uses minimization,
- For DFTs: weak bisimulation minimization,
- Can we use APHMIN instead?
- For FT-subtrees we can!
- 14 states instead of 21 states.



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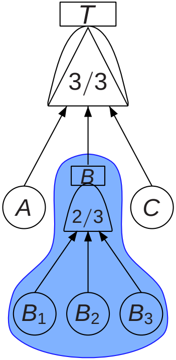

Case Study

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- FTPP case study (20 basic events, 21 gates),
- Three variants with 1,2, or 3 FT subtrees,
- Compare normal CA (CORAL) with enhanced CA (APHMIN),
- For FT subtrees: 21 states (CORAL) vs. 14 states (APHMIN).

#	Tool	States	Transitions	Time (s)	Unreliability
1	CORAL	1,672	12,303	10.37	$1.13 \cdot 10^{-7}$
	APHMIN	1,119	7,410	10.42	$1.13 \cdot 10^{-7}$
2	CORAL	59,739	598,524	24.52	$3.21 \cdot 10^{-4}$
	APHMIN	26,006	219,310	14.14	$3.21 \cdot 10^{-4}$
3	CORAL	1,777,955	21,895,068	14,047.99	0.209
	APHMIN	507,067	5,010,000	367.71	0.209

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Conclusion

We have seen...

- Describe BEs with APH distributions,
- Top event is also APH distributed,
- APH representations can be minimized, and
- APHMIN can be effectively used in compositional aggregation.

Future work

- Fully integrate APHMIN into CORAL,
- Identify dynamic fault trees that are APH distributed,
- Prove a conjecture about almost-sure minimality,
- Find APH-like structures in other Markovian models.

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