Analysis of an M/M/1 Queueing System with Impatient Customers and a Variant of Multiple Vacation Policy*

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ABSTRACT

In this paper, we consider an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy, where we examine the case that customer impatience is due to the servers' vacation. Whenever a system becomes empty, the server takes a vacation. However, the server is allowed to take a maximum number of vacations, denoted by K vacations, if the system remains empty after the end of a vacation. We derive the probability generating functions of the steady-state probabilities and obtain the closed-form expressions of the system sizes when the server is in different states. In addition, we obtain the closed-form expressions for other important performance measures. Finally, we present some numerical results.

Keywords

Queue, a variant of multiple vacation, impatience, probability generating function, mean system size.

1. INTRODUCTION

Queueing systems with server vacations that can be used in modeling numerous real world queueing situations have arisen in systems such as manufacturing systems, communication systems, and production-inventory systems. There is now growing interest in the analysis of queueing systems with impatient customers. This is due to the potential application of such communication systems, call centers, production-inventory systems, and many other related areas; see for instance [1], [2] and the references therein.

Recently, Altman and Yechiali [3] presented a comprehensive analysis of M/M/1, M/G/1 and M/M/c queueing models with server vacations and customer impatience, where customers became impatient only when the servers were on vacation. They analyzed both the single and the multiple vacation cases, and obtained various closed-form results. Altman and Yechiali [4] investigated the M/M/∞ queuing model with impatient customers and vacations. They derived the probability generating function of the number of customers in the system and calculated values of key performance measures such as the mean queue size, the mean length of a busy period, and the proportion of customers being served without abandoning the system. Yue et al. [5] extended the M/M/1 model in [3] by analyzing an M/M/1 queueing model with customer impatience and working vacations.

In this paper, we extend the M/M/1 model in [3] by considering a variant of the multiple vacation policy which includes both a single vacation and multiple vacations. For this variant of multiple vacation policy, the server is allowed to take at a maximum number K of consecutive vacations if the system remains empty after the end of a vacation. This vacation schedule is the kind proposed by Zhang and Tian [6], with multiple adaptive vacations, where K is not a random variable, rather a preassigned fixed positive integer. This type of vacation is called a variant of the multiple vacation schedule used by Banik [7]. Literatures related to this type of vacation can be found in papers by Ke [8], Ke et al. [9] and references therein.

The rest of paper is organized as follows: In Section 2, we

*This work is supported by the National Natural Science Foundation of China (No. 71071133), and is supported in part by GRANT-IN-AID FOR SCIENCE RESEARCH (No. 21500086) and MEXT, Japan.

describe the model. In Section 3, we carry out the stationary analysis of the system. We first develop the differential equations for the probability generating functions of the steady-state probabilities. Then, by solving the differential equations, we obtain the closed-form expressions of the mean system sizes when the server is in different states. We also obtain the closed-form expressions of other important performance measures. Some numerical results are presented in Section 4. Conclusions are given in Section 5.

2. MODEL DESCRIPTION

We consider an M/M/1 queueing system with impatient customers and a variant of a multiple vacation policy. Customers arrive according to a Poisson process at rate λ. The service is provided by a single server, who serves the customers on a first-come first-served (FCFS) basis. The service times follow an exponential distribution with a service rate μ.

When the server finishes serving a customer and finds the system empty, the server leaves for a vacation. If the server finds a customer at a vacation completion instant, the server returns to serve customers immediately. Otherwise, the server will take vacations consecutively until the server has taken a maximum number of vacations, denoted by K vacations, and then the server stays idle and waits to serve the next arrival. The vacation times are assumed to be exponentially distributed with vacation rate γ.

During the vacation, customers become impatient. That is, whenever a customer arrives at the system, it activates an "impatience timer" T, which is exponentially distributed with parameter ξ. If the customer's service has not been completed before the customer's timer expires, the customer abandons the queue, never to return.

Remark 1. If $K = 1$, the current model reduces to the single vacation model. If $K = \infty$, the current model reduces to the multiple vacation model. Both the single vacation and multiple vacation models have been studied by Altman and Yechiali [3].

3. STATIONARY ANALYSIS

In this section, we present a stationary analysis for the model described in the last section. We first develop the differential equations for the probability generating functions of the steady-state probabilities. Then, by solving the differential equations, we obtain the closed-form expressions of the mean system sizes when the server is in different states. We also obtain the closed-form expressions of other important performance measures.

3.1 Generating Functions

Let $L(t)$ denote the number of customers in the system at time t, and let $J(t)$ denote the status of the server at time t, which is defined as follows: $J(t) = j$ denotes that the server is taking the $(j+1)$th vacation at time t for $j = 0, 1, ..., K-1$, while $J(t) = K$ denotes that the server is idle or busy at time t. Then, the process $\{(L(t), J(t)), t \geq 0\}$ defines a continuous-time Markov process with state space $\Omega = \{(n, j) : n \geq 0, j = 0, 1, ..., K\}$.

Let $P_{nj} = \lim_{t\to\infty} P\{L(t) = n, J(t) = j\}$, $n \geq 0$, $j = 0, 1, ..., K$, denote the steady-state probabilities of the process $\{(L(t), J(t)), t \geq 0\}$. Then, the set of balance equations is given as follows:

$$(\lambda+\gamma)P_{00} = \xi P_{10} + \mu P_{1K}, \tag{1}$$

$$(\lambda+\gamma+n\xi)P_{n0} = \lambda P_{n-10} + (n+1)\xi P_{n+10},\ n \geq 1, \tag{2}$$

$$(\lambda+\gamma)P_{0j} = \xi P_{1j} + \gamma P_{0j-1},\ j = 1, 2, ..., K-1, \tag{3}$$

$$(\lambda+\gamma+n\xi)P_{nj} = \lambda P_{n-1j} + (n+1)\xi P_{n+1j},\quad j = 1, 2, ..., K-1,\ n \geq 1, \tag{4}$$

$$\lambda P_{0K} = \gamma P_{0K-1}, \tag{5}$$

$$(\lambda+\mu)P_{nK} = \lambda P_{n-1K} + \mu P_{n+1K} + \gamma \sum_{j=0}^{K-1} P_{nj},\ n \geq 1 \tag{6}$$

and the normalizing condition:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K} P_{nj} = 1. \tag{7}$$

Define the probability generating functions (PGFs) as

$$G_j(z) = \sum_{n=0}^{\infty} P_{nj} z^n,\ 0 \leq z \leq 1,\ j = 0, 1, ..., K.$$

Define $G'_j(z) = \frac{d}{dz} G_j(z)$, $j = 0, 1, ..., K$.

Then, multiplying each equation for n in Eqs. (1), (2), (3) and (4) by z^n, and summing all possible values of n and re-arranging terms, we get

$$\xi(1-z)G'_0(z) - [\lambda(1-z)+\gamma]G_0(z) = -\mu P_{1K} \tag{8}$$

and

$$\xi(1-z)G'_j(z) - [\lambda(1-z)+\gamma]G_j(z) = -\gamma P_{0j-1},\quad j = 1, 2, ..., K-1. \tag{9}$$

Similarly, using Eqs. (5) and (6) we obtain

$$(1-z)(\lambda z-\mu)G_K(z) = \gamma z \sum_{j=0}^{K-1} G_j(z) + (z-1)\mu P_{0K} - [\mu P_{1K} + \gamma \sum_{j=0}^{K-2} P_{0j}]z. \tag{10}$$

In next subsection, we solve the differential equations (8) and (9) by following the method used in Altman and Yechiali [3] .

3.2 Solutions of the Differential Equations

Eq. (8) can be written as follows:

$$G'_0(z) - \left[\frac{\lambda}{\xi} + \frac{\gamma}{\xi(1-z)}\right] G_0(z) = -\frac{\mu P_{1K}}{\xi(1-z)}. \tag{11}$$

In order to solve the differential Eq. (11), we multiply both sides of Eq. (11) by

$$e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\gamma}{\xi}}.$$

Then, we get

$$\frac{d}{dz}\left[e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\gamma}{\xi}}G_0(z)\right] = -\frac{\mu}{\xi}P_{1K}e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\gamma}{\xi}-1}.$$

Integrating from 0 to z, we have

$$G_0(z) = \frac{e^{\frac{\lambda}{\xi}z}\left\{G_0(0) - \frac{\mu}{\xi}P_{1K}\int_0^z(1-x)^{\frac{\gamma}{\xi}-1}e^{-\frac{\lambda}{\xi}x}dx\right\}}{(1-z)^{\frac{\gamma}{\xi}}}. \quad (12)$$

Since $G_0(1) = \sum_{n=0}^{\infty}P_{n0} < \infty$ and $z=1$ is the root of the denominator of the right hand side of Eq. (12), we have that $z=1$ must be the root of the numerator of the right hand side of Eq. (12). So, we obtain

$$G_0(0) = \frac{\mu}{\xi}CP_{1K} \quad (13)$$

where

$$C = \int_0^1 e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\gamma}{\xi}-1}dx. \quad (14)$$

Noting $G_0(0) = P_{00}$, Eq. (13) implies

$$P_{1K} = \frac{\xi}{\mu C}P_{00}. \quad (15)$$

Substituting Eq. (15) into Eq. (12), we obtain

$$G_0(z) = \frac{e^{\frac{\lambda}{\xi}z}}{(1-z)^{\frac{\gamma}{\xi}}}\left[1 - \frac{1}{C}\int_0^z(1-x)^{\frac{\gamma}{\xi}-1}e^{-\frac{\lambda}{\xi}x}dx\right]P_{00}. \quad (16)$$

Eq. (9) can be written as

$$G_j'(z) - \left[\frac{\lambda}{\xi} + \frac{\gamma}{\xi(1-z)}\right]G_j(z) = -\frac{\gamma P_{0j-1}}{\xi(1-z)}. \quad (17)$$

In a similar manner used for solving Eq. (11), we get

$$G_j(z) = \frac{e^{\frac{\lambda}{\xi}z}\left\{G_j(0) - \frac{\gamma}{\xi}P_{0j-1}\int_0^z(1-x)^{\frac{\gamma}{\xi}-1}e^{-\frac{\lambda}{\xi}x}dx\right\}}{(1-z)^{\frac{\gamma}{\xi}}}, \quad j = 1,2,...,K-1. \quad (18)$$

Since $G_j(1) = \sum_{n=0}^{\infty}P_{nj} < \infty$ and $z=1$ is the root of the denominator of the right hand side of Eq. (18), we have that $z=1$ must be the root of the numerator of the right hand side of Eq. (18). So, we obtain

$$P_{0j} = G_j(0) = \frac{\gamma}{\xi}CP_{0j-1}, \; j = 1,2,...,K-1 \quad (19)$$

where C is defined by Eq. (14). Using Eq. (19) repeatedly, we obtain

$$P_{0j} = A^jP_{00}, \; j = 1,2,...,K-1 \quad (20)$$

where $A = \frac{\gamma}{\xi}C$. Substituting Eq. (20) into Eq. (18), we obtain

$$G_j(z) = \frac{e^{\frac{\lambda}{\xi}z}A^j}{(1-z)^{\frac{\gamma}{\xi}}}\left\{1 - \frac{1}{C}\int_0^z(1-x)^{\frac{\gamma}{\xi}-1}e^{-\frac{\lambda}{\xi}x}dx\right\}P_{00}, \quad j = 1,2,...,K-1. \quad (21)$$

Using Eqs. (5) and (20), we obtain

$$P_{0K} = \frac{\gamma}{\lambda}A^{K-1}P_{00}. \quad (22)$$

Remark 2. It is easy to check that $\xi - \gamma C > 0$, see also Altman and Yechiali [3] (see p. 263). Thus, we have $0 < A < 1$.

For $j = 0,1,...,K-1$, Eqs. (16) and (21) expres $G_j(z)$ in terms of P_{00}. Eqs. (15), (20) and (22) show that P_{1K}, P_{0j}, $j = 1,2,...,K$ are all expressed in terms of P_{00}. Thus, from Eq. (10), $G_K(z)$ can also be expressed in terms of P_{00}. Therefore, once P_{00} is calculated, $G_j(z)$, for $j = 0,1,...,K$, are completely determined.

In the next subsection, we derive the probability P_{00} and the mean system sizes when the server is in different states.

3.3 Mean System Sizes

For $j = 0,1,...,K$, let L_j be the system size when the server is in the state j. Then, $E(L_j)$ is the mean system size when the server is in the state j, which is defined by

$$E(L_j) = G_j'(1) = \sum_{n=1}^{\infty}nP_{nj}, \; j = 0,1,...,K.$$

That is, for $j = 0,1,...,K-1$, $E(L_j)$ represents the mean system size when the server is taking the $(j+1)$th vacation, and $E(L_K)$ represents the mean system size when the server is busy or idle. We first derive $E(L_j)$ for $j = 0,1,...,K-1$.

From Eq. (11), using L'Hopital rule, we get

$$\begin{aligned} G_0'(1) &= \lim_{z\to 1}\frac{[\lambda(1-z)+\gamma]G_0(z) - \mu P_{1K}}{\xi(1-z)} \\ &= \frac{-\lambda G_0(1) + \gamma G_0'(1)}{-\xi}. \end{aligned}$$

Thus, we get

$$(\gamma+\xi)G_0'(1) = \lambda G_0(1). \quad (23)$$

Similarly, from Eq. (17), we get

$$(\gamma+\xi)G_j'(1) = \lambda G_j(1), \; j = 1,2,...,K-1. \quad (24)$$

Eqs. (23) and (24) imply

$$E(L_j) = \frac{\lambda}{\gamma+\xi}G_j(1), \; j = 0,1,...,K-1. \quad (25)$$

For $j = 0,1,...,K$, let $P_{\cdot j} = G_j(1) = \sum_{n=0}^{\infty}P_{jn}$. Then, for $j = 0,1,...,K-1$, $P_{\cdot j}$ represents the probability that the server is taking the $(j+1)$th vacation, and $P_{\cdot K}$ represents the probability that the server is busy or idle.

From Eqs. (16) and Eq. (21), using L'Hopital rule, we get

$$P_{\cdot j} = G_j(1) = A^{j-1}P_{00}, \; j = 0,1,...,K-1. \quad (26)$$

Using Eq. (26), Eq. (25) can be written as

$$E(L_j) = \frac{\lambda}{\gamma+\xi}A^{j-1}P_{00}, \; j = 0,1,...,K-1. \quad (27)$$

Remark 3. From Eq. (27) and $0 < A < 1$, it is easy to see that the mean system size $E(L_j)$ is a decreasing convex function of j for $j = 0,1,...,K-1$.

Furthermore, the mean system size when the server is on vacation, denoted by $E(L_V)$, is obtained as follows:

$$E(L_V) = \sum_{j=0}^{K-1} E(L_j) = \frac{\lambda}{\gamma+\xi} \cdot \frac{1-A^K}{A(1-A)} P_{00}. \tag{28}$$

Next, we derive $P_{\cdot K}$ and P_{00}. From Eqs. (15), (20) and (26), we have $\mu P_{1K} = \gamma P_{\cdot 0}$ and $P_{0j-1} = P_{\cdot j}$, $j = 1, 2, ..., K-1$. Thus, we have

$$\mu P_{1K} + \gamma \sum_{j=0}^{K-2} P_{0j} = \gamma \sum_{j=0}^{K-1} P_{\cdot j}. \tag{29}$$

Using Eq. (29), Eq. (10) can be written as

$$G_K(z) = \frac{\gamma z}{\lambda z - \mu} \cdot \frac{\sum_{j=0}^{K-1}[G_j(z) - P_{\cdot j}]}{1-z} - \frac{\mu P_{0K}}{\lambda z - \mu}. \tag{30}$$

Applying L'Hopital rule, we get

$$G_K(1) = \frac{\gamma \sum_{j=0}^{K-1} G'_j(1) + \mu P_{0K}}{\mu - \lambda}. \tag{31}$$

Noting $G_K(1) = P_{\cdot K}$ and $G'_j(1) = E(L_j)$, $j = 0, 1, ..., K-1$, from Eq. (31), we obtain

$$P_{\cdot K} = \frac{\gamma \sum_{j=0}^{K-1} E(L_j) + \mu P_{0K}}{\mu - \lambda} \tag{32}$$

Substituting Eqs. (22) and (28) into Eq. (32), we get

$$P_{\cdot K} = \frac{\gamma}{\mu - \lambda}\left[\frac{\lambda}{\gamma+\xi} \cdot \frac{1-A^K}{A(1-A)} + \frac{\mu}{\lambda}A^{K-1}\right] P_{00}. \tag{33}$$

Using the definition of $P_{\cdot j}$, it is easy to see that the normalizing condition (7) can also be written as

$$\sum_{j=0}^{K} P_{\cdot j} = 1. \tag{34}$$

Substituting Eqs. (26) and (33) into Eq. (34), we get

$$P_{00} = \left\{\frac{\mu\gamma + (\mu-\lambda)\xi}{(\mu-\lambda)(\gamma+\xi)} \cdot \frac{1-A^K}{A(1-A)} + \frac{\mu\gamma}{\lambda(\mu-\lambda)}A^{K-1}\right\}^{-1}. \tag{35}$$

Remark 4. Obviously, from Eq. (32), the inequality $P_{\cdot K} > 0$ is equivalent to $\lambda < \mu$. So, $\lambda < \mu$ is a necessary condition for the stability of our system. Therefore, we assume thereafter that $\lambda < \mu$.

Substituting Eq. (35) into Eq. (28), we obtain

$$E(L_V) = \frac{\lambda^2(\mu-\lambda)}{\mu\gamma[\lambda + (\gamma+\xi)H(K)] + \lambda\xi(\mu-\lambda)} \tag{36}$$

where

$$H(K) = \frac{A^K(1-A)}{1-A^K}. \tag{37}$$

Remark 5. It is easy to see that $H(K)$ is a decreasing function of K, which implies that $E(L_V)$ increases with K.

Now, we derive $E(L_K)$. From Eq. (30), using L'Hopital rule, we derive

$$E(L_K) = \frac{\gamma}{2(\mu-\lambda)} \sum_{j=0}^{K-1} G''_j(1) + \frac{\mu\gamma}{(\mu-\lambda)^2} \sum_{j=0}^{K-1} G'_j(1) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K} \tag{38}$$

where $G''_j(1)$ is obtained by differentiating twice $G_j(z)$ at $z = 1$ for $j = 0, 1, ..., K-1$. Differentiating twice Eqs. (8) and (9), respectively, we obtain

$$-2\xi G''_j(z) + \xi(1-z)\frac{d^3}{dz^3}G_j(z) = [\lambda(1-z) + \gamma]G''_j(z) - 2\lambda G'_j(z),\ j = 0, 1, ..., K-1. \tag{39}$$

Letting $z = 1$ in Eq. (39), we get

$$G''_j(1) = \frac{2\lambda}{\gamma + 2\xi} G'_j(1),\ j = 0, 1, ..., K-1. \tag{40}$$

Substituting Eq. (40) into Eq. (38), we obtain

$$E(L_K) = \frac{\gamma}{\mu-\lambda}\left(\frac{\mu}{\mu-\lambda} + \frac{\lambda}{\gamma+2\xi}\right) E(L_V) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K} \tag{41}$$

where $E(L_V)$ is calculated by Eq. (36), and P_{0K} is calculated by using Eqs. (22) and (35) as follows:

$$P_{0K} = \frac{\gamma(\mu-\lambda)(\gamma+\xi)H(K)}{\mu\gamma[\lambda + (\gamma+\xi)H(K)] + \lambda\xi(\mu-\lambda)} \tag{42}$$

where $H(K)$ is given by Eq. (37).

Let L be the number of customers in the system. Then, the mean system size $E(L) = E(L_V) + E(L_K)$ can be calculated from Eqs. (36) and (41).

3.4 Special Cases

The single vacation and the multiple vacation are two special cases of the variant vacation policy discussed in this paper.

Case 1. Multiple vacation model. If $K = \infty$, then $H(\infty) = 0$. From Eqs. (36) and (41), we have

$$E(L_V) = \frac{\lambda(\mu-\lambda)}{\mu\gamma + \xi(\mu-\lambda)}$$

and

$$E(L_K) = \frac{\gamma}{\mu-\lambda}\left(\frac{\mu}{\mu-\lambda} + \frac{\lambda}{\gamma+2\xi}\right)\frac{\lambda\mu}{\mu\gamma + \xi(\mu-\lambda)}.$$

These results agree with the results given by Altman and Yechiali [3].

Case 2. Single vacation model. If $K = 1$, then $H(1) = A$. From Eqs. (36) and (41), we have

$$E(L_V) = \frac{\lambda^2(\mu-\lambda)}{\mu\gamma[\lambda + (\gamma+\xi)A] + \lambda\xi(\mu-\lambda)}$$

and

$$E(L_K) = \frac{\gamma}{\mu-\lambda}\left(\frac{\mu}{\mu-\lambda}+\frac{\lambda}{\gamma+2\xi}\right)E(L_V) + \frac{\lambda\mu}{(\mu-\lambda)^2}\times\frac{\gamma(\mu-\lambda)(\gamma+\xi)A}{\mu\gamma[\lambda+(\gamma+\xi)A]+\lambda\xi(\mu-\lambda)}.$$

These results agree with the results given by Altman and Yechiali [3].

3.5 Other Performance Measures

In this subsection, we derive some other important performance measures.

(1) Probability that the server is on vacation

The probability that the server is on vacation is given by

$$P_v = \sum_{j=0}^{K-1} P_{\cdot j}. \tag{43}$$

Substituting Eq. (26) into Eq. (43), we obtain

$$P_v = \frac{1-A^K}{A(1-A)}P_{00}.$$

Using Eq. (28), we get

$$P_v = \frac{\gamma+\xi}{\lambda}E(L_V) \tag{44}$$

where $E(L_V)$ is given by Eq. (36).

(2) Probability that the server is busy

The probability that the server is busy is given by

$$P_b = \sum_{n=1}^{\infty} P_{nK} = 1 - P_{0K} - P_v. \tag{45}$$

Substituting Eqs. (42) and Eq. (44) into Eq. (45) and using Eq. (36), we obtain

$$P_b = \frac{\lambda\gamma[\lambda+(\gamma+\xi)H(K)]}{\mu\gamma[\lambda+(\gamma+\xi)H(K)]+\lambda\xi(\mu-\lambda)}. \tag{46}$$

Using a continuous variable x instead of the integer K in the right hand side of Eq. (46), we get a function of x, denoted by $Q(x)$. Taking the derivative of $Q(x)$ with respect to x, we obtain

$$Q'(x) = \frac{\lambda^2\gamma\xi(\mu-\lambda)(\gamma+\xi)H'(x)}{\{\mu\gamma[\lambda+(\gamma+\xi)H(x)]+\lambda\xi(\mu-\lambda)\}^2} < 0.$$

The inequality follows from the fact that $H'(x) < 0$. So, $Q(x)$ is a decreasing function. Therefore, P_b decreases with K.

(3) Proportion of customers served

Clearly, the expected number of customers served per unit of time is μP_b, implying that the proportion of customers served is given by

$$P_s = \frac{\mu P_b}{\lambda} \tag{47}$$

where P_b is given by Eq. (46).

(4) Average rate of abandonment due to impatience

When the system is in state $(0, n)$, $n \geq 1$, the rate of abandonment of a customer due to impatience is $n\xi$. Thus, the average rate of abandonment due to impatience is given by

$$R_a = \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} n\xi P_{nj} = \xi E(L_V) \tag{48}$$

where $E(L_V)$ is given by Eq. (36).

Remark 6. Clearly, from Eqs. (44) and (48), both P_v and R_a, as functions of K, have the same monotonicity as $E(L_V)$. Since $E(L_V)$ increases with K, we have that both P_v and R_a increase with K.

4. NUMERICAL RESULTS

For multiple vacation policy model, Altman and Yechiali [3] consider the monotonicity of some performance measures with respect to the parameter ξ. In this section, we investigate numerically the effects of the parameter ξ and K on some performance measures.

We choose parameters: $\lambda = 4$, $\mu = 5$, and $\gamma = 2$. In Fig. 1 and Fig. 2, the effects of parameters ξ and K on $E(L_V)$ and $E(L_K)$ are presented, where $E(L_V)$ is the mean system size when the server is on vacation and $E(L_K)$ is the mean system size when the server is busy or idle. In Table 1 and Table 2, the variations of some performance measures with K are presented for various ξ. In Table 1, the values of ξ are chosen to be small, i.e., $\xi = 0.5$, 1.0, and 1.5, and in Table 2, the values of ξ are chosen to be large, i.e., $\xi = 2.5$, 3.0, and 3.5.

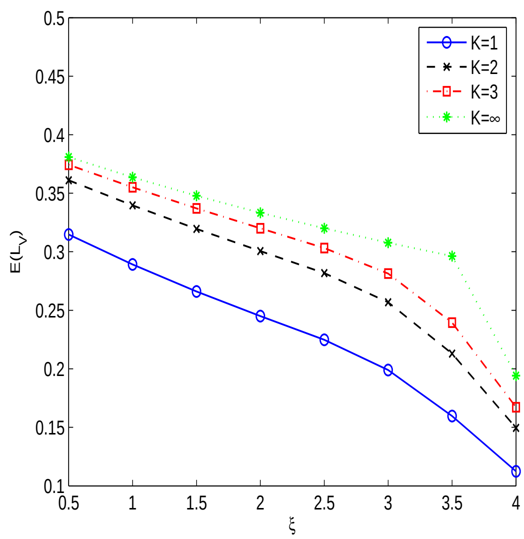

Figure 1: Effects of ξ and K on the mean system size $E(L_V)$ when the server is on vacation.

For multiple vacation policy model, Altman and Yechiali [3] show that the probability P_v that the server is on vacation is an increasing convex function of ξ and the probability P_b that the server is working is a decreasing concave function

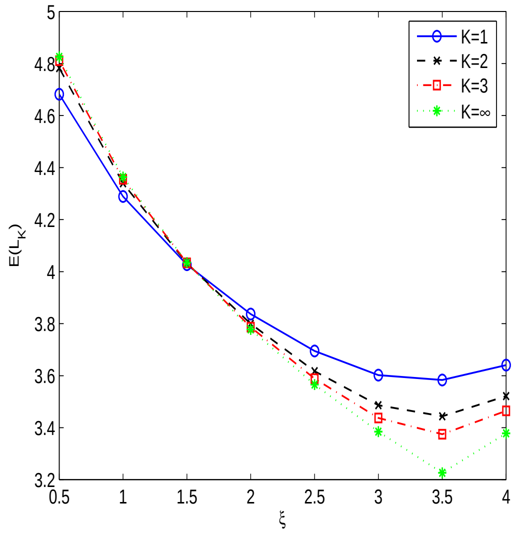

Figure 2: Effects of ξ and K on the mean system size $E(L_K)$ when the server is busy or idle.

of ξ. They also show that $E(L_V)$ behaves similar to P_b with respect to ξ. These results are also observable from Fig. 1 and Fig. 2.

Table 1: Performance Measures with Variations of ξ and K for $\xi = 0.5, 1.0, 1.5$.

K	ξ	$E(L_V)$	$E(L_K)$	$E(L)$	P_v	P_b	R_a
1	0.5	0.3147	4.6818	4.9964	0.1967	0.7685	0.1573
	1.0	0.2892	4.2892	4.5784	0.2169	0.7422	0.2892
	1.5	0.2660	4.0266	4.2926	0.2328	0.7202	0.3991
2	0.5	0.3611	4.7823	5.1434	0.2257	0.7639	0.1805
	1.0	0.3397	4.3397	4.6794	0.2548	0.7321	0.3397
	1.5	0.3196	4.0320	4.3515	0.2796	0.7041	0.4793
3	0.5	0.3743	4.8109	5.1852	0.2339	0.7626	0.1871
	1.0	0.3551	4.3551	4.7102	0.2663	0.7290	0.3551
	1.5	0.3370	4.0337	4.3707	0.2949	0.6989	0.5055
4	0.5	0.3786	4.8204	5.1990	0.2366	0.7621	0.1893
	1.0	0.3605	4.3605	4.7209	0.2704	0.7279	0.3605
	1.5	0.3436	4.0344	4.3779	0.3006	0.6969	0.5153
5	0.5	0.3801	4.8236	5.2038	0.2376	0.7620	0.1901
	1.0	0.3624	4.3624	4.7249	0.2718	0.7275	0.3624
	1.5	0.3461	4.0346	4.3807	0.3029	0.6962	0.5192
∞	0.5	0.3810	4.8254	5.2063	0.2381	0.7619	0.1905
	1.0	0.3636	4.3636	4.7273	0.2727	0.7273	0.3636
	1.5	0.3478	4.0348	4.3826	0.3043	0.6957	0.5217

From Table 1 and Table 2, we observe that $E(L_K)$ and the mean system size $E(L)$ all decrease with ξ for any finite K. However, from Table 2, we observe that P_v and P_b neither increase nor decrease with ξ when $K = 2$ and $K = 3$. That means that P_v and P_b are not monotone functions of ξ when $K \neq \infty$.

From Remark 5 in Subsection 3.3, we know that $E(L_V)$ is a increasing function of K. This property can also be observed from Fig. 1. However, Fig. 2 shows that $E(L_K)$ increases initially with K and then decreases with K. So, a threshold value ξ_0 may exist such that $E(L_K)$ increases with K if $0 < \xi < \xi_0$ and decreases with K if $\xi > \xi_0$. We also observe that $E(L)$ increases with K when ξ is less than a threshold value and then decreases with K when ξ is larger than this threshold value. That is, $E(L)$ as a function of K behaves similar to $E(L_K)$.

Table 2: Performance Measures with Variations of ξ and K for $\xi = 2.5, 3.0, 3.5$.

K	ξ	$E(L_V)$	$E(L_K)$	$E(L)$	P_v	P_b	R_a
1	2.5	0.2248	3.6949	4.9197	0.2529	0.6876	0.5620
	3.0	0.1989	3.6022	3.8011	0.2486	0.6807	0.5967
	3.5	0.1596	3.5833	3.7429	0.2195	0.6883	0.5586
2	2.5	0.2818	3.6175	4.8994	0.3170	0.6591	0.7045
	3.0	0.2568	3.4864	3.7432	0.3210	0.6459	0.7703
	3.5	0.2130	3.4439	3.6569	0.2928	0.6509	0.7454
3	2.5	0.3032	3.5885	4.8917	0.3411	0.6484	0.7580
	3.0	0.2814	3.4372	3.7186	0.3517	0.6312	0.8442
	3.5	0.2394	3.3749	3.6143	0.3292	0.6324	0.8379
5	2.5	0.3164	3.5705	4.8870	0.3560	0.6418	0.7911
	3.0	0.2998	3.4003	3.7002	0.3748	0.6201	0.8995
	3.5	0.2651	3.3077	3.5728	0.3646	0.6144	0.9279
7	2.5	0.3192	3.5668	4.8860	0.3591	0.6404	0.7980
	3.0	0.3052	3.3896	3.6948	0.3815	0.6169	0.9156
	3.5	0.2773	3.2760	3.5533	0.3813	0.6059	0.9705
∞	2.5	0.3200	3.5657	3.8857	0.3600	0.6400	0.8000
	3.0	0.3077	3.3846	3.6923	0.3846	0.6154	0.9231
	3.5	0.2963	3.2263	3.5226	0.4074	0.5926	1.3070

From Table 1 and Table 2, it is observed that P_v and R_a increase with K, while P_b decreases with K. This agrees with the results we obtained in Subsection 3.5.

5. CONCLUSIONS

In this paper, we have studied an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy, where the customer impatience is due to the server's vacation. We have obtained the closed-form expressions of the system sizes when the server is in different states. We have also obtained the closed-form expressions for other important performance measures. The effects of the parameters ξ and K on some performance measures have been investigated numerically.

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