ANALYSIS OF $BMAP/GI/1$ PHASE DEPENDENT VACATION MODEL WITH DECREMENTING-K DISCIPLINE

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Abstract

In this paper we present the analysis of the $BMAP/GI/1$ phase dependent vacation model with decrementing-K service discipline. The exhaustive discipline is a special case of the decrementing-K discipline, where $K = \infty$. The generalization of the service discipline independent expressions of the vacation model [7] for phase dependent vacation enables the treatment of the $BMAP/GI/1$ queue as special case of the $BMAP/GI/1$ vacation model with exhaustive discipline. The contributions of this paper are the unified analysis of these models and the specific analysis of the decrementing-K discipline.

Keywords: queueing theory, vacation model, BMAP, service discipline.

1. INTRODUCTION

In vacation models the server occasionally takes a vacation period in which no customer is served. For details on vacation models we refer to the comprehensive survey of Doshi [1] and to the excellent book of Takagi [2].

In the last decades queueing models with batch Markovian arrival process ($BMAP$) (Lucantoni [3]) have been intensively studied. Most of the analysis of the $BMAP/G/1$ queueing models are based on the standard matrix analytic-method pioneered by Neuts [4] and further extended by many others (see the tutorial of Lucantoni [5]). Recently Chang and Takine [6] analyzed several $BMAP/G^B/1$ exhaustive models with or without vacations. However their analysis has a separate treatment for each model variant.

The principal goal of this paper is the unified analysis of the $BMAP/G/1$ vacation model with exhaustive discipline and the $BMAP/G/1$ queue as special cases of the corresponding vacation model with decrementing-K service discipline. The exhaustive discipline is a special case of the decrementing-K discipline, where $K = \infty$. We extend the model of [7] and generalize the service discipline independent expressions for the phase dependent vacation to enable the treatment of the $BMAP/GI/1$ queue as special case of the corresponding vacation model with exhaustive discipline.

The new contributions of this paper are the unified analysis of the $BMAP/G/1$ queue and the corresponding vacation model with exhaustive discipline through the whole analysis as well as the specific analysis of the decrementing-K discipline.

The rest of the paper is organized as follows. In section II we describe the vacation model and the notations. The service discipline independent relations are generalized in Section III. In section IV we present the specific analysis of the vacation model with decrementing-K discipline. A discussion on the special cases of the model closes the paper in section V.

2. THE $BMAP/GI/1$ QUEUE WITH SERVER VACATION

Batch of customers arrive to the infinite buffer queue according to a $BMAP$. For details on the $BMAP$ we refer to [3]. $\Lambda(t)$ denotes the number of $BMAP$ arrivals in $(0,t]$ and $J(t)$ is the state of the phase process of $BMAP$ at time t for $(\Lambda(t) \in \{0,1,\ldots\}, J(t) \in \{1,2,\ldots,L\})$. The $BMAP$ is specified by the matrix generating function (matrix GF) $\widehat{\mathbf{D}}(z) = \sum_{k=1}^{\infty} \mathbf{D}_k z^k$. The stationary arrival rate of a $BMAP$, λ is positive and finite, $0 < \lambda < \infty$. The service times are independent and identically distributed. B, $B(t)$, b denote the service time r.v., its cumulated distribution function and its mean, respectively. The mean service time is positive and finite, $0 < b < \infty$.

The server occasionally takes vacations, in which no customer is served. After finishing the vacation the server continues to serve the queue. If no customer is present in the queue after finishing the vacation, the server immediately takes the next vacation. We define the *cycle time* as the service period and the vacation period together. The server utilization is $\rho = \lambda b$ and the system is stable. The arrival process, the customer service times and the length of the vacation periods are independent. The service is nonpreemtive. In addition the length of the vacation period can depend on the phase of the arrival process at the start of the vacation. We refer to this property as phase dependent vacation.

In the following $[Y]_{i,j}$ and $[Y]_i$ stand for the i,j-th element and of the i-th row of matrix $\mathbf{Y}$, respectively. Similarly $[y]_j$ and $\mathbf{y}^T$ denote the j-th element and the transpose of row vector $\mathbf{y}$, respectively.

We define matrix $\mathbf{A}_k$, whose (i,j)-th element denotes the conditional probability that during a customer service time the number of arrivals is k and the initial and final phases of the $BMAP$ are i and j, respectively. That is, for $k \geq 0$, $1 \leq i,j \leq L$, $[\mathbf{A}_k]_{i,j} = P\{\Lambda(B) = k, J(B) = j | J(0) = i\}$. The matrix GF $\widehat{\mathbf{A}}(z)$ is defined as $\widehat{\mathbf{A}}(z) = \sum_{k=1}^{\infty} \mathbf{A}_k z^k$ and it can be expressed explicitly as $\widehat{\mathbf{A}}(z) = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dB(t)$ (see [3]).

Let V_i, $V_i(t)$ denote the conditional vacation time r.v. and its cumulated distribution function, respectively, given that the phase of $BMAP$ is i at the start of the vacation, for $1 \leq i \leq L$. The vector of the mean conditional vacation time, $\mathbf{v}$, is defined as

$$\mathbf{v}^T = \sum_{i=1}^{L} \mathbf{e_i}^T E(V_i), \tag{1}$$

where $\mathbf{e_i}$ stands for the row vector, whose i-th element equals to 1 and its other elements are 0. The mean conditional vacation time is positive and finite, $0 < [\mathbf{v}]_i < \infty, 0 \leq i \leq L$.

To describe the arrivals during a vacation period we define matrix $\mathbf{U}_k$, whose (i,j)-th element, for $k \geq 0$, $1 \leq i,j \leq L$, is given as $[\mathbf{U}_k]_{i,j} = P\left\{\Lambda(V_i) = k, J\left(V_i\right) = j | J\left(0\right) = i\right\}$. The matrix GF, $\widehat{\mathbf{U}}\left(z\right) = \sum_{k=0}^{\infty} \mathbf{U}_k z^k$, is given as

$$\widehat{\mathbf{U}}\left(z\right) = \sum_{i=1}^{L} \mathbf{e_i}^T \left[\int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dV_i(t)\right]_i = \sum_{i=1}^{L} \mathbf{e_i}^T \left[E\left(e^{\widehat{\mathbf{D}}(z)V_i}\right)\right]_i. \tag{2}$$

Vacation models are distinguished by their service disciplines. The service discipline determines the begin and the end of the vacation. It can be, e.g., exhaustive, gated, limited-K, decrementing-K, etc [2].

3. SERVICE DISCIPLINE INDEPENDENT STATIONARY RELATIONS

In this section we present general relations which do not depend on the specific service discipline.

3.1. Stationary number of customers in the vacation period. Let $N(t)$ denote the number of customers in the system at time t. Furthermore t_k^m denotes the start of vacation in the k-th cycle for $k \geq 1$. We define the vector generating function (vector GF) of the stationary number of customers during the vacation period $\widehat{\mathbf{q}}^v\left(z\right)$ and at the start of vacation $\widehat{\mathbf{m}}\left(z\right)$ by their elements as $\left[\widehat{\mathbf{q}}^v\left(z\right)\right]_j = \lim_{t\to\infty} \sum_{n=0}^{\infty} P\left\{N\left(t\right) = n, J\left(t\right) = j \mid t \in \textit{vacation period}\right\} z^n, |z| \leq 1$ and $\left[\widehat{\mathbf{m}}\left(z\right)\right]_j = \lim_{k\to\infty} \sum_{n=0}^{\infty} P\left\{N(t_k^m) = n, J(t_k^m) = j\right\} z^n, |z| \leq 1$, respectively. As usual $\mathbf{I}$ stands for the identity matrix.

Theorem 1. *In the vacation model the following relation holds for the vector GF of the stationary number of customers in the vacation period:*

$$\widehat{\mathbf{q}}^v\left(z\right) \widehat{\mathbf{D}}(z) = \frac{\widehat{\mathbf{m}}\left(z\right)\left(\widehat{\mathbf{U}}\left(z\right) - \mathbf{I}\right)}{\widehat{\mathbf{m}}\left(1\right) \mathbf{v}^T}. \tag{3}$$

Proof. The proof is similar to the proof of the corresponding theorem in [7]. The vector GF of the stationary number of customers in the system at instant τ in the vacation period is $\widehat{\mathbf{m}}(z)\, e^{\widehat{\mathbf{D}}(z)\tau}$. We get $\widehat{\mathbf{q}}^v\left(z\right)$ by averaging $\widehat{\mathbf{m}}(z)\, e^{\widehat{\mathbf{D}}(z)\tau}$ over the duration of the phase dependent vacation period, $\left[\widehat{\mathbf{q}}^v\left(z\right)\right]_j = \frac{\sum_{i=1}^{L} [\widehat{\mathbf{m}}(z)]_i E\left(\int_{\tau=0}^{V_i} [e^{\widehat{\mathbf{D}}(z)\tau}]_{ij} d\tau\right)}{\sum_{i=1}^{L} [\widehat{\mathbf{m}}(1)]_i E(V_i)}$. Multiplying both sides of $\widehat{\mathbf{q}}^v(z)$ by $\widehat{\mathbf{D}}(z)$ yields

$$\widehat{\mathbf{q}}^v(z)\, \widehat{\mathbf{D}}(z) = \frac{\widehat{\mathbf{m}}(z) \sum_{i=1}^{L} \mathbf{e_i}^T \left[E\left(\int_{\tau=0}^{V_i} e^{\widehat{\mathbf{D}}(z)\tau} \widehat{\mathbf{D}}(z) d\tau\right)\right]_i}{\widehat{\mathbf{m}}(1) \sum_{i=1}^{L} \mathbf{e_i}^T E\left(V_i\right)}. \tag{4}$$

The integral term can be rewritten [7] as $\int_{\tau=0}^{V_i} e^{\widehat{\mathbf{D}}(z)\tau}\,\widehat{\mathbf{D}}(z)\,d\tau = e^{\widehat{\mathbf{D}}(z)V_i} - \mathbf{I}$. Substituting the integral into (4), using (1), (2) and $\sum_{i=1}^{L} \mathbf{e_i}^T \left[\mathbf{I}\right]_i = \mathbf{I}$ results in the theorem. □

3.2. The vector GF and the mean of the stationary number of customers. We define the vector GF of the stationary number of customers at an arbitrary instant $\widehat{\mathbf{q}}\,(z)$ by its elements as $\left[\widehat{\mathbf{q}}\,(z)\right]_j = \lim_{t\to\infty}\sum_{n=0}^{\infty} P\left\{N\left(t\right)=n, J\left(t\right)=j\right\} z^n, |z| \le 1$. Furthermore we introduce the notations $\mathbf{D}^{(i)}$, $\mathbf{A}^{(i)}$, $\mathbf{U}^{(i)}$, $\mathbf{q}^{(i)}$ and $\mathbf{m}^{(i)}$, $i \ge 1$ for the i-th derivatives of $\widehat{\mathbf{D}}\,(z)$, $\widehat{\mathbf{A}}\,(z)$, $\widehat{\mathbf{U}}\,(z)$, $\widehat{\mathbf{q}}\,(z)$ and $\widehat{\mathbf{m}}\,(z)$ at $z=1$, respectively. We also use the notations $\mathbf{D} = \widehat{\mathbf{D}}\,(1)$, $\mathbf{A} = \widehat{\mathbf{A}}\,(1)$, $\mathbf{U} = \widehat{\mathbf{U}}\,(1)$, $\mathbf{q} = \widehat{\mathbf{q}}\,(1)$ and $\mathbf{m} = \widehat{\mathbf{m}}\,(1)$. $\boldsymbol{\pi}$ stands for the stationary probability vector of the phase process of the $BMAP$ ($\boldsymbol{\pi}\mathbf{D} = \boldsymbol{\pi}$).

Theorem 2. *In the vacation model the following service discipline independent relations hold:*

- *the vector GF of the stationary number of customers at an arbitrary instant can be expressed as*

$$\widehat{\mathbf{q}}\,(z)\,\widehat{\mathbf{D}}\,(z)\left(z\mathbf{I} - \widehat{\mathbf{A}}(z)\right) = \frac{\widehat{\mathbf{m}}\,(z)\left(\widehat{\mathbf{U}}\,(z) - \mathbf{I}\right)}{\widehat{\mathbf{m}}(1)\mathbf{v}^T}\,(1-\rho)\,(z-1)\,\widehat{\mathbf{A}}(z), \tag{5}$$

- *the mean of the stationary number of customers at an arbitrary instant is given by*

$$\begin{aligned}
\mathbf{q}^{(1)} \;=\; & \frac{\mathbf{m}^{(1)}}{\lambda\mathbf{m}\mathbf{v}^T}\left(\mathbf{U}^{(1)}\mathbf{e}\boldsymbol{\pi} + (\mathbf{U}-\mathbf{I})\left(\mathbf{A}^{(1)} - \mathbf{A}\,(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\,\mathbf{D}^{(1)}\right)\mathbf{e}\boldsymbol{\pi}\right) \\
+\; & \frac{\mathbf{m}}{\lambda\mathbf{m}\mathbf{v}^T}\left(\frac{1}{2}\mathbf{U}^{(2)}\mathbf{e}\boldsymbol{\pi} + \frac{1}{2}\,(\mathbf{U}-\mathbf{I})\,\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi} + \mathbf{U}^{(1)}\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\right) \\
-\; & \frac{\mathbf{m}}{\lambda\mathbf{m}\mathbf{v}^T}\left(\mathbf{U}^{(1)}\mathbf{A} + (\mathbf{U}-\mathbf{I})\,\mathbf{A}^{(1)}\right)(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\,\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi} \\
+\; & \frac{\mathbf{m}}{\lambda\mathbf{m}\mathbf{v}^T}\left(\mathbf{U}^{(1)}\mathbf{A}\mathbf{e}\boldsymbol{\pi} + (\mathbf{U}-\mathbf{I})\,\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\right)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\,\mathbf{C_1}\right) \\
+\; & \frac{\mathbf{m}}{\lambda\mathbf{m}\mathbf{v}^T}\,(\mathbf{U}-\mathbf{I})\,\mathbf{A}\,(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\lambda\mathbf{I} - \mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}\right)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\,\mathbf{C_1}\right) \\
+\; & \boldsymbol{\pi}\left(\frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2\,(1-\rho)} - \left(\mathbf{I} - \mathbf{A}^{(1)}\right)\mathbf{C_1}\right),
\end{aligned} \tag{6}$$

where matrices $\mathbf{C_1}$ and $\mathbf{C_2}$ are defined as

$$\mathbf{C_1} = (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1}\left(\frac{\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}}{(1-\rho)} + \mathbf{I}\right), \quad \mathbf{C_2} = \mathbf{D}^{(1)}\,(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\,\mathbf{D}^{(1)} - \frac{1}{2}\mathbf{D}^{(2)}.$$

Proof. Starting from relation (3) and applying the arguments of the corresponding theorems in [7] gives the theorem. □

Note that the contribution of the concrete service discipline to relations (5) and (6) is incorporated by the quantities $\widehat{\mathbf{m}}\,(z)$, $\mathbf{m}^{(1)}$ and $\mathbf{m}$.

4. ANALYSIS OF VACATION MODEL WITH DECREMENTING-K DISCIPLINE

We define the homogenous bivariate Markov chain $\left\{(N(t_k^d), J(t_k^d)); k \in \{1, \ldots\}\right\}$ on the state space $(N(t_k^d), J(t_k^d))$. where t_k^d denotes the k-th customer departure epoch of the corresponding $BMAP/GI/1$ queue (having the same arrival and departure processes but without vacation) for $k \geq 1$. We define matrix $\mathbf{G}$, whose (i, j)-th elements is given as the probability that starting from $(n+1, i)$ the first state visited in level n is (n, j), $n \in 1, 2, \ldots$, $1 \leq i, j \leq L$.

Let t_k^f denotes the end of vacation in the k-th cycle for $k \geq 1$. The vectors $\mathbf{f}_n$ and $\mathbf{m}_n$, $n \geq 0$ are defined by their elements as $[\mathbf{f}_n]_j = \lim_{k\to\infty} P\left\{N(t_k^f) = n, J(t_k^f) = j\right\}$ and $[\mathbf{m}_n]_j = \lim_{k\to\infty} P\left\{N(t_k^m) = n, J(t_k^m) = j\right\}$, respectively.

To obtain the unknowns $\mathbf{m}$, $\mathbf{m}^{(1)}$ in (6), we setup the system equations and compute the stationary probability vectors $\mathbf{m}_n$, $n \geq 0$ numerically.

Theorem 3. *In the vacation model with decrementing-K discipline the probability vectors $\mathbf{m}_n$, $n \geq 0$ are determined by the following system of equations:*

$$\sum_{n=0}^{K}\sum_{k=0}^{n} \mathbf{m}_k \mathbf{U}_{n-k} \mathbf{G}^n z^0 + \sum_{n=K+1}^{\infty}\sum_{k=0}^{n} \mathbf{m}_k \mathbf{U}_{n-k} \mathbf{G}^K z^{n-K} = \sum_{n=0}^{\infty} \mathbf{m}_n z^n. \tag{7}$$

Proof. According to the decrementing-K discipline either the server continues serving until the number of customers is decremented by K compared to those present at the beginning of service period or the queue becomes empty before. The phase change of the $BMAP$ during each decrement of the number of customers present at the beginning of the service is described by matrix $\mathbf{G}$. Using it the governing relation for transition $f \to m$ of the vacation model can be given as

$$\sum_{n=0}^{K} \mathbf{f}_n \mathbf{G}^n z^0 + \sum_{n=K+1}^{\infty} \mathbf{f}_n \mathbf{G}^K z^{n-K} = \sum_{n=0}^{\infty} \mathbf{m}_n z^n. \tag{8}$$

The number of customers at the end of the vacation equals the sum of those present at the start of the vacation and those who arrived during the vacation period. Therefore the governing relation for transition $m \to f$ of the vacation model can be expressed as

$$\sum_{k=0}^{n} \mathbf{m}_k \mathbf{U}_{n-k} = \mathbf{f}_n. \tag{9}$$

Combining (8) and (9) results in the statement. $\square$

Let $I_{\{x=0\}}$ denote the indicator of $x = 0$. Matrix $\mathbf{G}$ in (7) can be computed e.g. by applying the standard algorithm of Lucantoni [3]. To get the probability vectors $\mathbf{m}_n$, $n \geq 0$ a numerical method can be developed by setting a ρ dependent upper limit X for n in (7). Taking the x-th derivatives of (7) at $z = 1$ for $x = 0, \ldots, X$ leads to a system of linear equations

$$I_{\{x=0\}} \sum_{n=0}^{K-1} \sum_{k=0}^{n} \mathbf{m}_k \mathbf{U}_{n-k} \mathbf{G}^n + \sum_{n=K+x}^{X} \sum_{k=0}^{n} \mathbf{m}_k \mathbf{U}_{n-k} \mathbf{G}^K \frac{(n-K)!}{(n-K-x)!} = \sum_{n=x}^{X} \mathbf{m}_n \frac{n!}{(n-x)!}, \quad (10)$$

in which the number of equations and the number of unknowns is $L\,(X+1)$.

5. SPECIAL CASES

The exhaustive discipline is a special case of the decrementing-K discipline. Thus (7) and (10) can be applied also for the *vacation model with exhaustive discipline* by setting $K = \infty$.

Due to the phase dependent vacation property the *idle period* of a $BMAP/G/1$ queue can also be treated as a vacation. Therefore the $BMAP/G/1$ *queue* is a special case of the corresponding *vacation model with exhaustive discipline*. It can be shown by applying (16) of [3] that for the *idle period* $\widehat{\mathbf{V}}(z)$ is given explicitly as

$$\widehat{\mathbf{V}}(z) = \mathbf{I} - (\mathbf{D}_0)^{-1}\,\widehat{\mathbf{D}}(z)\,. \quad (11)$$

Hence the results (7) and (10) can be applied by setting $K = \infty$ and using (11).

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