On-off Markov Reward Models *

Khalid Begain[1], László Jereb[2], Antonio Puliafito[3], Miklós Telek[2]

[1] Department of Computer Science
Mu'tah University, 61710 Mu'tah, Jordan

[2] Department of Telecommunications
Technical University of Budapest, 1521 Budapest, Hungary

[3] Istituto di Informatica e Telecomunicazioni
Università di Catania, 95125 Catania, Italy

Abstract

The analysis of Markov Reward Models with preemptive resume policy usually results in a double transform expression, whose solution is based on an inverse transformation, both in the time and in the reward variable domains. This paper discusses the case when the reward rates can be described only by 0 *or positive c values. These on-off Markov Reward Models are analyzed and a symbolic solution is presented, from which numerical solution can be obtained by a computationally effective method. The mean completion time and the probability distribution of states at the completion are evaluated.*

Key words: Markov Reward Models, Preemptive resume policy, Completion time, Accumulated reward.

1 Introduction

The properties of stochastic reward processes have been studied for a long time [4]. However, the class of Stochastic Reward Models (SRMs) only recently has drawn attention as a modelling tool in the performability (performance and reliability) evaluation. Indeed, the possibility of associating a reward variable to each state increases the descriptive power and the flexibility of the model. Different interpretations of the underlying process and of the associated reward structure can be used to describe different applications [8]. Based on the distribution of the accumulated reward in a given period or on the distribution of the time needed to accumulate a predefined (possible random) amount of reward, a great number of measures can be introduced and used in performability evaluations. Usual assignments of the reward rates are: execution rates of tasks in computing systems, number of active processors, throughput, and so on. Moreover, the most important measures of the classical reliability theory [1] can be derived as particular cases of *SRMs* constraining the reward rates to be binary variables.

Kulkarni et al. [6] derived the closed form Laplace transform equations for the time to accumulate a given amount of reward in the case when the underlying stochastic process is a Continuous Time Markov Chain (*CTMC*). We refer these cases as Markov Reward Models (*MRM*s).

Various numerical techniques for the evaluation of the performability have been investigated in recent papers: [3, 5, 7]. In this paper, we propose a computationally effective approach to

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the analysis of the class of *MRMs* with on-off reward accumulation. These models play crucial role in the description of the maintenance aspects of some computer/communication systems.

The paper is organized as follows. Section 2 provides a formal definition of *SRM*s, and introduces the studied subset of *MRM*s. In Section 3 the proposed analysis of the on-off *MRM*s is presented. Finally, the paper is concluded.

2 Markov Reward Models

The adopted modeling framework consists of describing the behaviour of the system configuration in time by means of a stochastic process, and by associating a non-negative real number to each state of the underlying process. The real variable associated to each state is called the *reward rate* [4] and it represents the effective working capacity, the performance level, the cost or the stress of the system in the given state.

Let the underlying stochastic process $Z(t)$ $(t \geq 0)$ referred as *structure-state process* be a stochastic process defined over a discrete and finite state space Ω of cardinality n. Let f be a non-negative real-valued function defined as:

$$f[Z(t)] = r_i \geq 0 \ , \qquad \text{if} \quad Z(t) = i \tag{1}$$

where $f[Z(t)]$ represents the instantaneous reward rate associated to state i.

Definition 1 *The* **accumulated reward** $B(t)$ *is a random variable which represents the accumulation of reward in time,*

$$B(t) = \int_0^t f[Z(u)]du = \int_0^t r_{Z(u)}du$$

$B(t)$ is a stochastic process that depends on $Z(u)$ for $0 \leq u \leq t$. According to Definition 1, this paper restricts the attention to the class of models in which no state transition can entail to a loss of the accumulated reward. A *SRM* of this kind is called *preemptive resume* (prs) model. Let us define the distribution of the accumulated reward:

$$B(t,w) = Pr\{B(t) < w\}$$

Definition 2 *The* **completion time** C *is the random variable representing the time to accumulate a reward requirement being equal to a random variable* W*:*

$$C = min\,[t \geq 0 \, : \, B(t) = W]$$

With other words, C is the time at which the reward accumulated by the system reaches the value W for the first time.

Assume, in general, that W is a random variable with distribution $G(w)$ with support on $(0, \infty)$. Obviously, this definition captures the degenerate case, when W is deterministic as well. For a given sample of $W = w$, the completion time $C(w)$ is defined as:

$$C(w) = \min\,[t \geq 0 \, : \, B(t) = w] \ . \tag{2}$$

Let $C(t,w)$ be the *Cdf* (Cumulative distribution function) of the completion time when the work requirement is w:

$$C(t,w) = Pr\,\{C(w) \leq t\} \tag{3}$$

The completion time C of a *prs* SRM is characterized by the following distribution:

$$CD(t) = Pr\,\{\,C \leq t\} = \int_0^\infty C(t,w)\,dG(w) \tag{4}$$

In case of *prs* reward accumulation the distribution of the completion time is closely related to the distribution of the accumulated reward by means of the following relation:

$$B(t,w) = Pr\{B(t) \leq w\} = Pr\{C(w) \geq t\} = 1 - C(t,w) \tag{5}$$

Finally, we introduce the following matrix functions $\mathbf{P}(t,w)$ and $\mathbf{F}(t,w)$:

$$P_{ij}(t,w) = Pr\{Z(t) = j\, ,\, B(t) \leq w \mid Z(0) = i\} \tag{6}$$

$$F_{ij}(t,w) = Pr\{Z(C(w)) = j\, ,\, C(w) \leq t \mid Z(0) = i\} \tag{7}$$

where

- $P_{ij}(t,w)$ defines the (possibly defective) state dependent distribution of the accumulated reward at time t before completion supposed that the initial state of the structure state process is i,
- $F_{ij}(t,w)$ means the (possibly defective) state dependent distribution of the completion time supposed that the initial state of the structure state process is i.

From (6) and (7), it follows for any t:

$$\sum_{j \in \Omega} [\, P_{ij}(t,w) + F_{ij}(t,w) \,] = 1$$

Given that $G(w)$ is the cumulative distribution function of the random work requirement W, the distribution of the completion time can be written as:

$$CD(t) = \int_{w=0}^{\infty} \left[\sum_{i \in \Omega} \sum_{j \in \Omega} P_i(0) F_{ij}(t,w) \right] dG(w) = \int_{w=0}^{\infty} P(0)\, \mathbf{F}(t,w)\, h\, dG(w) \tag{8}$$

where $P(0)$ is the row vector of the initial probabilities, and h is the column vector with all elements being equal to 1.

2.1 Markov Reward Models

If the structure-state process $Z(t)$ is a *CTMC* with the infinitesimal generator $\mathbf{A}$, the above introduced matrix functions can be described in the double transform domain. The detailed derivations of these functions are presented in [2, 6, 10], and the final expressions obtained are as follows:

$$(s + vr_k)\, F_{ij}^{\sim *}(s,v) = \delta_{ij}\; r_i \;+ \sum_{k \in R} a_{ik}\, F_{kj}^{\sim *}(s,v) \tag{9}$$

$$(s + vr_k)\, P_{ij}^{\sim *}(s,v) = \delta_{ij}\; \frac{s}{v} \;+ \sum_{k \in R} a_{ik}\, P_{kj}^{\sim *}(s,v) \tag{10}$$

where $^{\sim}$ denotes the Laplace-Stieltjes transform with respect to $t \to s$ and * denotes the Laplace transform with respect to $w \to v$. Equations (9) and (10) can be rewritten in matrix form:

$$\mathbf{F}^{\sim *}(s,v) = (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R} \tag{11}$$

$$\mathbf{P}^{\sim *}(s,v) = \frac{s}{v}\, (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \tag{12}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{R}$ is the diagonal matrix of the reward rates $\{r_k\}$. The dimensions of $\mathbf{I}$, $\mathbf{R}$, $\mathbf{A}$, $\mathbf{F}$ and $\mathbf{P}$ are $(n \times n)$.

Starting from Equations (9-12), a natural way to evaluate the reward measures of a *MRM* consists of the following steps:

1. derivation of the entries of the $\mathbf{P}^{\sim *}(s, v)$ and $\mathbf{F}^{\sim *}(s, v)$ matrices symbolically in the double transform domain by (11) and (12);
2. symbolical inverse Laplace-stieltjes transformation of $\mathbf{P}^{\sim *}(s, v)$ and/or $\mathbf{F}^{\sim *}(s, v)$ with respect to s;
3. numerical inverse Laplace transformation with respect to v;
4. unconditioning the result according to the Cdf of the work requirement (8).

However, the analysis contains some computationally intensive steps, and the whole procedure can be applied only to very small scale problems (5-10 states).

3 Analysis of on-off Markov Reward Models

Definition 3 *The subclass of MRMs in which the reward rates can only be 0 or a positive value c is called on-off MRM.*

There are several practical examples which results in an on-off MRM and moreover, most of the classical reliability measures based on Markovian models can be described with on-off MRMs.

The analysis of an on-off MRM can always be transformed into the analysis of an on-off MRM with binary reward rates (i.e. all the positive reward rates equal 1), by means of the following equation:

$$CD(t) \;=\; Pr\{\, C \le t\} \;=\; \int_0^\infty C^b(t, w/c)\, dG(w) \qquad (13)$$

where $C^b(t, w)$ is the distribution of the completion time of the same MRM with binary reward rates, and the superscript b referes the binary reward rates case in the following.

According to the associated reward rates the states of a on-off MRM can be devided into two parts, namely S and $\Omega - S$, where S contains the states with positive reward rates. Suppose that S contains m states out of n. Thus, we can renumber the states in $\mathbf{\Omega}$ in such a way that the states indexed $1, 2, \ldots, m$ belong to S and the states numbered $m+1, m+2, \ldots, n$ belong to $\Omega - S$. Using this ordering of states, $\mathbf{A}$ can be partitioned into the following matrix blocks $\mathbf{A} = \begin{array}{|c|c|} \hline \mathbf{A_1} & \mathbf{A_2} \\ \hline \mathbf{A_3} & \mathbf{A_4} \\ \hline \end{array}$, where $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$, and $\mathbf{A_4}$ describes the transitions inside S, from S to $\Omega - S$, from $\Omega - S$ to R, and inside $\Omega - S$, respectively.

3.1 Mean completion time of on-off Markow Reward Models

Using (13), the mean completion time of an on-off MRM can be evaluated based on $\mathrm{E}(C^b(w))$.

Theorem 1 *The expected time while an on-off MRM with binary reward rates completes w amount of work is:*

$$\mathrm{E}(C^b(w)) = P(0) \begin{bmatrix} L(w) & -L(w)\mathrm{A}_2\mathrm{A}_4{}^{-1} \\ -\mathrm{A}_4{}^{-1}\mathrm{A}_3 L(w) & \mathrm{A}_4{}^{-1} + \mathrm{A}_4{}^{-1}\mathrm{A}_3 L(w)\mathrm{A}_2\mathrm{A}_4{}^{-1} \end{bmatrix} h \qquad (14)$$

where

$$\beta = \mathrm{A}_1 - \mathrm{A}_2\mathrm{A}_4{}^{-1}\mathrm{A}_3 \quad , \quad L(w) = \int_0^w e^{\beta w} dw.$$

Proof:

$$\mathrm{E}(C^b(w)) = \int_{t=0}^{\infty} 1 - C^b(t,w)\,dt = \int_{t=0}^{\infty} B^b(t,w)\,dt = \lim_{s\to 0} \frac{1}{s} B^{b\sim}(s,w) =$$

$$\lim_{s\to 0} P(0)^T \mathbf{P}^{b\sim}(s,w) h = P(0)\ \mathrm{LT}^{-1}\left[\frac{1}{v}(v\mathbf{R}-\mathbf{A})^{-1}\right]\ h$$

Let us consider the term $\mathrm{LT}^{-1}\left[\frac{1}{v}(v\mathbf{R}-\mathbf{A})^{-1}\right]$ separately. Based on the numbering of the states $\mathbf{R}$ has the form $\mathbf{R} = \begin{array}{|c|c|}\hline I_1 & 0 \\ \hline 0 & 0 \\ \hline\end{array}$, where I_1 is the identity matrix of dimension $(m \times m)$. Using this and the inverse of a partitioned matrix [9], the inverse Laplace transform is as follows:

$$\mathrm{LT}^{-1}\left[\frac{1}{v}(v\mathbf{R}-\mathbf{A})^{-1}\right] = \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} v\mathrm{I}_1 - \mathrm{A}_1 & -\mathrm{A}_2 \\ -\mathrm{A}_3 & -\mathrm{A}_4 \end{bmatrix}^{-1}\right\} =$$

$$\mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} (v\mathrm{I}_1-\beta)^{-1} & -(v\mathrm{I}_1-\beta)^{-1}\mathrm{A}_2{\mathrm{A}_4}^{-1} \\ -{\mathrm{A}_4}^{-1}\mathrm{A}_3(v\mathrm{I}_1-\beta)^{-1} & {\mathrm{A}_4}^{-1} + {\mathrm{A}_4}^{-1}\mathrm{A}_3(v\mathrm{I}_1-\beta)^{-1}\mathrm{A}_2{\mathrm{A}_4}^{-1} \end{bmatrix}\right\} = \quad (15)$$

$$\begin{bmatrix} L(w) & -L(w)\mathrm{A}_2{\mathrm{A}_4}^{-1} \\ -{\mathrm{A}_4}^{-1}\mathrm{A}_3 L(w) & {\mathrm{A}_4}^{-1} + {\mathrm{A}_4}^{-1}\mathrm{A}_3 L(w)\mathrm{A}_2{\mathrm{A}_4}^{-1} \end{bmatrix}$$

From 15 the theorem follows. □

3.2 State probability distribution at completion

A further important analysis problem of SRMs is the probability distribution of the structure state process when it completes. The required maintenance after a mission of the system can be estimated based on this performance measure. Define

$$P^c_{ij} = Pr\{Z(C) = j | Z(0) = i\}$$

Theorem 2 *The probability of being in state j at completion given that the process started from state i is:*

$$P^c_{ij} = \int_{w=0}^{\infty} \begin{bmatrix} e^{\beta w} & 0 \\ -{\mathrm{A}_4}^{-1}\mathrm{A}_3 e^{\beta w} & 0 \end{bmatrix}_{ij} dG(cw) \quad (16)$$

Proof:

$$P^c_{ij} = \lim_{t\to\infty}\int_{w=0}^{\infty} F^b_{ij}(t,w)\,dG(cw) = \lim_{s\to 0}\int_{w=0}^{\infty} F^b_{ij}{}^{\sim}(s,w)\,dG(cw) =$$

$$\int_{w=0}^{\infty} \mathrm{LT}^{-1}\left[(v\mathbf{R}-\mathbf{A})^{-1}\mathbf{R}\right]_{ij}\ dG(cw) \quad (17)$$

Let us consider the term $\mathrm{LT}^{-1}\left[(v\mathbf{R}-\mathbf{A})^{-1}\mathbf{R}\right]$ separately. By the partitioned form of $\mathbf{R}$ and $\mathbf{A}$ the inverse Laplace transform satisfies the following equation:

$$\mathrm{LT}^{-1}\left[(v\mathbf{R}-\mathbf{A})^{-1}\mathbf{R}\right]=\mathrm{LT}^{-1}\left\{\begin{bmatrix} v\mathrm{I}_1-\mathrm{A}_1 & -\mathrm{A}_2 \\ -\mathrm{A}_3 & -\mathrm{A}_4 \end{bmatrix}^{-1}\mathbf{R}\right\}=$$

$$\mathrm{LT}^{-1}\begin{bmatrix} (v\mathrm{I}_1-\beta)^{-1} & 0 \\ -\mathrm{A}_4{}^{-1}\mathrm{A}_3(v\mathrm{I}_1-\beta)^{-1} & 0 \end{bmatrix}=\begin{bmatrix} e^{\beta w} & 0 \\ -\mathrm{A}_4{}^{-1}\mathrm{A}_3 e^{\beta w} & 0 \end{bmatrix} \tag{18}$$

From which the Theorem comes. □
We can see that P^c_{ij} equals to 0 if $j \in \Omega - S$, since the accumulated reward does not increase in those states.

4 Conclusions

Markov Reward Models are widely used to model performability of computer/communication systems. The on-off *MRMs* represent a sub-class of MRMs of practical interest in many real situations. Focusing on an on-off assignment of reward variables the analitical description of MRMs is discussed. Performance indices such as the mean completion time and the probability distribution of states at the completion instant are derived and a numerically effective computational method is described

The complexity of the algorithm is related to the evaluation of $e^{\beta t}$, where β is a $(m \text{ x } m)$ matrix, and to the evaluation of the inverse of matrix A_4 with cardinality $((n-m) \text{ x } (n-m))$.

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