AN INTRODUCTION TO CLASSICAL CYCLIC POLLING MODEL

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ABSTRACT

An alternative analysis of the classical $M/G/1$ cyclic order polling model is presented. We introduce the new Markov regenerative process framework, which provides a way to perform the analysis of the polling model on a unified, i.e. service discipline independent way. We derive the decomposition property and the general expressions of the stationary number of customers as well as the stationary waiting time. We show a unified way to reduce the analysis for finding the stationary number of customers at the polling epochs and demonstrate it on several known service disciplines. The tutorial character of the approach makes it appropriate to be used for an introduction.

KEYWORDS:

polling model, service discipline, decomposition, zero and nonzero-switchover-times model.

INTRODUCTION

In this paper we study the classical polling model, where the server attends the queues in a cyclic order. For a summary on such systems see (Takagi 1986). An early work of (Kuehn 1979) has shown essential service discipline independent formulas of the stationary mean cycle time, station service time and intervisit time. A basic property of such systems is an $M/G/1$ decomposition property. It means, that the stationary number of the customer who arrives to queue i can be decomposed as the sum of two independent random variables.

The $M/G/1$ decomposition property in a more general setting was presented first by (Fuhrmann and Cooper 1985). They have proved it for the $M/G/1$ queue with generalized vacation. In the literature on polling models, traditionally the nonzero-switchover-times and the zero-switchover-times models are considered separately. Consequently considerable research effort has been taken to relate the zero- and nonzero-switchover-times models to each other, see e.g. (Srinivasan et al. 1995). Later (Borst and Boxma 1997) has unified the derivation of the stochastic decomposition for both the nonzero- and the zero-switchover-times models.

The principial goal of this paper is to present a general introduction to the classical cyclic polling model, that unifies the above mentioned results.

The contribution of this paper is two-fold. The first contribution is the alternative analysis of the classical $M/G/1$ cyclic order polling model. We provide a unified way to reduce the analysis for finding the stationary number of customers at the polling epochs. The second contribution is the introduction of the Markov regenerative process (MRP) framework. This provides a way to perform the analysis of the polling model on a unified, i.e. service discipline independent way.

The rest of this paper is organized as follows. In the next section we introduce the classical cyclic polling model. Section III gives a brief description of the MRP framework. The analysis of the polling model follows in section IV. Section V supplements the results. In section VI we demonstrate on some waiting time formulas, how the results can be used to derive several known expressions in a simple and unified way. Our conclusion is given in section VII.

THE CLASSICAL CYCLIC POLLING MODEL

Consider a continuous-time asymmetric polling model. A single server attends a sequence of N stations in cyclic order. Each station has an infinite buffer queue, which is served when the server attends that queue. The arrival process of the customers is Poisson at each station. λ_i denotes the stationary arrival rate at station i. The customer who arrives to station i is called i-customer. The customer service times have general distributions, but no bulk departure is allowed. B_i, b_i and $b_i^{(2)}$ denote the service time at station i and the first two moments of it, respectively. The switchover times have general distributions. R_i denotes the switchover time following the service of station i. On the classical cyclic polling model we impose the following conditions:

C.1 LAA condition. The "lack of anticipation" (LAA) assumption holds for the arrival process at each station. It ensures to have the "Poisson arrivals see times averages" (PASTA) property (Wolff 1982).

C.2 Mutual independency condition. The arrival processes, the service times and the switchover times are mutually independent.

C.3 Mixed-discipline system condition. Each station can use different service disciplines.

Definition 1: Cycle time: The cycle time of a given station is defined as the time ellapsed between two consecutive server visits to that station. The (polling) cycle time of station i is called i-polling cycle. C_i and c_i denote the i-polling cycle, and its mean, respectively.

Definition 2: Polling epoch, departure epoch: The arrival time of the server to a given station is called polling epoch. Similarly the time when the server finishes the service at the given station and departs is called departure epoch. The polling epoch of station i is called i-polling epoch. Similarly i-departure epoch is

the name of departure epoch of station i. F_i, f_i, $f_i^{(2f)}$ denotes the number of customers at station i, seen at the i-polling epoch, and the first two factorial moments of it, respectively. Analogously M_i, m_i, $m_i^{(2f)}$ denotes the number of customers at station i, seen at the i-departure epoch, and the first two factorial moments of it, respectively,

Definition 3: Service discipline: The service discipline gives a condition on the beginning and the end of the service at the given station. It does not depend on the history of the system and it is work-conserving as well as nonpreemptive.

The most commonly known disciplines are exhaustive, gated, non-exhaustive, semi-exhaustive, binomial-gated, binomial-exhaustive, limited-N and nonpreemtive limited-T (for more details see (Takagi 1986)).

To obtain the expression of the LST of the stationary waiting time still supplementary conditions are necessary on the model.

SC.1 Independence condition. The arrival processes, the service processes, the switchover times and the service disciplines of the model are mutually independent.

SC.2 FIFO condition. The queueing discipline is the First-In-First-Out (FIFO) order at each queue.

Now the properties needed to obtain the expression of the LST of the stationary waiting time can be defined as follows:

W.1 A new arriving i-customers do not affect the time in the system of the previously arrived i-customers, i.e. their waiting and service time. This holds due to the conditions **C.1**, **SC.1** and **SC.2**.

W.2 The waiting time of a tagged customer and the service time of it are independent. This follows from the condition **SC.1**.

As usual the server utilization at station i is denoted by ρ_i, and ρ stands for the total server utilization. The following simple relations hold: $\rho_i = \lambda_i b_i$, and $\rho = \sum_{i=1}^{N} \rho_i$. The Laplace-Stieljes transform (LST) of a nonnegative random variable A is denoted as $A^*(s)$, and for the discrete random variable A its probability-generating function (PGF) is denoted by $\widehat{A}(z)$.

MARKOV REGENERATIVE PROCESS FRAMEWORK

Observe that in our polling model the number of customers at every station, seen at any time can be interpreted as a multivariate $Z(t)$ Markov regenerative process. There exists an underlying $\{(Y(m), U(m)), m \geqq 0\}$ Markov renewal sequence with multidimensional countable state space Ω, where $U(0) = 0$ by definition. Here $Y(m)$ is the number of customers at every stations at the m-th i-polling epoch, while $U(m)$ is the m-th i-polling epoch, i.e. $Z\left(U(m)^{+}\right) = Y(m)$. The intervals between two consecutive regenerative points correspond to the i-polling cycles. $C(m)$ denotes the m-th i-polling cycle, for every $m > 0$. We call these i-polling cycles simply cycles. Given the state of $Z(t)$ at the regenerative point the stochastic behavior of the system repeats itself independent of the ellapsed polling cycles.

During the analysis of polling models the subject of interest are often quantities taking their values only in some embedded time points. This is the case e.g. when we are interested in the number of customers at each station only at the arrival times of i-customers. They are a kind of embedded processes.

Consider an $\{X(m,l), m \geqq 1, l \geqq 1\}$ stochastic process embedded in the $Z(t)$ Markov regenerative process with state space Ω, where m $(m \geqq 1)$ is the index of the polling cycle, and l $(l \geqq 1)$ is the index of the embedded time points inside the given polling cycle. We define the following quantities:

$Y \stackrel{def}{=} lim_{m \longrightarrow \infty} Y(m)$,

$C \stackrel{def}{=} lim_{m \longrightarrow \infty} C(m)$,

H_m - the number of the embedded time points inside the m-th polling cycle of $Z(t)$,

$H \stackrel{def}{=} lim_{m \longrightarrow \infty} H_m$,

$X(l) \stackrel{def}{=} lim_{m \longrightarrow \infty} X(m,l)$, in other words it is the value of $X(m,l)$ at the l-th embedded time points $(l \geqq 1)$ inside of a random cycle,

$X \stackrel{def}{=} lim_{M \longrightarrow \infty} \frac{\sum_{m=1}^{M} \sum_{l=1}^{H_m} X(m,l)}{\sum_{m=1}^{M} H_m}$, in other words it is the value of $X(m,l)$ at random occurrence inside of a random cycle,

$X_i(l)$- the i-th element of the vector $X(l)$, and finally

X_i - the i-th element of the vector X.

Theorem 1: Assume, that (i) the Y limiting distribution exists, (ii) $E[C] < \infty$, and (iii) $E[H] < \infty$. Then the X_i limiting distribution exists, and the following relation holds:

$$E\left[z^{X_i}\right] = \frac{E\left[\sum_{l=1}^{H} z^{X_i(l)}\right]}{E[H]}. \tag{1}$$

Proof. Due to (i), the assumption (ii) is equivalent with $E\{C \mid Y = j\} < \infty$ for every $j \in \Omega$. According to theorem 9.30 of (Kulkarni 1995) these ensure the existence of the limiting distribution of $Z(t)$. Due to the markov regenerative property of $Z(t)$ it follows, that a stationary version of $Z(t)$ also exists. In this version of the process the stochastic behavior of each polling cycle is the same, consequently H, for every $l \geqq 1$ the $X(l)$ and X limiting distributions exist, as well as all are stationary distributions. For the limiting distribution of X we can write:

$$P\{X = k\} = \frac{E\left[\sum_{l=1}^{H} P\{X(l) = k\}\right]}{E[H]}.$$

The assumption (iii) ensures, that the distribution is not degenerate. This expression is also valid for X_i, for the PGF of X_i from which we directly get the statement of the theorem.□

Remark 1: The statement of the theorem 1 suggests thinking in stationary random cycles. The PGF (or distribution) of X_i is the average of the "partial" PGFs (or distributions) of $X_i(l)$ over a random cycle. The number of occurrences of $X_i(l)$ in the random cycle equals with the limiting distribution of the number of embedded time points inside of a cycle.

In a stable polling system (i) and (ii) hold. Therefore the stationary distributions of all the necessary measures exist. From now on we assume that the polling system is stable.

ANALYSIS

We introduce several notations for the analysis (they are illustrated on figure 1).

G_i, g_i - the number of customers served at station i during an i-polling cycle, and its mean,

A_i - the number of customers arriving to station i during an i-polling cycle,

$T_i(n)$ - the service completion time of the n-th i-customer during an i-polling cycle ($n = 0, \ldots, G_i$, where $T_i(0)$ is the i-polling epoch by definition),

$A_i(n)$ - the number of customers arriving to station i during the service time of the n-th i-customer ($n = 1, \ldots, G_i + 1$, where $A_i(G_i + 1)$ is the number of customers arriving to station i during the intervisit time of the station i),

$T_i(n, k)$ - the arriving time of the k-th i-customer during the service time of the n-th i-customer ($n = 1, \ldots, G_i + 1$, where $T_i(G_i + 1, k)$ is the arriving time of the k-th i-customer during the intervisit time of the i-polling cycle, $k = 1, \ldots, A_i(n)$).

Theorem 2: (Expression of $\widehat{Q}_i(z)$.) Consider the stable classical nonzero-switchover-times cyclic polling model satisfying conditions **C.1** - **C.3.** For this model the stationary PGF of the number of i-customers at a random point in time is

$$\widehat{Q}_i(z) = \frac{(1-\rho_i)(1-z)B_i^*(\lambda_i - \lambda_i z)}{B_i^*(\lambda_i - \lambda_i z) - z} \cdot \frac{\widehat{M}_i(z) - \widehat{F}_i(z)}{(f_i - m_i)(1-z)}. \tag{2}$$

Proof. Let Q_i, Q_i^d, and Q_i^a denote the number of customers at station i seen at any time, seen by a departing i-customer, and seen by an arriving i-customer, respectively. It is well known, that in our model the stationary Q_i^a and Q_i^d are the same (see e.g. chapter 5. in (Kleinrock 1975)). Additionally we have PASTA (model condition **C.1**), so we can summarize:

$$\widehat{Q}_i(z) = \widehat{Q}_i^a(z) = \widehat{Q}_i^d(z). \tag{3}$$

In the stable model $E[G_i] < \infty$ and $E[A_i] < \infty$ hold, so the relationship (1) can be applied to express $\widehat{Q}_i^d(z)$ and $\widehat{Q}_i^a(z)$. We get:

$$\widehat{Q}_i^d(z) = \frac{E\left[\sum_{n=1}^{G_i} z^{Q_i(T_i(n))}\right]}{E[G_i]}, \tag{4}$$

$$\widehat{Q}_i^a(z) = \frac{E\left[\sum_{n=1}^{G_i+1}\sum_{k=1}^{A_i(n)} z^{Q_i(T_i(n,k))}\right]}{E\left[\sum_{n=1}^{G_i+1} A_i(n)\right]}. \tag{5}$$

We decompose the nominator of (5) as follows:

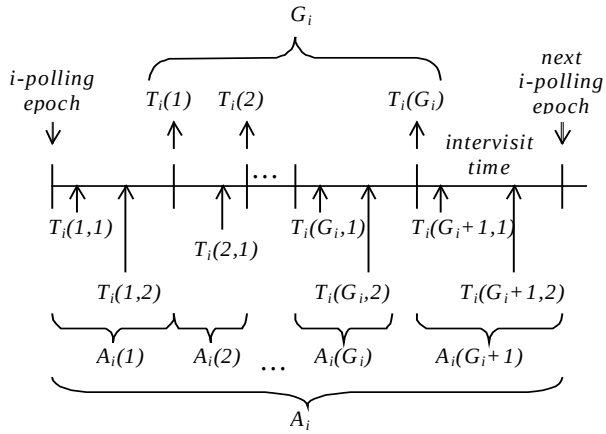

Fig. 1. Notations

$$\begin{aligned} & E\left[\sum_{n=1}^{G_i+1}\sum_{k=1}^{A_i(n)} z^{Q_i(T_i(n,k))}\right] \\ = \; & E\left[\sum_{n=1}^{G_i}\sum_{k=1}^{A_i(n)} z^{Q_i(T_i(n,k))}\right] \\ & + E\left[\sum_{k=1}^{A_i(G_i+1)} z^{Q_i(T_i(G_i+1,k))}\right]. \end{aligned} \tag{6}$$

During the service of the n-th i-customer the k-th arriving i-customer sees, on one hand, the i-customers left at the service completion time of the $(n-1)$-th i-customer and, on the other hand, $(k-1)$ previously arrived i-customers. Note, that the service starting time of the first i-customer is $T_i(0)$ due to work-conservation property of service disciplines. So the first term of the right side of (6) is:

$$\begin{aligned} & E\left[\sum_{n=1}^{G_i}\sum_{k=1}^{A_i(n)} z^{Q_i(T_i(n,k))}\right] \\ = \; & E\left[\sum_{n=1}^{G_i} z^{Q_i(T_i(n-1))}\sum_{k=1}^{A_i(n)} z^{k-1}\right] \\ = \; & E\left[\sum_{n=1}^{G_i} z^{Q_i(T_i(n-1))}\frac{1-z^{A_i(n)}}{1-z}\right]. \end{aligned} \tag{7}$$

Due to Poisson arrivals and general independent service times the terms $Q_i(T_i(n-1))$ and $A_i(n)$ are independent. The arrival process and customer service time are mutual independent (condition **C.2**), so $E\left[z^{A_i(n)}\right] = B_i^*(\lambda_i - \lambda_i z)$ for $n = 1 \ldots G_i$. Using these, and (4) it follows:

$$\begin{aligned} & E\left[\sum_{n=1}^{G_i} z^{Q_i(T_i(n-1))}\frac{1-z^{A_i(n)}}{1-z}\right] \\ = \; & \frac{1-B_i^*(\lambda_i-\lambda_i z)}{1-z}. \end{aligned} \tag{8}$$

$$E\left[\sum_{n=1}^{G_i} z^{Q_i(T_i(n))} + z^{Q_i(T_i(0))} - z^{Q_i(T_i(G_i))}\right]$$
$$= \frac{1-B_i^*(\lambda_i-\lambda_i z)}{1-z}\left(g_i\widehat{Q}_i^d(z) + \widehat{F}_i(z) - \widehat{M}_i(z)\right).$$

We consider now the second term of the right side of (6). Using the same argument as in (7), and noting that $A_i(G_i+1)$ can depend on $Q_i(T_i(G_i))$, we have:

$$\begin{aligned} & E\left[\sum_{k=1}^{A_i(G_i+1)} z^{Q_i(T_i(G_i+1,k))}\right] \quad (9) \\ = & E\left[z^{Q_i(T_i(G_i))}\sum_{k=1}^{A_i(G_i+1)} z^{k-1}\right] \\ = & \frac{1}{1-z}E\left[z^{Q_i(T_i(G_i))} - z^{Q_i(T_i(0))}\right] \\ = & \frac{\widehat{M}_i(z) - \widehat{F}_i(z)}{1-z}. \end{aligned}$$

The stationary mean number of served and arriving i-customers in a polling cycle are equal:

$$g_i = E[G_i] = E[A_i] = E\left[\sum_{n=1}^{G_i+1} A_i(n)\right]. \quad (10)$$

Substituting (8) and (9) into (6) and then into (5), as well as using (10) and (3) we get:

$$\begin{aligned} \widehat{Q}_i(z) = & \frac{(1-B_i^*(\lambda_i-\lambda_i z))}{g_i(1-z)} \quad (11) \\ & \cdot\left(g_i\widehat{Q}_i(z) + \widehat{F}_i(z) - \widehat{M}_i(z)\right) \\ & + \frac{\widehat{M}_i(z) - \widehat{F}_i(z)}{g_i(1-z)}. \end{aligned}$$

Solving (11) we have:

$$\widehat{Q}_i(z) = \frac{1}{g_i}\frac{B_i^*(\lambda_i-\lambda_i z)}{B_i^*(\lambda_i-\lambda_i z) - z}\left(\widehat{M}_i(z) - \widehat{F}_i(z)\right). \quad (12)$$

We need the following relationship, where s_i denotes the stationary mean service time of the station i:

$$(1-\rho_i)g_i = g_i - \lambda_i b_i g_i = g_i - \lambda_i s_i = f_i - m_i. \quad (13)$$

Now multiplying both the nominator and the denominator of (12) by $(1-\rho_i)(1-z)$, and using (13) we get the statement of the theorem.□

Theorem 3: (Decomposition of Q_i.) Consider the stable classical nonzero-switchover-times cyclic polling model satisfying conditions **C.1** - **C.3.** For this model the stationary number of i-customers in the system can be decomposed as the sum of two independent random variables, as follows:

$$Q_i = Q_i^r + Q_i^{aI}. \quad (14)$$

Proof. Let Q_i^{aI} denotes the number of customers at station i, seen by an i-customer arriving in the intervisit time of station i. Additionally Q_i^r denotes the number of customers in the corresponding M/G/1 queue of station i (having the same arrival rate and customer service time, as station i of the polling system). The first term on the right side of (2) is the Pollaczek-Khinchin formula for Q_i^r (chapter 5. in (Kleinrock 1975)). To interpret the right term we write the PGF of Q_i^{aI} by applying (1):

$$\widehat{Q}_i^{aI}(z) = \frac{E\left[\sum_{k=1}^{A_i(G_i+1)} z^{Q_i(T_i(G_i+1,k))}\right]}{E[A_i(G_i+1)]}. \quad (15)$$

The denominator is the mean number of i-customers arriving in the intervisit time. It is the same as $(f_i - m_i)$. Together with (9) we get:

$$\widehat{Q}_i^{aI}(z) = \frac{\widehat{M}_i(z) - \widehat{F}_i(z)}{(f_i - m_i)(1-z)}. \quad (16)$$

Now it can be seen, that the second term on the right side of (2) equals with (15). So we have:

$$\widehat{Q}_i(z) = \widehat{Q}_i^r(z)\,\widehat{Q}_i^{aI}(z). \quad (17)$$

□

Corollary 1: For the stable classical nonzero-switchover-times cyclic polling model satisfying conditions **C.1** - **C.3**, the mean stationary number of i-customers:

$$E[Q_i] = \rho_i + \frac{\lambda_i^2 b_i^{(2)}}{2(1-\rho_i)} + \frac{f_i^{(2f)} - m_i^{(2f)}}{2(f_i - m_i)}. \quad (18)$$

Proof. It can be derived from (2).□

Theorem 4: (Expression of $W_i^*(s)$.) Consider the stable classical nonzero-switchover-times cyclic polling model satisfying conditions **C.1** - **C.3** and supplementary conditions **SC.1** - **SC.2.** For this model the LST of the stationary waiting time of the customers at station i is

$$\begin{aligned} W_i^*(s) = & \frac{s(1-\rho_i)}{s - \lambda_i + \lambda_i B_i^*(s)} \quad (19) \\ & \cdot\frac{\widehat{M}_i\left(1-\frac{s}{\lambda_i}\right) - \widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)}{\frac{s}{\lambda_i}(f_i - m_i)}. \end{aligned}$$

Proof. We argue, that due to FIFO queueing discipline the number of i-customers left in the system at service completion of a tagged i-customers is equal with the number of i-customers arrived during the sojourn time of that i-customer in the system. Due to the mutual independency condition **C.2,** and properties **W.1** and **W.2** for $\widehat{Q}_i^d(z)$ we get:

$$\widehat{Q}_i^d(z) = W_i^*(\lambda_i - \lambda_i z)\,B_i^*(\lambda_i - \lambda_i z). \quad (20)$$

Substituting $s = \lambda_i - \lambda_i z$ and rearranging (20) gives the basic relationship for $W_i^*(s)$:

$$W_i^*(s) = \frac{\widehat{Q}_i^d\left(1-\frac{s}{\lambda_i}\right)}{B_i^*(s)}.$$

Taking into account (3) and by substituting (2) we get the statement of the theorem.□

Theorem 5: (Decomposition of W_i.) Consider the stable classical nonzero-switchover-times cyclic polling model satisfying conditions **C.1** - **C.3** and supplementary conditions **SC.1** - **SC.2.** For this model the LST of the stationary waiting time of the customers at station i can be decomposed as the sum of two independent random variables, one being the stationary waiting time in the corresponding standard M/G/1 queue, and the other relates to the stationary number of the i-customers seen by the i-customers arriving in the intervisit time of the station i.

Proof. Let W_i^r denote the waiting time in the corresponding M/G/1 queue of station i. The left term on the right side of (19) is the Pollaczek-Khinchin formula for the waiting time of the customers in the standard M/G/1 system (Kleinrock 1975), i.e. $W_i^{r*}(s)$. Using this and (16):

$$W_i^*(s) = W_i^{r*}(s)\,\widehat{Q}_i^{aI}\left(1-\frac{s}{\lambda_i}\right). \tag{21}$$

□

Corollary 2: For the stable classical nonzero-switchover-times cyclic polling model satisfying conditions **C.1** - **C.3** and supplementary conditions **SC.1** - **SC.2**, the mean stationary waiting time is

$$E[W_i] = \frac{\lambda_i b_i^{(2)}}{2(1-\rho_i)} + \frac{f_i^{(2f)} - m_i^{(2f)}}{2\lambda_i\,(f_i - m_i)}\ . \tag{22}$$

Proof. It can be derived from (19).□

SUPPLEMENTS

Corollary 3: Consider the stable classical nonzero-switchover-times polling model satisfying conditions **C.1** - **C.3** and the supplementary conditions **SC.1** - **SC.2**. If the

$$\widehat{F}_i(z) \Rightarrow \widehat{M}_i(z)$$

relation can be expressed explicitly, then the analysis reduces to the one of finding the stationary number of customers at the i-polling epochs.

Proof. Having $\widehat{M}_i(z)$ as an expression of $\widehat{F}_i(z)$, substituting it and it's derivatives in the (2), (18), (19) and (22) results, that only $\widehat{F}_i(z)$ and it's derivatives remains unknown. After solving the system for $\widehat{F}_i(z)$ or it's derivatives, we can substitute them in that equalities to get $\widehat{Q}_i(z)$, $E[Q_i]$, $W_i^*(s)$ and $E[W_i]$.□

Theorem 6: Consider the stable classical zero-switchover-times polling model satisfying conditions **C.1** - **C.3** and optionally the supplementary conditions **SC.1** - **SC.2**. Then all the previous results, i.e. the theorems 2, 3, 4 and 5 and the corollaries 1, 2, and 3 are also valid for the zero-switchover-times polling model .

Proof. Let Ω^1 denote the nonzero-switchover-times polling model. The polling model, where all the input parameters are the same except the switchover times is referred as corresponding zero-switchover-times polling model. This is denoted by Ω^0. Let us introduce a modified zero-switchover-times polling model, say Ω^{0+},

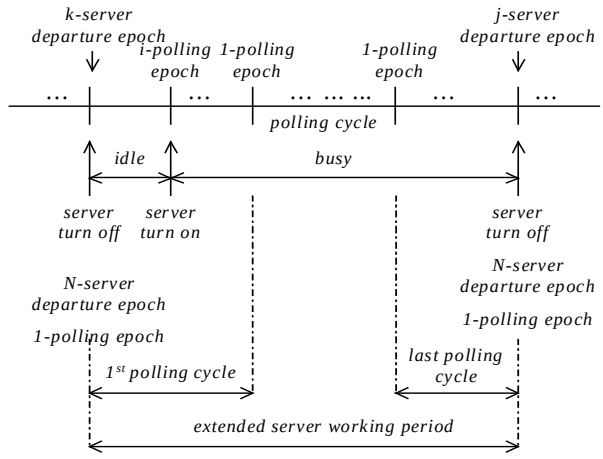

Fig. 2. Modified zero-switchover-times model

where the server working period is virtually extended before the server turn on and after the server turn off (figure 2). This is done on such a way, that the idle period is assigned with the first polling cycle of the next server working period. Additionally some server visits are inserted, if necessary. So the server starts with visiting the first queue (before the idle period), and finishes by visiting the N-th queue (again before the next idle period). Due to these extensions the extended server working period composes a number of complete polling cycles. Any change in the number of customers in the Ω^{0+} model has a corresponding event at the same time in the Ω^0 model, because during the extended visits and the idle time period no arrival and no service occurs. Therefore in any time between the server turn on and next turn off the number of customers in the system are also the same in both the Ω^0 and Ω^{0+} models. It is true for $\widehat{Q}_i^d(z)$, $\widehat{Q}_i^{aI}(z)$ and $\widehat{Q}_i^a(z)$, therefore also for $\widehat{Q}_i(z)$ due to PASTA.

Unfortunately the distributions of the random variables F_i and M_i in the Ω^{0+} model differs from their counterpairs in the Ω^0 model. This is exactly because of the extended server visits, i.e. there can be more number of occurences of i-polling and i-departure epochs in the extended server working period. It follows, that in these epochs the number of occurences of no customer present is also different in the Ω^0 and Ω^{0+} models. However this difference in the Ω^0 and Ω^{0+} models affects $\widehat{F}_i(z)$, $\widehat{M}_i(z)$, f_i, and m_i on the same way. Hence due to the substraction and division this difference also vanishes in $\frac{\widehat{M}_i(z)-\widehat{F}_i(z)}{(f_i-m_i)}$. Therefore the meaning of this term of the right side of (2) in the Ω^{0+} model is the same as that one in the the Ω^0 model.□

Corollary 4: For nonzero-switchover-times model the formula (19) can be further simplified as follows:

$$W_i^*(s) = \frac{1}{c_i}\,\frac{\widehat{M}_i\left(1-\frac{s}{\lambda_i}\right) - \widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)}{s-\lambda_i+\lambda_i B_i^*(s)}. \tag{23}$$

Proof. The statement follows by applying the equilibrium relation: $\frac{g_i}{\lambda_i} = c_i$, and (13).

WAITING TIME EXAMPLES

In the next we demonstrate the use of corollary 3.

Example *1:* Exhaustive service discipline

For exhaustive service discipline $\widehat{M}_i(z) = 1$, and $m_i = 0$. Substituting these values in (19) results:

$$W_i^*(s) = \frac{s(1-\rho_i)}{s-\lambda_i+\lambda_i B_i^*(s)} \frac{1-\widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)}{\left(\frac{s}{\lambda_i}\right) f_i}. \quad (24)$$

This equation also can be found as equation 3 in (Srinivasan et al. 1995) for the zero-switchover-times model. For the nonzero-switchover-times model applying from (23) we get:

$$W_i^*(s) = \frac{1}{c_i} \frac{1-\widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)}{s-\lambda_i+\lambda_i B_i^*(s)}. \quad (25)$$

This equation also can be found as equation 4.32, p. 80, in (Takagi 1986).

Example *2:* Gated service discipline

Due to the definition of the service strategy the service time of the station i is $S_i^*(s) = \widehat{F}_i(B_i^*(s))$. Additionally M_i equals with the number of i-customers arriving during S_i. In other words:

$$\widehat{M}_i(z) = \widehat{F}_i(B_i^*(\lambda_i - \lambda_i z)). \quad (26)$$

Similarly $m_i = \rho_i f_i$. Substituting these values in (19), for the zero-switchover-times model we get:

$$W_i^*(s) = \frac{s(1-\rho_i)}{s-\lambda_i+\lambda_i B_i^*(s)} \frac{\widehat{F}_i(B_i^*(s)) - \widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)}{\left(\frac{s}{\lambda_i}\right)(1-\rho_i) f_i}. \quad (27)$$

For the nonzero-switchover-times model using (26) in (23) gives:

$$W_i^*(s) = \frac{1}{c_i} \frac{\widehat{F}_i(B_i^*(s)) - \widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)}{s-\lambda_i+\lambda_i B_i^*(s)}. \quad (28)$$

This equation also can be found as equation 5.45, p. 110, in (Takagi 1986).

Example *3:* Non-exhaustive service discipline

At the i-departure epoch, the server just completed one service with probability $P(F_i \geqq 1)$, or no service with probability $1 - P(F_i \geqq 1)$, because the discipline has only these two possibilities. That implies, that during an i-polling cycle $g_i = P(F_i \geqq 1)$. Conditioning $\widehat{F}_i(z)$ on $F_i \geqq 1$, and denoting it by F_i^+, that is $P\left(F_i^+ = k\right) = \frac{P(F_i=k)}{P(F_i \geqq 1)}$ $(k = 1, 2, ...)$, it follows that $\widehat{F}_i^+(z) = \frac{\widehat{F}_i(z)-(1-g_i)}{g_i}$.

If $F_i \geqq 1$, then the PGF of the arriving i-customers during the service time of the one i-customer served is $B_i^*(\lambda_i - \lambda_i z)$. This is independent from $\widehat{F}_i^+(z)$, so after some straightforward manipulation we get:

$$\widehat{M}_i(z) = \frac{B_i^*(\lambda_i - \lambda_i z)}{z}\left(\widehat{F}_i(z) - (1-g_i)\right) + (1-g_i). \quad (29)$$

Similarly for m_i we get: $m_i = g_i f_i - g_i(1-\rho_i)$. Substituting these values in (19) gives:

$$W_i^*(s) = \frac{s(1-\rho_i)}{s-\lambda_i+\lambda_i B_i^*(s)} \cdot \frac{\frac{B_i^*(s)-\left(1-\frac{s}{\lambda_i}\right)}{1-\frac{s}{\lambda_i}}\left(\widehat{F}_i\left(1-\frac{s}{\lambda_i}\right)-(1-g_i)\right)}{\left(\frac{s}{\lambda_i}\right)((1-g_i) f_i - g_i(1-\rho_i))}. \quad (30)$$

CONCLUSION

We have presented an alternative analysis of the classical $M/G/1$ cyclic polling system. The analysis was based on a new Markov regenerative framework, which enabled to carry out the analysis on unified, service discipline independent way. The MRP framework provides a general way of thinking in stationary random cycles and of relating stationary measures.

We have obtained the general expressions for the stationary number of customers (expression (2) and (18)), and for the stationary waiting time (expressions (19), (22) and (23)). These results except (23) are valid for both the nonzero- and the zero-switchover-times models. The expressions can be used to simplify the analysis of the relating polling models (corollary 3). We have demonstrated this methodology on several known service disciplines in the previous chapter.

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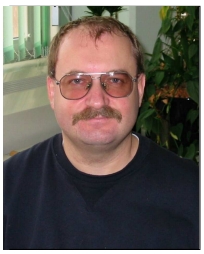

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