A Modeling Framework to Implement Preemption Policies in non-Markovian SPNs

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Abstract

Petri nets represent a useful tool for performance, dependability and performability analysis of complex systems. Their modeling power can be increased even more if non-exponentially distributed events are considered. However, the inclusion of non-exponential distributions destroys the memoryless property and requires to specify how the marking process is conditioned upon its past history. In this paper, we consider, in particular, the class of stochastic Petri nets whose marking process can be mapped into a Markov regenerative process.

An adequate mathematical framework is developed to deal with the considered class of Markov Regenerative Stochastic Petri Nets *(MRSPN)*. An unified approach for the solution of *MRSPN*s where different preemption policies can be defined in the same model is presented. The solution is provided both in steady-state and in transient condition. An example concludes the paper.

Key words: Stochastic Petri Nets, Markov regenerative processes, preemptive policies, transient and steady-state analysis.

1 Introduction

Stochastic Petri Nets (*SPN*) provide a well known specification language for the modeling and analysis of stochastic systems. Over the years, many extensions to the basic model

have appeared in the literature. Some of these extensions are a matter of convenience, mainly regarding graphical representation, and some others increase the modeling power. While the usual definition of Stochastic Petri Nets (*SPN*) is based on the assumption that all the firing times are exponentially distributed, in this paper we consider the implication of associating generally distributed firing times (including the deterministic) to the timed transitions.

Dealing with non-exponentially distributed events widens the field of applicability of *PN*-based modeling tools to real situations, but destroys the memoryless property of the underlying marking process.

There are a great number of circumstances in which deterministic or generally distributed events occur. Events such as timeouts in a protocol, service times in a manufacturing system, hard deadlines in real-time systems, memory access or instruction execution in a low-level hardware or software have durations which are constant or have a very low coefficient of variation. Continuous [14] or Discrete [9] Phase-type distributions can be used to approximate the occurrence time of a generally distributed event, but this method leads to a prohibitive size for the expanded state space and the error incurred in the final performance parameters can not be estimated; furthermore, one of the possible preemption mechanisms cannot be captured.

Choi et al. have shown in [7, 8] that the marking process underlying a *Stochastic Petri Net (SPN)*, where at most one generally distributed transition is enabled in each marking, belongs to the class of Markov Regenerative Processes *(MRGP)*. For this reason, they referred to this new class of *PN* as *Markov Regenerative Stochastic Petri Net (MRSPN)*. Various contributions have recently followed this line of research [16, 10, 17, 23, 24, 27].

The analysis technique proposed for this class of models, consists in identifying a sequence of time points, indicated as *regeneration time points (RTP)*, at which the marking process enjoys the Markov property: i.e. the future evolution depends only on the state entered at a given *RTP*. Based on the sequence of the regeneration time points, an analytical formulation of the process is available [13, 22].

All the mentioned references on *MRSPN*s implicitly assume an *enabling memory policy* [1] for the non-exponential transitions, and the resampling of the firing time each time the corresponding transition is disabled or fires. This policy is also known as the *preemptive repeat different (prd)* policy. The authors have enlarged the previously considered class of *MRSPN*s by introducing the concept of marking processes with non-overlapping dominant transitions [5]. In this framework, new preemption policies [30, 3, 32] can be accommodated. With the *preemptive resume (prs)* policy an interrupted event can be restarted by resuming the work already done before the interruption. This policy was referred to as *age memory* policy in [1]. With a *preemptive repeat identical (pri)* policy an interrupted event is restarted with an *identical* firing time.

A natural objection to the implementation of *PN* models with generally distributed events and complex combinations of preemption policies is that they are very hard to formulate and solve. The authors reply is based on the following argumets:

i) - the world is not necessarily exponential: the use of exponential distributions is often matter of analytical convenience rather than of motivated modeling assumptions.

ii) - theoretical research work is preliminary to the discovery of practical results and the successive implementation of tools.

iii) - simulation approaches require also the definition of a well established and clearly specified modeling environment [18].

iv) - effective numerical methods, presented in this paper, are already available for the steady state analysis of *MRSPN*s, and are ready to be integrated into a tool.

The present paper is an effort to offer a contribution in the above directions and is an attempt to synthesize the recent research activity of the authors in the area of *MRSPN*s by providing a common formalism and a common solution technique.

The paper is organized as follows. Section 2 discusses the inclusion of generally distributed transitions into a *SPN* and defines the concept of *execution policy*. Section 3 defines the class of *MRSPN*s examined in this paper. The analysis of this class, based on the theory of Markov regenerative processes, is then considered. It is shown that the underlying process can be decomposed into independent subproblems consisting in considering the evolution of the marking process between two consecutive regeneration time points. The analysis of a single subordinated process is carried on in Section 4, by a proper partitioning of the state space. Moreover, particular cases are examined, when the subordinated process is a *Continuous Time Markov Chain (CTMC)* or a *Semi-Markov Process (SMP)*. The steady-state analysis is dealt with in Section 5, and a computationally effective method is derived in the case of subordinated *CTMC*. An example with mixed preemption policies is evaluated in Section 6. Section 7 discusses the complexity of the presented methodology. Finally, Section 8 concludes the paper.

2 The individual memory model

A non-Markovian *SPN* is a stochastically timed *PN* in which the time evolution of the marking process can be more general than a *CTMC*. In the spirit of many modeling formalisms [19], in which the complexity of the solution must be hidden to the modeler, the way in which the future evolution of the marking process depends on its past history needs to be specified at the *PN* level.

We adhere to the model with generally distributed firing times and with individual memory policies proposed in [1]. We refer to this model as *Generally Distributed Transition-SPN (GDT-SPN)*. Formally, a *GDT-SPN* is a tuple $PN = (P, Tr, I, O, H, G, \mu, M)$, where:

- P (of cardinality $||P||$) is the set of places (drawn as circles);
- Tr (of cardinality $||Tr||$) is the set of transitions (drawn as bars);
- I, O and H are the input, the output and the inhibitor functions, respectively. The input function I provides the multiplicities of the input arcs from places to transitions; the output function O provides the multiplicities of the output arcs from transitions to places; the inhibitor function H provides the multiplicity of the inhibitor arcs from places to transitions.

- G (of cardinality $||Tr||$) is the set of random variables γ_k associated to each transition tr_k, being $G_k(t)$ the corresponding cdf.
- μ (of cardinality $||Tr||$) is the set of execution policies[1] u_k associated to each transition tr_k. $\mu_k = (a_k, \iota_k)$ is composed by two elements: the age variable a_k and the indicator resampling variable ι_k.
- M (of cardinality $||P||$) is the marking. The generic entry m_i is the number of tokens (drawn as black dots) in place p_i, in marking M.

Input and output arcs have an arrowhead on their destination, inhibitor arcs have a small circle. A transition is enabled in a marking if each of its ordinary input places contains at least as many tokens as the multiplicity of the input function I and each of its inhibitor input places contains fewer tokens than the multiplicity of the inhibitor function H. An enabled transition fires by removing as many tokens as the multiplicity of the input function I from each ordinary input place, and adding as many tokens as the multiplicity of the output function O to each output place. The number of tokens in an inhibitor input place is not affected. The reachability set $\mathcal{R}(M_0)$ is the set of all the markings that can be generated from an initial marking M_0 by repeated application of the above rules in an untimed net.

For the sake of simplicity, in the present formulation, the set Tr contains only timed transitions. However, immediate transitions could be easily accommodated in the proposed framework for the analysis of *MRSPN*s, as it will be indicated in the sequel of the paper.

In a stochastically timed *PN*, a natural choice to select the next timed transition to fire among those enabled in a given marking is according to a *race policy*: if more than one timed transition is enabled, the transition fires whose associated delay is the minimum.

However, in addition to the *race policy*, also an *execution policy* must be specified. The *execution policy* consists in a set of specifications for univocally defining the stochastic process underlying a *SPN*. Two elements characterize the *execution policy*: a criterion to keep memory of the past history of the process (the *memory policy*), and an indicator of the resampling status of the firing time. The *memory policy* defines how the process is conditioned upon the past. An *age variable* a_g associated to the timed transition tr_g keeps track of the time in which the transition has been enabled. A timed transition fires as soon as the memory variable a_g reaches the value of the firing time γ_g. The *activity period* of a transition is the period of time in which its age variable is not 0.

The random firing time γ_g of a transition tr_g can be sampled in a time instant antecedent to the beginning of an activity period. To keep track of the resampling condition of the random firing time associated to a timed transition, we assign to each timed transition tr_g a binary indicator variable ι_g that is equal to 1 when the firing time is sampled and equal to 0 when the firing time is not sampled. ι_g is set to 1 each time tr_g is enabled and its reset depends on the *execution policy*. We refer to ι_g as the *resampling indicator variable*. Hence, in general, the (continuous) memory of a transition tr_g is

[1] A formal definition of *execution policy* will be provided in the following.

indicated by the tuple (a_g, ι_g). At any time epoch t, transition tr_g has memory (its firing process depends on the past) if either a_g or ι_g are different from zero.

At the entrance in a new marking, the remaining firing time ($rft_g = \gamma_g - a_g$) is computed for each enabled transition given its currently sampled firing time γ_g and the age variable a_g. According to the race policy, the next firing is determined by the minimal of the rft's.

Adopting the previous formalism, the following individual execution policies can be introduced. A timed transition tr_g can be:

- *Preemptive repeat different (prd)*:
  If both the age variable a_g and the resampling indicator ι_g are reset each time tr_g is disabled or fires.

- *Preemptive resume (prs)*:
  If both the age variable a_g and the resampling indicator ι_g are reset only when tr_g fires.

- *Preemptive repeat identical (pri)*:
  If the age variable a_g is reset each time tr_g is disabled or fires but the resampling indicator ι_g is reset only when tr_g fires.

Figure 1 gives a pictorial description of the introduced preemption policies with respect to a single transition tr_g. In the figure, the time instants marked with E, D and F indicate the enabling, disabling and firing time points of tr_g, respectively. Each preemption policy is illustrated via the evolution of the age variable a_g associated with the considered transition tr_g and of its remaining firing time ($rft_g = \gamma_g - a_g$). The horizontal lines below the diagrams indicate the period of time while $\iota_g = 1$.

Transition tr_g is prd - Each time a *prd* transition is disabled or fires, its memory variable a_g is reset and its indicator resampling variable ι_g is set to 0 (the firing time must be resampled from the same distribution as tr_g becomes enabled again). With reference to Figure 1a, tr_g is enabled for the first time at $t = 0$, its memory variable a_g starts increasing linearly, ι_g is set to 1 and the firing time is sampled from its distribution to a value, say, γ_1. At time D, tr_g is disabled and the memory is reset ($a_g = 0$, $\iota_g = 0$). At the next enabling time instant E, a_g restarts from zero, ι_g is set to 1 and the firing time is resampled from the same distribution assuming a different value, say, γ_2. When tr_g fires (point F) both a_g and ι_g are reset. At the successive enabling point E, a_g restarts and the firing time is resampled (γ_3). From the above, it follows that a *prd* transition looses its memory at any D and F points. The memory of the transition is confined to the period of time in which tr_g is continuously enabled.

Transition tr_g is prs - With reference to Figure 1b, when tr_g is disabled (in point D), its associated age variable a_g is not reset but maintains its constant value until tr_g is enabled again and $\iota_g = 1$. In the successive enabling point E, a_g restarts from the previously retained value. When tr_g fires, both a_g and ι_g are reset so that the firing time must be

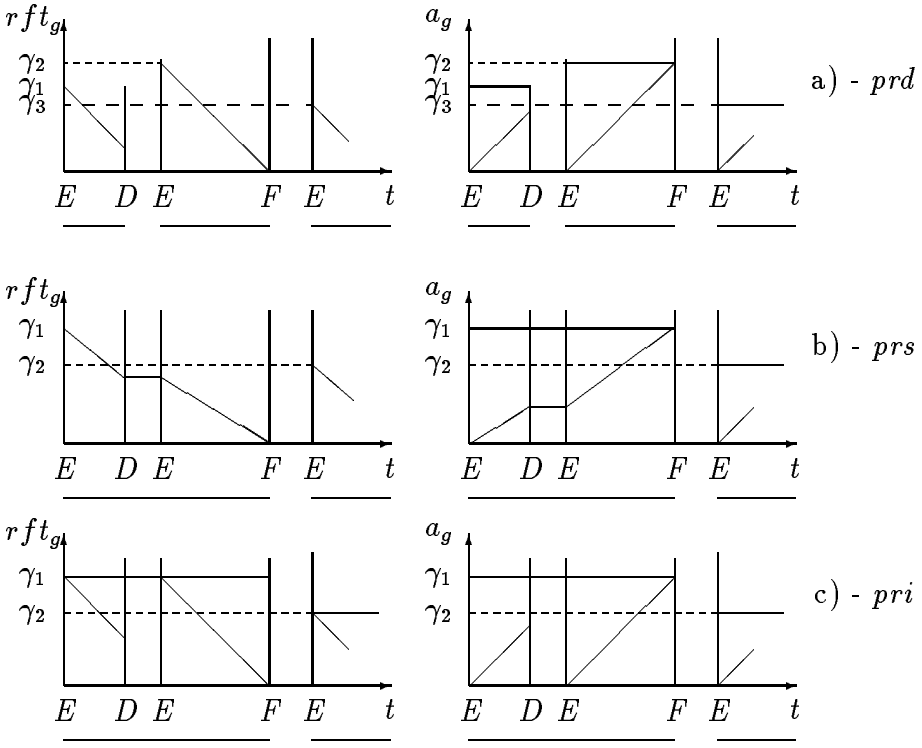

Figure 1: Pictorial representation of different firing time sampling policies

resampled at the successive enabling point (γ_2). The memory of tr_g is reset only when the transition fires.

Transition tr_g is pri - Under this policy (Figure 1c), each time tr_g is disabled, its age variable is reset, but ι_g remains equal to 1, and the firing time value γ_1 remains active, so that in the next enabling period an identical firing time should be accomplished. In Figure 1c, the same value (γ_1) is maintained over different enabling periods up to the firing of tr_g. Only when tr_g fires both a_g and ι_g are reset and the firing time is resampled (γ_2). Hence, also in this case, the memory is lost only upon firing of tr_g.

It is clear from Figure 1 that the instant of firing of a transition, under any execution policy, can be obtained as the first instant of time at which the age variable equals the sampled value of the firing time. Moreover, with any distribution, the three different preemption policies behave differently only if the corresponding transition can be disabled before firing. In this situation, the following particular cases can be mentioned. If the firing time is exponentially distributed both the *prd* and *prs* policy behave in the same way and can be omitted. However, the *pri* policy does not enjoy the memoryless property [3]. Thus, the marking process of a *PN* with only exponentially distributed firing times is not a *CTMC* if at least a single transition exists (that can be disabled before firing) with assigned *pri* policy. If the firing time is deterministic, both the *prd* and *pri* policy

behave in the same way (indeed, resampling a deterministic variable provides always an identical value).

According to the previous discussion, transitions can be classified as EXP (or memoryless) if they have associated an exponentially distributed firing time with either *prd* or *prs* policy, or MEM (non-memoryless) if they have associated an exponentially distributed firing time with *pri* policy or any non-exponential distribution. Only MEM transitions need to be assigned an execution policy. An usual graphical representation to distinguish between an EXP and a MEM transition is to draw the former as an empty rectangle an the latter as a filled rectangle.

The memory of the global marking process is considered as the superposition of the individual memories of the transitions.

3 Markov Regenerative Stochastic Petri Nets

Definition 1 *The stochastic process underlying a GDT_SPN is called the marking process $\mathcal{M}(t)$ $(t \geq 0)$. $\mathcal{M}(t)$ is the marking of the GDT_SPN at time t.*

A single realization of the marking process $\mathcal{M}(t)$ can be written as:

$$\mathcal{R} = \{(\tau_0, M_0); (\tau_1, M_1); \ldots; (\tau_i, M_i); \ldots\}$$

where M_{i+1} is a marking directly reachable from M_i, and $\tau_{i+1} - \tau_i$ is the sojourn time in marking M_i. With the above notation, $\mathcal{M}(t) = M_i$ for $\tau_i \leq t < \tau_{i+1}$. In the following τ_i will be referred to as a *Regenerative Time Point (RTP).*

In the following, we restrict our analysis to *SPN*s in which the random firing times have continuous cdfs. With this assumption, the marking process $\mathcal{M}(t)$ is a right-continuous, piecewise constant, continuous-time discrete-state stochastic process whose state space is isomorphic to the reachability graph of the untimed *PN*. Intrigued semantic interpretations related to the possibility of contemporary firings are avoided [25, 20, 9, 12].

A formal definition of a class of Markov Regenerative Stochastic Petri Nets *(MRSPN)* has been presented in [7]:

Definition 2 *A SPN is called a Markov Regenerative Stochastic Petri Net (MRSPN) if its marking process $\mathcal{M}(t)$ is a Markov Regenerative Process (MRGP)*[2].

*MRGP*s [22] (or Semi Regenerative Processes [13]) are discrete-state continuous-time stochastic processes with an embedded sequence of *Regenerative Time Points (RTP)* [33], at which the process enjoys the Markov property. The relevance of Definition 2 comes from the fact that *MRSPN*s can be studied by resorting to the techniques available for *MRGP*s [13, 22]. Only MEM transitions affect the search for the *RTP*s, since EXP transitions do not have memory. Based on the concept of memory introduced in the previous section, *RTP*s can be defined as follows:

[2] *MRSPN*s are referred to as Semi Regenerative *SPN*s in [11].

Definition 3 *A regenerative time point (RTP) in the marking process $\mathcal{M}(t)$ underlying a SPN is an instant of time where all the transitions do not have memory; i.e. all the memory variables a_k and the resampling indicator variables ι_k ($k = 1, 2, \ldots, ||Tr||$) are equal to zero.*

The time interval between two consecutive *RTP*s is indicated as a *regeneration interval*. The framework in which a *SPN*, with mixed preemption policies [32], generates a *MRGP* marking process is based on the notion of *non-overlapping dominant MEM transition* [5].

Definition 4 *A MEM transition is a unique dominant transition over a regeneration interval if it becomes enabled in the marking entered at the initial RTP and its memory is reset at the successive RTP.*

Definition 5 *A SPN is said to be non-overlapping if a unique dominant transition can be associated to each regeneration period. A non-overlapping SPN is a MRSPN.*

If in a marking entered at a *RTP* all the enabled transitions are EXP, any firing results in the successive *RTP*, so that no state transition is possible in between. The evolution of the marking process $\mathcal{M}(t)$ during a regeneration period between two consecutive *RTP*s is called the process *subordinated* to the MEM dominant transition. The subordinated process can include any number of EXP transitions, but also MEM transitions provided that their memory cycle is completely contained into the regeneration period of the unique dominant transition (Definition 4). However, a complete analytical characterization of a *MRSPN* is possible if all the subordinated processes are restricted to be a *SMP* or a *CTMC*.

3.1 Analysis by Markov Regenerative Theory

Given a *PN*, let $\mathcal{R}_{TP}(M_0) \in \mathcal{R}(M_0)$ be the subsets of markings which determine *RTP*s, according to Definition 3. Let us further define $N = ||\mathcal{R}(M_0)||$ and $N' = ||\mathcal{R}_{TP}(M_0)||$. Hence, $N' \leq N$. By the memoryless property of the *MRGP* at the *RTP*s, the analysis of a *MRSPN* can be split into N' independent subproblems each one represented by the restriction of the marking process $\mathcal{M}(t)$ starting at any state $i \in \mathcal{R}_{TP}(M_0)$, and before the occurrence of the successive *RTP*.

Let us denote by $\mathcal{M}^i(t)$ the subordinated process starting from state $i \in \mathcal{R}_{TP}(M_0)$:

$$\mathcal{M}^i(t) = \{\mathcal{M}(t) : \mathcal{M}(\tau_0) = i, t < \tau_1^*\} \tag{1}$$

where $\tau_0 = 0$ and τ_1^* are successive *RTP*s.

The probabilistic functions that must be evaluated for the transient analysis of a *MRSPN* are commonly referred to as global and local kernels [13, 22]. The global kernel is a $(N' \times N')$ matrix $\mathbf{K}(t) = [K_{ij}(t)]$ that describes the occurrence of the next *RTP*:

$$K_{ij}(t) = Pr\{M_{(1)} = j \in \mathcal{R}_{TP}(M_0),\ \tau_1^* \leq t \,|\, \mathcal{M}(0) = i \in \mathcal{R}_{TP}(M_0)\}$$

where $M_{(1)}$ is the right continuous state hit by the marking process at the next *RTP*. The local kernel is a $(N' \times N)$ matrix $\mathbf{E}(t) = [E_{ij}(t)]$ that describes the state transition probabilities inside a regeneration period, before the next *RTP* occurs:

$$E_{ij}(t) = Pr\left\{\mathcal{M}(t) = j \in \mathcal{R}(M_0)\,,\ \tau_1^* > t \,|\, \mathcal{M}(0) = i \in \mathcal{R}_{TP}(M_0)\right\}$$

By these definitions $\sum_{j\in\mathcal{R}_{TP}(M_0)} K_{ij}(t) + \sum_{j\in\mathcal{R}(M_0)} E_{ij}(t) = 1,\ \ \forall i \in \mathcal{R}_{TP}(M_0),\ \ \forall t \geq 0.$

In the particular case in which the marking process is a semi-Markov process all the reachable states must be *RTP*s hence $N' = N$, and the local kernel $\mathbf{E}(t)$ results to be a square $(N \times N)$ diagonal matrix, because no state transition is possible between consecutive *RTP*s. The conditions under which a *SPN* generates a semi-Markov marking process have been studied in [15].

The entries of the ith row $(i \in \mathcal{R}_{TP}(M_0))$ of the kernel matrices $\mathbf{E}(t)$ and $\mathbf{K}(t)$ depends only on the subordinated process $\mathcal{M}^i(t)$ starting from state i, and on the execution policy of the single MEM transition dominating the considered regeneration period. For a *prd* dominant MEM transition the analysis is given in [8], for a *prs* dominant MEM transition in [5, 30] and for a *pri* dominant MEM transition in [3].

Let $\mathbf{V}(t) = [V_{ij}(t)]$ denote the $(N' \times N)$ transition probability matrix over $(0, t)$, i.e.:

$$V_{ij}(t) = Pr\left\{\mathcal{M}(t) = j \in \mathcal{R}(M_0) \,|\, \mathcal{M}(0) = i \in \mathcal{R}_{TP}(M_0)\right\} \tag{2}$$

Note that the initial state i in any entry of (2) must be a regeneration state, since the analysis based on the kernel matrices is valid only for that case. With *PN*, the initial marking at $t = 0$ is memoryless and hence is always a *RTP* (Definition 3).

Based on the global and the local kernels the transient analysis can be carried out in the time domain by solving the following generalized Markov renewal equation [13, 22]:

$$V_{ij}(t) \;=\; E_{ij}(t) \;+\; \sum_{k\in\mathcal{R}_{TP}(M_0)} \int_0^t d\,K_{ik}(y)\; V_{kj}(t-y) \tag{3}$$

or in the transform domain:

$$\mathbf{V}^{\sim}(s) \;= [\mathbf{I} \;-\; \mathbf{K}^{\sim}(s)]^{-1}\; \mathbf{E}^{\sim}(s) \tag{4}$$

where the superscript $^{\sim}$ indicates the Laplace-Stieltjes transform (*LST*) and s the complex transform variable of t (i.e.: $F^{\sim}(s) = \int_0^{\infty} e^{-st} dF(t)$).

A time domain solution for the transition probability matrix $\mathbf{V}(t)$ can be obtained by numerically integrating Equation (3). Alternatively, starting from the *LST* Equation (4) a combination of symbolic and numeric computation is needed to obtain measures in the time domain [6].

For the purpose of the steady-state analysis of a *MRSPN*, the following measures of the subordinated processes should be evaluated:

$$\begin{aligned} \alpha_{ij} &= \mathbb{E}\left[\int_0^{\infty} I_{\mathcal{M}^i(t)=j}\; dt\right] \\ \phi_{ij} &= Pr\left\{M_{(1)} = j \;|\; \mathcal{M}(0) = i\right\} \end{aligned} \tag{5}$$

where $I_{(\cdot)}$ is a binary indicator function, α_{ij} is the expected time the subordinated process $\mathcal{M}^i(t)$ spends in state j, and ϕ_{ij} is the probability that the subordinated process $\mathcal{M}^i(t)$ is followed by a regeneration period starting from state j. Indeed, the matrix $\boldsymbol{\phi} = [\phi_{ij}]$ is the transition probability matrix of the *DTMC* embedded at the *RTP*s. The measures in Equation (5) can be obtained from the global and local kernels either in the time or in the transform domain:

$$\alpha_{ij} = \int_{t=0}^{\infty} E_{ij}(t)\, dt = \lim_{s \to 0} \frac{1}{s} E_{ij}^{\sim}(s) \tag{6}$$

$$\phi_{ij} = \lim_{t \to \infty} K_{ij}(t) = \lim_{s \to 0} K_{ij}^{\sim}(s) \tag{7}$$

It is clear from the above equations that $\boldsymbol{\alpha} = [\alpha_{ij}]$ is a $(N' \times N)$ matrix and $\boldsymbol{\phi} = [\phi_{ij}]$ is a $(N' \times N')$ matrix.

The evaluation of the measures in (5) is also dependent on the nature of the execution policy associated to the transition dominating the subordinated process. For a *prd* dominant MEM transition the analysis is given in [2], for a *prs* dominant MEM transition in [31] and for a *pri* dominant MEM transition in [4].
The steady-state analysis of an *MRSPN* requires three steps:

Step 1: Evaluate the $\boldsymbol{\alpha} = [\alpha_{ij}]$ and $\boldsymbol{\phi} = [\phi_{ij}]$ matrices based on the results of Section 5 [2, 31, 4] and compute:

$$\alpha_i = \sum_{j \in \mathcal{R}(M_0)} \alpha_{ij}$$

where α_i is the expected duration of $\mathcal{M}^i(t)$ before the next *RTP*.

Step 2: Evaluate the N'-dimensional vector $D = [D_i]$, whose elements are the stationary state probabilities of the *DTMC* embedded at the *RTP*s. D is the unique solution of:

$$D = D\boldsymbol{\phi}\ ; \qquad \sum_{i \in \mathcal{R}_{TP}(M_0)} D_i = 1 \tag{8}$$

Step 3: The steady-state probabilities of the *MRGP* are given by:

$$v_j = \lim_{t \to \infty} Pr\{\mathcal{M}(t) = j \in \mathcal{R}(M_0)\} = \frac{\sum_{k \in \mathcal{R}_{TP}(M_0)} D_k\, \alpha_{kj}}{\sum_{k \in \mathcal{R}_{TP}(M_0)} D_k\, \alpha_k} \tag{9}$$

The following section shows how the previous equations can be derived by means of an independent analysis of each subordinated process.

4 Analysis of a single subordinated process

Let us concentrate on the analysis of a single subordinated process $\mathcal{M}^i(t)$ starting from a generic *RTP* identified as state i. This analysis provides all the entries of the ith row of the kernel matrices $\mathbf{E}(t)$ and $\mathbf{K}(t)$. In order to completely evaluate the kernel matrices, the analysis presented in this section must be iterated for any state $i \in \mathcal{R}_{TP}(M_0)$.

When only exponential transitions are enabled in state $i \in \mathcal{R}_{TP}(M_0)$ and when a MEM transition is exclusively enabled so that the next firing results in a new *RTP* with probability 1, the elements of the ith row of the kernel matrices can be directly obtained from their definition [30, 15]. In the following, we focus our attention on the subordinated processes with possible intermediate state transitions.

By Definition 4, $\mathcal{M}^i(t)$ is dominated by a single transition tr_g with firing time distribution $G_g(w)$ and associated either a *prd*, or a *prs* or a *pri* policy.

Theorem 1 *Given a MRSPN with non-overlapping dominant transitions, the state space $\mathcal{R}^i$ of a generic subordinated process $\mathcal{M}^i(t)$ starting from state $i \in \mathcal{R}_{TP}(M_0)$, can be generated from the original untimed PN by removing the dominant transition tr_g, and assuming marking i as the initial marking.*

Proof - The second condition (assuming marking i as the initial marking) is implicit in the definition of subordinated process given in Equation (1). The first condition (removing the dominant transition) is equivalent to generating the subset of the original reachability graph consisting in all the possible firing sequences but the one involving the firing of the dominant transition. Hence, the generated subset is equal to the one stopped by the firing of the dominant transition. □

It follows from Theorem 1 that $\mathcal{R}^i$ is strictly contained in $\mathcal{R}(M_0)$. The set $\mathcal{R}^i$ can be divided into two disjoint exhaustive subsets $\mathcal{R}^i = \mathcal{E}^i \cup \mathcal{D}^i$ (Figure 2), where:

- $\mathcal{E}^i$ groups the states of $\mathcal{R}^i$ in which tr_g is enabled (a_g strictly increases in $\mathcal{E}^i$);
- $\mathcal{D}^i$ groups the states of $\mathcal{R}^i$ in which tr_g is not enabled (a_g does not increase in $\mathcal{D}^i$).

Note that in the case of a *prd* dominant transition, any transition to states in $\mathcal{D}^i$ concludes the subordinated process.

Let $N^i = ||\mathcal{R}^i||$, $N^i_{\mathcal{E}} = ||\mathcal{E}^i||$, and $N^i_{\mathcal{D}} = ||\mathcal{D}^i||$, so that $N^i = N^i_{\mathcal{E}} + N^i_{\mathcal{D}}$.

The following analysis is developed in the case in which the firing time associated to the dominant MEM transition is deterministic. If, however, the distribution of the firing time is not deterministic, the analysis proceeds in two steps [30]:

1. Fix a value for the random firing time $w = \gamma_g$ and perform the analysis as in the deterministic case. Let $A(\bullet \mid w)$ be the calculated probability measure.
2. Uncondition the obtained results with respect to the firing time distribution $G_g(w)$ of γ_g; i.e. $A(\bullet) = \int_{w=0}^{\infty} A(\bullet \mid w)\, dG_g(w)$.

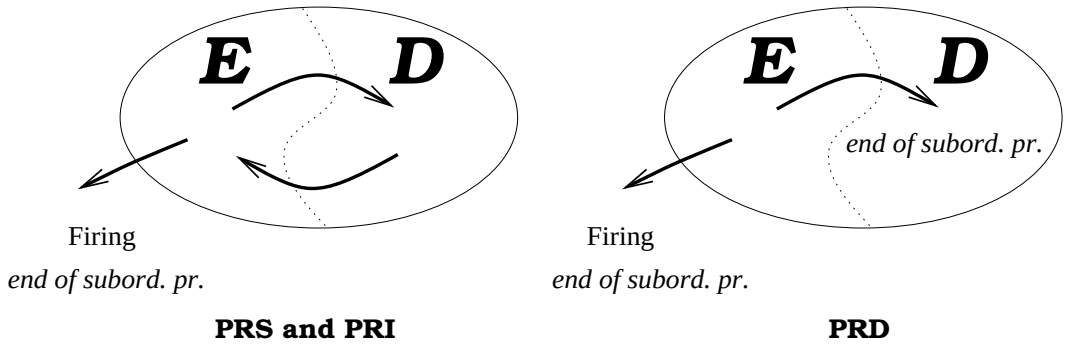

Figure 2: Partitioned state space of the subordinated process

In order to avoid unnecessarily large matrices during the analysis of $\mathcal{M}^i(t)$, the states in $\mathcal{R}^i$ are renumbered. The states numbered as 1, 2, ..., $N^i_{\mathcal{E}}$ belong to the subset $\mathcal{E}^i$ and the ones numbered $N^i_{\mathcal{E}}+1$, $N^i_{\mathcal{E}}+2$, ..., $N^i_{\mathcal{E}}+N^i_{\mathcal{D}}$ belong to $\mathcal{D}^i$.

In order to keep reference to the original numbering of the same states in the state space $\mathcal{R}(M_0)$ of the complete *PN*, we introduce a $(N^i \times N)$ shuffel matrix $\mathbf{S}^i = [S_{kj}]$. The generic row k of $\mathbf{S}^i$ is a N-dimensional vector with all the entries equal to 0 but entry j equal to 1, to indicate that state k in $\mathcal{R}^i$ corresponds to state j in $\mathcal{R}(M_0)$. Without loss of generality, we can always suppose that state i in $\mathcal{R}(M_0)$ (originating the subordinated process under exam) corresponds to state 1 in $\mathcal{R}^i$. With this assumption, the initial probability vector of $\mathcal{M}^i(t)$ is always in the form $U^i = [U^i_{\mathcal{E}}, U^i_{\mathcal{D}}] = (1, 0, 0, \ldots, 0)$.

At any time t the subordinated process $\mathcal{M}^i(t)$ can be in one of the following three exhaustive and disjoint conditions:

- $\mathcal{M}^i(t)$ is not concluded yet;
- $\mathcal{M}^i(t)$ is concluded by the firing of tr_g;
- $\mathcal{M}^i(t)$ is concluded by the disabling of tr_g (this case holds only for *prd* dominant MEM transition).

Let us fix a value of the firing requirement $w = \gamma_g$, and define the following matrix functions $\mathbf{P}^i(t,w)$, $\mathbf{F}^i(t,w)$ and $\mathbf{C}^i(t,w)$ with dimensions $(N^i_{\mathcal{E}} \times N^i)$, $(N^i_{\mathcal{E}} \times N^i_{\mathcal{E}})$, and $(N^i_{\mathcal{E}} \times N^i_{\mathcal{D}})$, respectively, which provide a formal description of the above conditions.

$$P^i_{k\ell}(t,w) = Pr\{\mathcal{M}^i(t) = \ell \in \mathcal{R}^i\,,\, \tau^*_1 > t \,|\, \mathcal{M}^i(0) = k \in \mathcal{E}^i\,,\, \gamma_g = w\} \tag{10}$$

$$F^i_{k\ell}(t,w) = Pr\{\mathcal{M}^i(\tau^{*-}_1) = \ell \in \mathcal{E}^i\,,\, \tau^*_1 \le t,\, tr_g \text{ fires} \,|\, \mathcal{M}^i(0) = k \in \mathcal{E}^i\,,\, \gamma_g = w\} \tag{11}$$

If the dominant MEM transition is *prd*, we define also:

$$C^i_{k\ell}(t,w) = Pr\{\mathcal{M}^i(\tau^*_1) = \ell \in \mathcal{D}^i\,,\, \tau^*_1 \le t,\, tr_g \text{ did not fire} \,|\, \mathcal{M}^i(0) = k \in \mathcal{E}^i\,,\, \gamma_g = w\} \tag{12}$$

otherwise $C^i_{k\ell}(t,w) = 0$.
By the above definitions

- $P^i_{k\ell}(t,w)$ is the probability that $\mathcal{M}^i(t)$ is in state $\ell \in \mathcal{R}^i$ at time t before the age variable of the dominant transition reaches the value w, starting in state $k \in \mathcal{E}^i$ at $t = 0$.

- $F^i_{k\ell}(t,w)$ is the probability that tr_g fires from state $\ell \in \mathcal{E}^i$ (the age variable of the dominant transition reaches the value w in ℓ) before t, starting in state $k \in \mathcal{E}^i$ at $t = 0$.

- $C^i_{k\ell}(t,w)$ with *prd* dominant MEM transition is the probability that a transition to $\ell \in \mathcal{D}^i$ occurs (resetting a_g) before the firing of tr_g and before time t, starting in state $k \in \mathcal{E}^i$ at $t = 0$.

The conditions covered by Equations (10), (11) and (12), represent all the possible outcomes of $\mathcal{M}^i(t)$ at a given time t. Hence, for any $t \geq 0$ and $k \in \mathcal{E}^i$:

$$\sum_{\ell \in \mathcal{R}^i} P^i_{k\ell}(t,w) + \sum_{\ell \in \mathcal{E}^i} F^i_{k\ell}(t,w) + \sum_{\ell \in \mathcal{D}^i} C^i_{k\ell}(t,w) = 1 \tag{13}$$

Let us further introduce the $(N^i_{\mathcal{E}} \times N)$ branching probability matrix $\mathbf{\Delta}^i = [\Delta^i_{kj}]$. The generic entry Δ^i_{kj} represents the probability that the firing of the dominant transition tr_g in state $k \in \mathcal{E}^i$ leads to a marking $j \in \mathcal{R}(M_0)$.

$$\Delta^i_{kj} = Pr\{\text{next marking is } j \in \mathcal{R}(M_0) \,|\, \text{current marking is } k \in \mathcal{E}^i,\ tr_g \text{ fires}\}$$

If immediate transitions are excluded from the original *PN* definition (as in the present setting), each row of $\mathbf{\Delta}^i$ contains one and only one entry equal to 1 being all the other entries equal to 0. However, if immediate transitions are allowed, their effect could be accounted for by properly modifying the entries of $\mathbf{\Delta}^i$ [8]. In this case, the probability of jumping from a tangible state k to any possible tangible state j through vanishing markings, must be located in the proper Δ^i_{kj} entry of $\mathbf{\Delta}^i$ (being the sum of each row equal to 1).

Remembering the initial probability vector U^i of the subordinated process $\mathcal{M}^i(t)$, the elements of the ith row of matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$ can be expressed as a function of the elements of the first row of the matrices $\mathbf{P}^i(t,w)$, $\mathbf{F}^i(t,w)$ and $\mathbf{C}^i(t,w)$ in the following way:

$$\underline{K_i}(t) = \int_{w=0}^{\infty} U^i_{\mathcal{E}} \left[\mathbf{F}^i(t,w)\,\mathbf{\Delta}^i + \mathbf{C}^i(t,w)\,\mathbf{S}^i_{\mathcal{D}}\right] dG_g(w) \tag{14}$$

$$\underline{E_i}(t) = \int_{w=0}^{\infty} U^i_{\mathcal{E}}\,\mathbf{P}^i(t,w)\,\mathbf{S}^i\,dG_g(w) \tag{15}$$

where the notation $\underline{A_i}(\cdot)$ refers to the ith row of matrix $\mathbf{A}(\cdot)$, and $\mathbf{S}^i_{\mathcal{D}}$ is the proper $(N^i_{\mathcal{D}} \times N)$ partition of matrix $\mathbf{S}^i$.

Equations (14) and (15) show how the local and global kernels can be evaluated from the knowledge of matrices $\mathbf{P}^i(t,w)$, $\mathbf{F}^i(t,w)$ and $\mathbf{C}^i(t,w)$. In the following section, the above matrices are derived from the analysis of the subordinated process over the partitioned state space $\mathcal{R}^i = \mathcal{E}^i \cup \mathcal{D}^i$.

4.1 Partitioned state space

In order to simply the notation, in the following derivation we eliminate the superscript i in all the symbols.

It is however tacitly intended, that all the quantities refer to the single specific process $\mathcal{M}^i(t)$ subordinated to the regeneration period starting from state i.

With reference to Figure 2, and with the adopted renumbering of the states, $\mathcal{M}^i(t)$ starts in state $1 \in \mathcal{E}$, then moves through states in $\mathcal{D}$ reentering $\mathcal{E}$ in any state $k \in \mathcal{E}$. However, the dominant transition of $\mathcal{M}^i(t)$ can only fire from states in $\mathcal{E}$. Let us denote by T_1 the random time point until $\mathcal{M}^i(t)$ visits $\mathcal{E}$ and by T_2 the random time point until $\mathcal{M}^i(t)$ visits $\mathcal{D}$. By enumerating all the possible exhaustive and mutually exclusive conditions in which the subordinated process $\mathcal{M}^i(t)$ can be in states belonging to $\mathcal{E}$ or $\mathcal{D}$, the following partitioned measures can be evaluated. For states in $\mathcal{E}$ we define:

$$P\mathcal{E}_{k\ell}(t,w) \;=\; Pr\{\mathcal{M}^i(t) = \ell \in \mathcal{E}\;,\; \tau_1^* > t\;,\; T_1 > t \,|\, \mathcal{M}^i(0) = k \in \mathcal{E}\;,\; \gamma_g = w\} \tag{16}$$

$$F\mathcal{E}_{k\ell}(t,w) \;=\; Pr\{\mathcal{M}^i(\tau_1^{*-}) = \ell \in \mathcal{E}\;,\; \tau_1^* \le t\;,\; T_1 > \tau_1^* \,|\, \mathcal{M}^i(0) = k \in \mathcal{E}\;,\; \gamma_g = w\} \tag{17}$$

$$P\mathcal{ED}_{k\ell}(t,w) \;=\; Pr\{\mathcal{M}^i(T_1) = \ell \in \mathcal{D}\;,\; \tau_1^* > T_1\;,\; T_1 < t \,|\, \mathcal{M}^i(0) = k \in \mathcal{E}\;,\; \gamma_g = w\} \tag{18}$$

Since tr_g can not fire from $\mathcal{D}$ we also define:

$$P\mathcal{D}_{k\ell}(t) \;=\; Pr\{\mathcal{M}^i(t) = \ell \in \mathcal{D}\;,\; T_2 > t \,|\, \mathcal{M}^i(0) = k \in \mathcal{D}\} \tag{19}$$

$$P\mathcal{DE}_{k\ell}(t) \;=\; Pr\{\mathcal{M}^i(T_2) = \ell \in \mathcal{E}\;,\; T_2 < t \,|\, \mathcal{M}^i(0) = k \in \mathcal{D}\} \tag{20}$$

By the above definitions it follows that:

- $\mathbf{P}\mathcal{E}(t,w)$ is a $(N_\mathcal{E} \times N_\mathcal{E})$ dimensional matrix whose generic element $P\mathcal{E}_{k\ell}(t,w)$ is the probability of being at time t in state $\ell \in \mathcal{E}$ starting in state $k \in \mathcal{E}$ at $t = 0$, without intermediate passage to $\mathcal{D}$ and before the firing of the dominant transition.
- $\mathbf{F}\mathcal{E}(t,w)$ is a $(N_\mathcal{E} \times N_\mathcal{E})$ dimensional matrix whose generic element $F\mathcal{E}_{k\ell}(t,w)$ is the probability that tr_g fires from state $\ell \in \mathcal{E}$ before t, starting in state $k \in \mathcal{E}$ at $t = 0$, and without intermediate passage to $\mathcal{D}$.
- $\mathbf{P}\mathcal{ED}(t,w)$ is a $(N_\mathcal{E} \times N_\mathcal{D})$ dimensional matrix whose generic element $P\mathcal{ED}_{k\ell}(t,w)$ is the probability that the subordinated process left $\mathcal{E}$ before time t and before the firing of the dominant transition, hitting state $\ell \in \mathcal{D}$, starting from state $k \in \mathcal{E}$ at $t = 0$.

- $\mathbf{P}\mathcal{D}(t,w)$ is a $(N_{\mathcal{D}} \times N_{\mathcal{D}})$ dimensional matrix whose generic element $P\mathcal{D}_{k\ell}(t)$ is the probability of being at time t in state $\ell \in \mathcal{D}$ starting in state $k \in \mathcal{D}$ at $t = 0$, and without intermediate passage to $\mathcal{E}$.

- $\mathbf{P}\mathcal{D}\mathcal{E}(t,w)$ is a $(N_{\mathcal{D}} \times N_{\mathcal{E}})$ dimensional matrix whose generic element $P\mathcal{D}\mathcal{E}_{k\ell}(t)$ is the probability that the subordinated process left $\mathcal{D}$ before time t hitting state $\ell \in \mathcal{E}$, starting in state $k \in \mathcal{D}$ at $t = 0$.

Given that the process started in a state $k \in \mathcal{E}$ at $t = 0$, the following equality holds:

$$\sum_{\ell \in \mathcal{E}} P\mathcal{E}_{k\ell}(t,w) + \sum_{\ell \in \mathcal{D}} P\mathcal{E}\mathcal{D}_{k\ell}(t,w) + \sum_{\ell \in \mathcal{E}} F\mathcal{E}_{k\ell}(t,w) = 1$$

The process starting from $\mathcal{D}$ is very similar to the one starting from $\mathcal{E}$. The only difference between the two is that the firing of tr_g is not possible in $\mathcal{D}$. Hence, these measures are independent of the firing time requirement (w) and:

$$\sum_{\ell \in \mathcal{D}} P\mathcal{D}_{k\ell}(t) + \sum_{\ell \in \mathcal{E}} P\mathcal{D}\mathcal{E}_{k\ell}(t) = 1$$

The functions (16) to (20), are defined without any specific reference to the particular execution policy of the dominant transition. However, this knowledge is now necessary to evaluate the matrix functions $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$ and $\mathbf{C}(t,w)$ from Equation (16) - (20), and then the kernel matrices of the *MRSPN*.

4.1.1 *prd* dominant MEM transition

Any transition out of subset $\mathcal{E}$ (either by firing or by disabling the dominant transition tr_g) terminates the subordinated process $\mathcal{M}^i(t)$.

Theorem 2 *The time-domain and the LST transform expressions of the probability matrices* $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$ *and* $\mathbf{C}(t,w)$ *satisfy:*

$$\begin{array}{lcllcl} \mathbf{P}(t,w) & = & [\,\mathbf{P}\mathcal{E}(t,w),\,\mathbf{0}\,] & \mathbf{P}^{\sim}(s,w) & = & [\,\mathbf{P}\mathcal{E}^{\sim}(s,w),\,\mathbf{0}\,] \\ \mathbf{F}(t,w) & = & \mathbf{F}\mathcal{E}(t,w) & \mathbf{F}^{\sim}(s,w) & = & \mathbf{F}\mathcal{E}^{\sim}(s,w) \\ \mathbf{C}(t,w) & = & \mathbf{P}\mathcal{E}\mathcal{D}(t,w) & \mathbf{C}^{\sim}(s,w) & = & \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w) \end{array} \qquad (21)$$

The proof follows directly from the definition of the functions [5]. The equality in the *LST* domain has been explicitly reported, because this form is derived directly in the next section.

In the first line of (21), $\mathbf{P}(t,w) = [\bullet_{\mathcal{E}\mathcal{E}}, \bullet_{\mathcal{E}\mathcal{D}}]$ is expressed in partitioned form, being the second partition the $(N_{\mathcal{E}} \times N_{\mathcal{D}})$ $\mathbf{0}$ matrix, since under the *prd* policy no transition is possible from $\mathcal{E}$ to $\mathcal{D}$ before the successive *RTP*.

4.1.2 *pri* type dominant MEM transition

The regeneration period can be concluded only by the firing of the dominant *pri* transition from a state in $\mathcal{E}$. However, the firing can occur after (0, 1, 2, ...) visits in $\mathcal{D}$. Any time the subordinated process enters or re-enters $\mathcal{E}$, an identical firing time requirement w has to be completed.

Theorem 3 *The LST transform of the probability matrices* $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$ *and* $\mathbf{C}(t,w)$ *satisfy:*

$$\mathbf{F}^{\sim}(s,w) = [\mathbf{I} - \mathbf{P}\mathcal{ED}^{\sim}(s,w)\,\mathbf{P}\mathcal{DE}^{\sim}(s)]^{-1}\,\mathbf{F}\mathcal{E}^{\sim}(s,w) \tag{22}$$

$$\mathbf{P}^{\sim}(s,w) = [\mathbf{I} - \mathbf{P}\mathcal{ED}^{\sim}(s,w)\,\mathbf{P}\mathcal{DE}^{\sim}(s)]^{-1}\,[\mathbf{P}\mathcal{E}^{\sim}(s,w),\ \mathbf{P}\mathcal{ED}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}^{\sim}(s)] \tag{23}$$

$$\mathbf{C}^{\sim}(s,w) = \mathbf{0} \tag{24}$$

where the second member of the r.h.s. of Equation (23) is expressed in the same partitioned form as in (21).

The proof is given in Appendix A.

4.1.3 *prs* type dominant MEM transition

The analysis of the process subordinated to a dominant *prs* transition is very similar to the *pri* case, examined in the previous subsection. Also in the *prs* case, the regeneration period can be concluded only by the firing of the dominant transition from a state in $\mathcal{E}$. The firing can occur after (0, 1, 2, ...) visits in $\mathcal{D}$, but any time the subordinated process enters or re-enters $\mathcal{E}$, only the residual firing time needs to be accomplished.

Theorem 4 *The double transform of the probability matrices* $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$ *and* $\mathbf{C}(t,w)$ *satisfy:*

$$\mathbf{F}^{\sim *}(s,v) = [\mathbf{I} - v\,\mathbf{P}\mathcal{ED}^{\sim *}(s,v)\,\mathbf{P}\mathcal{DE}^{\sim}(s)]^{-1}\,\mathbf{F}\mathcal{E}^{\sim *}(s,v) \tag{25}$$

$$\mathbf{P}^{\sim *}(s,v) = [\mathbf{I} - v\,\mathbf{P}\mathcal{ED}^{\sim *}(s,v)\,\mathbf{P}\mathcal{DE}^{\sim}(s)]^{-1}\,[\mathbf{P}\mathcal{E}^{\sim *}(s,v)\,,\ \mathbf{P}\mathcal{ED}^{\sim *}(s,v)\,\mathbf{P}\mathcal{D}^{\sim}(s)] \tag{26}$$

$$\mathbf{C}^{\sim *}(s,v) = \mathbf{0} \tag{27}$$

where superscript * *means Laplace transformation, and* v *is the complex transform variable of* w *(i.e.:* $F^*(v) = \int_0^\infty F(w)e^{-vw}dw$*).*

The proof of Theorem 4 is given in Appendix B by resorting to a renewal argument.

4.2 Subordinated process of specific structure

If the stochastic structure of the subordinated process $\mathcal{M}^i(t)$ is known, the measures derived in the previous sections can be expressed in closed form and solved. In the following two paragraphs, we consider the particular cases in which the subordinated process is a *SMP* or a *CTMC*.

4.2.1 Subordinated SMP

Let $\mathcal{M}^i(t)$ be a *SMP* whose probability transition matrix is a $(N^i \times N^i)$ matrix denoted by $\mathbf{Q}(t) = [Q_{k\ell}(t)]$. Let the sojourn time distribution of state k be $Q_k(t) = \sum_{\ell \in \mathcal{R}^i} Q_{k\ell}(t)$.

Theorem 5 *When the subordinated process is a SMP, the measures defined on the partitioned state space given in Equations (16) - (20), take the form:*

$$P\mathcal{E}^{\sim *}_{k\ell}(s,v) \;=\; \delta_{k\ell}\,\frac{s\,[1 \,-\, Q^{\sim}_{k}(s \,+\, v)\,]}{v(s \,+\, v)} \;+\; \sum_{u \in \mathcal{E}} Q^{\sim}_{ku}(s \,+\, v)\,P\mathcal{E}^{\sim *}_{u\ell}(s,v), \quad k,\ell \in \mathcal{E} \tag{28}$$

$$F\mathcal{E}^{\sim *}_{k\ell}(s,v) \;=\; \delta_{k\ell}\,\frac{1 \,-\, Q^{\sim}_{k}(s \,+\, v)}{s \,+\, v} \;+\; \sum_{u \in \mathcal{E}} Q^{\sim}_{ku}(s \,+\, v)\,F\mathcal{E}^{\sim *}_{u\ell}(s,v), \quad k,\ell \in \mathcal{E} \tag{29}$$

$$P\mathcal{E}\mathcal{D}^{\sim *}_{k\ell}(s,v) \;=\; \frac{1}{v}Q^{\sim}_{k\ell}(s \,+\, v) \;+\; \sum_{u \in \mathcal{E}} Q^{\sim}_{ku}(s \,+\, v)\,P\mathcal{E}\mathcal{D}^{\sim *}_{u\ell}(s,v), \quad k \in \mathcal{E}, \ell \in \mathcal{D} \tag{30}$$

$$P\mathcal{D}^{\sim}_{k\ell}(s) \;=\; \delta_{k\ell}\,[1 - Q^{\sim}_{k}(s)] \;+\; \sum_{u \in \mathcal{D}} Q^{\sim}_{ku}(s)\,P\mathcal{D}^{\sim}_{u\ell}(s), \quad k,\ell \in \mathcal{D} \tag{31}$$

$$P\mathcal{D}\mathcal{E}^{\sim}_{k\ell}(s) \;=\; Q^{\sim}_{k\ell}(s) \;+\; \sum_{u \in \mathcal{D}} Q^{\sim}_{ku}(s)\,P\mathcal{D}\mathcal{E}^{\sim}_{u\ell}(s), \quad k \in \mathcal{D}, \ell \in \mathcal{E} \tag{32}$$

The proof of Theorem 5 can be found in Appendix C.

The Equations (28) - (32) must be substituted in the expressions for the matrices $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$ and $\mathbf{C}(t,w)$ given in Theorems 2 - 4. Then Equation (14) and (15) can be applied. Further symbolical manipulation requires the knowledge of the particular functions in $\mathbf{Q}(t)$.

4.2.2 Subordinated CTMC

Let $\mathcal{M}^i(t)$ be a *CTMC* whose infinitesimal generator is a $(N^i \times N^i)$ matrix denoted by $\mathbf{A} = [a_{k\ell}]$. The infinitesimal generator can be expressed in the following partitioned form:

$$\mathbf{A} = \begin{array}{c|c|c} & N_{\mathcal{E}} & N_{\mathcal{D}} \\ \hline N_{\mathcal{E}} & \mathbf{A}_{\mathcal{E}} & \mathbf{A}_{\mathcal{ED}} \\ \hline N_{\mathcal{D}} & \mathbf{A}_{\mathcal{DE}} & \mathbf{A}_{\mathcal{D}} \end{array} \tag{33}$$

With respect to the *SMP* case, considered in the previous section, the following correspondence can be established:

$$Q^{\sim}_{k\ell}(s) = \begin{cases} \dfrac{a_{k\ell}}{s - a_{kk}} & \text{if}: k \neq \ell \\ 0 & \text{if}: k = \ell \end{cases} \tag{34}$$

where $a_{kk} = -\sum_{\ell \in \mathcal{R}^i, \ell \neq k} a_{k\ell}$. Applying a direct substitution of (34) into (28) - (32), the partitioned measures can be expressed in matrix form based on the block description (33) of the infinitesimal generator of the subordinated *CTMC* [5, 3]. The matrix form is given in the *LST* domain, being, as usual, s the transform variable of the time t and v the transform variable of the sampled firing time w.

$$\mathbf{P}\mathcal{E}^{\sim *}(s,v) = \frac{s}{v}((s+v)\mathbf{I} - \mathbf{A}_{\mathcal{E}})^{-1} \tag{35}$$

$$\mathbf{F}\mathcal{E}^{\sim *}(s,v) = ((s+v)\mathbf{I} - \mathbf{A}_{\mathcal{E}})^{-1} \tag{36}$$

$$\mathbf{P}\mathcal{ED}^{\sim *}(s,v) = \frac{1}{v}((s+v)\mathbf{I} - \mathbf{A}_{\mathcal{E}})^{-1}\mathbf{A}_{\mathcal{ED}} \tag{37}$$

$$\mathbf{P}\mathcal{D}^{\sim}(s) = s\ (s\mathbf{I} - \mathbf{A}_{\mathcal{D}})^{-1} \tag{38}$$

$$\mathbf{P}\mathcal{DE}^{\sim}(s) = (s\mathbf{I} - \mathbf{A}_{\mathcal{D}})^{-1}\mathbf{A}_{\mathcal{DE}} \tag{39}$$

After a symbolical inverse Laplace transformation according to the variable v, Equations (35), (36) and (37) become, respectively:

$$\mathbf{P}\mathcal{E}^{\sim}(s,w) = s\int_0^w e^{(-s\mathbf{I}+\mathbf{A}_{\mathcal{E}})z}dz \tag{40}$$

$$\mathbf{F}\mathcal{E}^{\sim}(s,w) = e^{(-s\mathbf{I}+\mathbf{A}_{\mathcal{E}})w} \tag{41}$$

$$\mathbf{P}\mathcal{ED}^{\sim}(s,w) = \int_0^w e^{(-s\mathbf{I}+\mathbf{A}_{\mathcal{E}})z}dz\ \mathbf{A}_{\mathcal{ED}} \tag{42}$$

5 Steady state analysis

The steady-state analysis is based on Equations (6) and (7) of Section 3.1. In general, the transient analysis of the kernel elements is needed, so that the computational complexity of the steady-state solution is the same as the one of the transient solution.

However, if the subordinated process is a *CTMC*, Equations (6) and (7) can be solved explicitly, and the elements of the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ can be expressed directly from the infinitesimal generator $\mathbf{A}$. Hence, in this case, the steady-state solution can be obtained by a computationally effective method. The proper expressions when the dominant transition is either *prd* or *prs* or *pri* are presented in the following subsections.

Let us now concentrate on the steady-state analysis of a single subordinated process $\mathcal{M}^i(t)$ starting from state $i \in \mathcal{R}_{TP}(M_0)$ under the hypothesis that $\mathcal{M}^i(t)$ is a *CTMC* with infinitesimal generator of the form (33). Combining Equations (6) and (7) with Equations (15) and (14), the ith row of matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ can be written as:

$$\underline{\alpha}_i = \lim_{s\to 0} \frac{1}{s}\,\underline{E}_i^{\sim}(s) = \lim_{s\to 0} \int_{w=0}^{\infty} U_{\mathcal{E}}^i \,\frac{1}{s}\,\mathbf{P}^{i\sim}(s,w)\,\mathbf{S}^i\, dG_g(w) \tag{43}$$

$$\underline{\phi}_i = \lim_{s\to 0} \underline{K}_i^{\sim}(s) = \lim_{s\to 0} \int_{w=0}^{\infty} U_{\mathcal{E}}^i \,[\,\mathbf{F}^{i\sim}(s,w)\,\boldsymbol{\Delta}^i + \mathbf{C}^{i\sim}(s,w)\,\mathbf{S}_{\mathcal{D}}^i\,]\, dG_g(w) \tag{44}$$

If the dominant transition is deterministic, Equations (43) and (44) simplify, since the integration $\int_{w=0}^{\infty}[\bullet]\, dG_g(w)$ is avoided. Equations (43) and (44) are now particularized according to the preemption policy of the dominant transition. In the sequel, the superscript i in the symbols is omitted.

5.1 The dominant MEM transition is prd

Theorem 6 *Given the dominant transition is prd, and the subordinated process is a CTMC with generator of the form (33), the* $\underline{\alpha}_i$ *and* $\underline{\phi}_i$ *row vectors are given by:*

$$\underline{\alpha}_i = \int_{w=0}^{\infty} U_{\mathcal{E}}\,[\,\mathbf{L}(w),\ \mathbf{0}\,]\,\mathbf{S}\, dG_g(w) \tag{45}$$

$$\underline{\phi}_i = \int_{w=0}^{\infty} U_{\mathcal{E}}\,[\,e^{w\mathbf{A}_{\mathcal{E}}}\ \boldsymbol{\Delta} + \mathbf{L}(w)\ \mathbf{A}_{\mathcal{E}\mathcal{D}}\,\mathbf{S}_{\mathcal{D}}\,]\, dG_g(w) \tag{46}$$

where $\mathbf{L}(w) = \int_{z=0}^{w} e^{z\,\mathbf{A}_{\mathcal{E}}}\, dz$.

Proof - From (21) and (40) we obtain:

$$\lim_{s\to 0} \frac{1}{s}\,\mathbf{P}^{\sim}(s,w) = \lim_{s\to 0} \frac{1}{s}\,[\,\mathbf{P}\mathcal{E}^{\sim}(s,w),\,\mathbf{0}\,] = [\,\mathbf{L}(w),\ \mathbf{0}\,] \tag{47}$$

substituting (47) into (43), Equation (45) is obtained. Furthermore, from (21), (41) and (42), we get:

$$\begin{aligned}&\lim_{s\to 0}[\,\mathbf{F}^{\sim}(s,w)\,\boldsymbol{\Delta} + \mathbf{C}^{\sim}(s,w)\,\mathbf{S}_{\mathcal{D}}] = \lim_{s\to 0}[\,\mathbf{F}\mathcal{E}^{\sim}(s,w)\ \boldsymbol{\Delta} + \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{S}_{\mathcal{D}}\,] = \\ &\mathbf{F}\mathcal{E}^{\sim}(0,w)\ \boldsymbol{\Delta} + \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(0,w)\,\mathbf{S}_{\mathcal{D}} = e^{w\mathbf{A}_{\mathcal{E}}}\ \boldsymbol{\Delta} + \mathbf{L}(w)\,\mathbf{A}_{\mathcal{E}\mathcal{D}}\,\mathbf{S}_{\mathcal{D}}\end{aligned} \tag{48}$$

Equation (48), combined with (44), provides (46) □.

Theorem 6 shows that the row elements of matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ can be directly evaluated from the infinitesimal generator of the underlying *CTMC* at the same cost of evaluating the transient solution (term $e^{w\mathbf{A}_\varepsilon}$ in Equation 46), or the integral solution (term $\mathbf{L}(w)$ in Equations 45 and 46), up to time w.

5.2 The dominant MEM transition is pri

Theorem 7 *Given the dominant transition is pri, and the subordinated process is a CTMC with generator of the form (33), the* $\underline{\alpha}_i$ *and* $\underline{\phi}_i$ *row vectors are given by:*

$$\underline{\alpha}_i = \int_{w=0}^{\infty} U_{\mathcal{E}} \left[\mathbf{I} + \mathbf{L}(w)\, \mathbf{A}_{\mathcal{ED}}\, \mathbf{A}_{\mathcal{D}}^{-1}\, \mathbf{A}_{\mathcal{DE}} \right]^{-1} \left[\mathbf{L}(w),\ -\mathbf{L}(w)\, \mathbf{A}_{\mathcal{ED}}\, \mathbf{A}_{\mathcal{D}}^{-1} \right] \mathbf{S}\, dG_g(w) \tag{49}$$

$$\underline{\phi}_i = \int_{w=0}^{\infty} U_{\mathcal{E}} \left[\mathbf{I} + \mathbf{L}(w)\, \mathbf{A}_{\mathcal{ED}}\, \mathbf{A}_{\mathcal{D}}^{-1}\, \mathbf{A}_{\mathcal{DE}} \right]^{-1} e^{w\mathbf{A}_\varepsilon}\, \boldsymbol{\Delta}\, dG_g(w) \tag{50}$$

where $\mathbf{L}(w) = \int_{z=0}^{w} e^{z\,\mathbf{A}_\varepsilon}\, dz$.

Proof - From (23), we obtain:

$$\begin{aligned} &\lim_{s\to 0} \frac{1}{s} \mathbf{P}^{\sim}(s,w) = \\ &\lim_{s\to 0} \frac{1}{s} \left[\mathbf{I} - \mathbf{P}\mathcal{ED}^{\sim}(s,w)\, \mathbf{P}\mathcal{DE}^{\sim}(s)\right]^{-1} \left[\mathbf{P}\mathcal{E}^{\sim}(s,w),\ \mathbf{P}\mathcal{ED}^{\sim}(s,w)\, \mathbf{P}\mathcal{D}^{\sim}(s)\right] \end{aligned} \tag{51}$$

remembering the explicit expressions from (35) to (42), expression (49) is easily obtained. Similarly, from (22), we have:

$$\lim_{s\to 0} \mathbf{F}^{\sim}(s,w) = \left[\mathbf{I} - \mathbf{P}\mathcal{ED}^{\sim}(s,w)\, \mathbf{P}\mathcal{DE}^{\sim}(s)\right]^{-1} \mathbf{F}\mathcal{E}^{\sim}(s,w) \tag{52}$$

from which Equation (50) can be obtained by substituting the explicit expressions (35) - (42) in the corresponding terms in (52). □

The steady-state solution in the *pri* case, involves the inversion of matrix $\mathbf{A}_{\mathcal{D}}$ and of the matrix term $\mathbf{I} + \mathbf{L}(w)\, \mathbf{A}_{\mathcal{ED}}\, \mathbf{A}_{\mathcal{D}}^{-1}\, \mathbf{A}_{\mathcal{DE}}$. The cardinality of these square matrices is equal to the number of states in $\mathcal{D}$ and $\mathcal{E}$, respectively. However, all the terms can be computed by the knowledge of the infinitesimal generator of the subordinated *CTMC*, only.

5.3 The dominant MEM transition is prs

Theorem 8 *Given the dominant transition is prs, and the subordinated process is a CTMC with generator of the form (33), the* $\underline{\alpha}_i$ *and* $\underline{\phi}_i$ *row vectors are given by:*

$$\underline{\alpha}_i = \int_{w=0}^{\infty} U_{\mathcal{E}} \left[\mathbf{L}_\beta(w),\ -\mathbf{L}_\beta(w)\, \mathbf{A}_{\mathcal{ED}}\, \mathbf{A}_{\mathcal{D}}^{-1} \right] dG_g(w) \tag{53}$$

$$\underline{\phi}_i = \int_{w=0}^{\infty} U_{\mathcal{E}} \left[e^{w\boldsymbol{\beta}} \right] \boldsymbol{\Delta}\, dG_g(w) \tag{54}$$

where

$$\boldsymbol{\beta} \;=\; \mathbf{A}_{\mathcal{E}} \;-\; \mathbf{A}_{\mathcal{ED}}\,\mathbf{A}_{\mathcal{D}}^{-1}\,\mathbf{A}_{\mathcal{DE}} \qquad \textit{and} \qquad \mathbf{L}_{\beta}(w) = \int_{z=0}^{w} e^{z\boldsymbol{\beta}}\; dz$$

Proof - We adopt the notation $LT_{v\to w}^{-1}\,A^{*}(v) \;=\; A(w)$ to indicate the inverse Laplace transformation with respect to the variable v. From (26) and (35) - (42), by successive manipulations, we can write:

$$\begin{aligned}
& LT_{v\to w}^{-1}\;\lim_{s\to 0}\frac{1}{s}\;\mathbf{P}^{\sim *}(s,v) \\
& = LT_{v\to w}^{-1}\;\lim_{s\to 0}\frac{1}{s}\;[\mathbf{I} \;-\; v\,\mathbf{P}_{\mathcal{ED}}^{\sim *}(s,v)\,\mathbf{P}_{\mathcal{DE}}^{\sim}(s)]^{-1} \\
& \qquad\qquad [\mathbf{P}_{\mathcal{E}}^{\sim *}(s,v)\,,\; \mathbf{P}_{\mathcal{ED}}^{\sim *}(s,v)\,\mathbf{P}_{\mathcal{D}}^{\sim}(s)] \\
& = LT_{v\to w}^{-1}[\;\mathbf{I} + (v\mathbf{I}-\mathbf{A}_{\mathcal{E}})^{-1}\;\mathbf{A}_{\mathcal{ED}}\;\mathbf{A}_{\mathcal{D}}^{-1}\;\mathbf{A}_{\mathcal{DE}}\;]^{-1} \\
& \qquad\qquad [\;\tfrac{1}{v}\,(v\mathbf{I}-\mathbf{A}_{\mathcal{E}})^{-1}\;,\; -\tfrac{1}{v}\,(v\mathbf{I}-\mathbf{A}_{\mathcal{E}})^{-1}\;\mathbf{A}_{\mathcal{ED}}\;\mathbf{A}_{\mathcal{D}}^{-1}\;] \\
& = LT_{v\to w}^{-1}[\;\frac{1}{v}\,(v\mathbf{I}-\boldsymbol{\beta})^{-1}\;,\; -(v\mathbf{I}-\frac{1}{v}\,\boldsymbol{\beta})^{-1}\;\mathbf{A}_{\mathcal{ED}}\;\mathbf{A}_{\mathcal{D}}^{-1}\;]
\end{aligned} \tag{55}$$

from which Equation (53) is obtained. Furthermore, from (25), we can write:

$$\begin{aligned}
& LT_{v\to w}^{-1}[\;\lim_{s\to 0}\;\mathbf{F}^{\sim *}(s,v)\;] \\
& = LT_{v\to w}^{-1}[\;\mathbf{I} + (v\mathbf{I}-\mathbf{A}_{\mathcal{E}})^{-1}\;\mathbf{A}_{\mathcal{ED}}\;\mathbf{A}_{\mathcal{D}}^{-1}\;\mathbf{A}_{\mathcal{DE}}\;]^{-1}\;\;(v\mathbf{I}-\mathbf{A}_{\mathcal{E}})^{-1} \\
& = LT_{v\to w}^{-1}[\;(v\mathbf{I}-\boldsymbol{\beta})^{-1}\;]
\end{aligned} \tag{56}$$

Equation (54) comes from (56). □

Matrix $\boldsymbol{\beta}$ is the infinitesimal generator of a *CTMC* defined over the states in $\mathcal{E}$. Hence, the computational complexity associated to the solution of Equations (53) and (54) is determined by the computation of $\boldsymbol{\beta}$ (that involves the inverse of $\mathbf{A}_{\mathcal{D}}$), and the evaluation of the transient solution of the *CTMC* with generator $\boldsymbol{\beta}$ up to time w (terms $[e^{w\boldsymbol{\beta}}]$) and its integral (term $[\mathbf{L}_{\beta}(w)]$). A fully developed example has been reported in [31].

6 Combined preemption policies: an example

A two processor system runs two types of jobs according to the following scheduling policy. Jobs of class 1 require both processors and have preemptive priority over jobs of class 2. Jobs of class 2 have lower priority and are scheduled to run on a single processor that is chosen according to a predefined switching probability.

A *PN* modeling the system operation according to the described scheduling policy is represented in Figure 3a. Place p_1 is the customer of class 2 thinking. The thinking time is exponentially distributed with a global rate λ. However, with a rate $c\cdot\lambda$ jobs are directed to processor 1 (transition t_1), and with a rate $(1-c)\cdot\lambda$ jobs are directed to

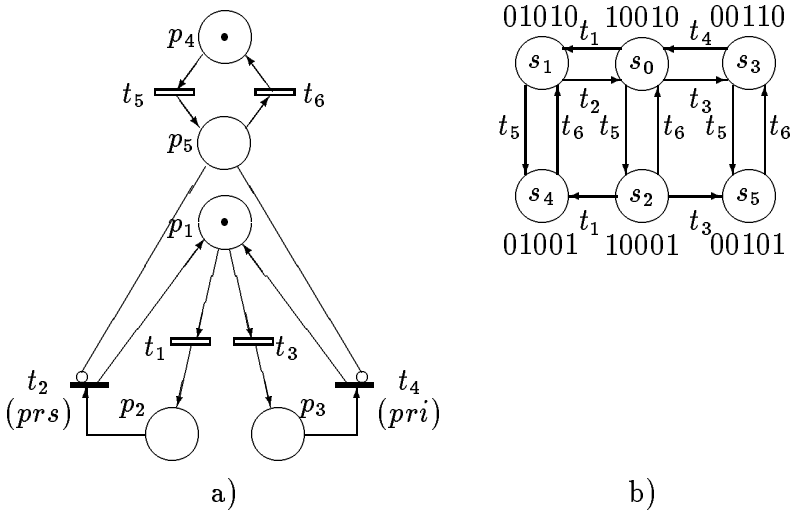

Figure 3: The Petri net and the reachability graph of the two processor system

processor 2 (transition t_3). Place p_2 (p_3) is processor 1 (processor 2) serving customer 2 with service time distribution modeled by transition t_2 (t_4). Place p_4 is customer 1 thinking, while place p_5 represents jobs of class 1 running on both processors while preempting customer 2 under service. (Inhibitor arcs from p_5 to both t_2 and t_4). Transitions t_5 and t_6 represent the arrival and the service of jobs of class 1 and have an exponentially distributed firing time with parameter λ_5 and λ_6, respectively.

We assume that the service time of customer 2 is generally distributed. Therefore, t_2 and t_4 are MEM transitions (filled rectangles) with distribution $G_2(t)$ and $G_4(t)$, respectively. We further assume that processor 1 has 'state saving' capabilities so that the execution of jobs is *prs*. Processor 2, instead, does not have 'state saving' capabilities so that a recovery of an interrupted job occurs according to a *pri* policy. To this end, we associate to transition t_2 a *prs* policy, and to transition t_4 a *pri* policy.

Inspection of the reachability graph, depicted in Figure 3b, leads to the following assertions:

- The instants of entrance into state s_0 and s_2 are always *RTP*s, and the associated subordinated processes are concluded by the next state transition due to the enabled EXP transitions;

- The instants of entrance into states s_1 and s_3 from s_0 are *RTP*s as well. According to the described system characteristics the subordinated process starting from state s_1 (s_3) is dominated by t_2 (t_4) with *prs* (*pri*) policy and is concluded by the firing of t_2 (t_4), only.

- The instants of entrance into states s_4 and s_5 from s_2 are *RTP*s. But when states s_4 and s_5 are entered from s_1 and s_3 they do not generate *RTP*s since the memory

variable associated to transitions t_2 and t_4, respectively, is never zero. The outgoing transitions from s_4 and s_5 are EXP transitions.

From the above assertions follows that all the states can become a regeneration state, i.e. $\mathcal{R}_{TP}(M_0) = \mathcal{R}(M_0)$ and $N' = N = 6$. The analysis proceeds by examining in isolation all the subordinated processes starting from each possible regeneration state.

Subordinated process starting from s_0 - From s_0, all the enabled transitions are EXP and the next regeneration marking can be either s_1, s_2 or s_3. The subordinated process is a single step *CTMC*.

Subordinated process starting from s_2 - From s_2, all the enabled transitions are EXP and the next regeneration marking can be s_0, s_4 or s_5. The subordinated process is a single step *CTMC*.

Subordinated process starting from s_4 - The EXP transition t_6 is the only enabled one, and the next regeneration marking is s_1. The subordinated process is a single step *CTMC*.

Subordinated process starting from s_5 - The EXP transition t_6 is the only enabled one, and the next regeneration marking is s_3. The subordinated process is a single step *CTMC*.

Subordinated process starting from s_1 - The subordinated process starting from s_1 is dominated by the MEM transition t_2 with associated *prs* policy and is a *CTMC* with state space $\mathcal{R}^{s_1} = \{s_1\,,\, s_4\}$. The next regeneration marking can be s_0, only.

Subordinated process starting from s_3 - The subordinated process starting from s_3 is dominated by the MEM transition t_4 with associated *pri* policy and is a *CTMC* with state space $\mathcal{R}^{s_3} = \{s_3\,,\, s_5\}$. The next regeneration marking can be s_0, only.

Based on the above considerations, the subordinated processes starting from states s_0, s_2, s_4 and s_5 can be evaluated by simple Markovian analysis, and the corresponding rows of the kernel matrices in the transient and the steady-state case can be filled in based on the knowledge of the transition rates of the enabled exponential transitions. The analysis of the subordinated processes starting from states s_1 and s_3 deserves a more detailed description.

6.1 The analysis of the subordinated process starting from s_1

The markings reachable during the subordinated process starting from state s_1 are generated (Theorem 1) by removing the dominant MEM transition t_2 from the original *PN*, and by assuming s_1 as initial marking. Figure 4 shows the reduced *PN* and its reachability graph corresponding to the state space $\mathcal{R}^{s_1}$ of the subordinated process dominated by the removed transition t_2. According to the definitions of Section 4, the state space is $\mathcal{R}^{s_1} = \{s_1, s_4\}$, where $\mathcal{E}^{s_1} = \{s_1\}$ and $\mathcal{D}^{s_1} = \{s_4\}$ ($N_{\mathcal{E}}^{s_1} = 1$ and $N_{\mathcal{D}}^{s_1} = 1$).

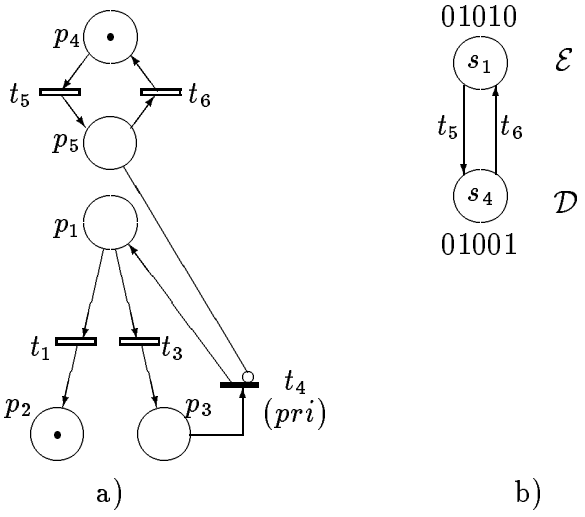

Figure 4: The subordinated process starting from s_1

Since only EXP transitions are enabled in this reduced *PN*, the subordinated process is a *CTMC* with generator

$$\mathbf{A}^{s_1} = \begin{bmatrix} -\lambda_5 & \lambda_5 \\ \lambda_6 & -\lambda_6 \end{bmatrix}$$

the non-overlapping condition with subordinated *SMP* as it is discussed, for example, in [5, 30]. In this example we assume t_6 to be EXP in order to apply the results available for the steady state analysis (Equation (45) - (56)).

During the generation of the reachable markings in the subordinated process the shuffel matrix can be generated based on the correspondence of the states:

$$\mathbf{S}^{s_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and the effect of the firing of t_2 is stored as:

$$\boldsymbol{\Delta}^{s_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Equations (35) - (39) for this example, we obtain the following (1×1) matrices:

$$P\mathcal{E}^{s_1\sim *}(s,v) = \frac{s}{v(s+v+\lambda_5)}$$

$$F\mathcal{E}^{s_1\sim *}(s,v) = \frac{1}{s+v+\lambda_5}$$

$$P\mathcal{E}\mathcal{D}^{s_1\sim *}(s,v) = \frac{\lambda_5}{v(s+v+\lambda_5)}$$

$$PD^{s_1\sim}(s) \;=\; \frac{s}{s+\lambda_6}$$

$$PDE^{s_1\sim}(s) \;=\; \frac{\lambda_6}{s+\lambda_6}$$

Applying Equation (25) and (26), we obtain:

$$P^{s_1\sim *}(s,v) \;=\; \left[\; \frac{s(s+\lambda_6)}{v[s^2+(\lambda_5+\lambda_6)s+(\lambda_6+s)v]} \;,\; \frac{s\lambda_5}{v[s^2+(\lambda_5+\lambda_6)s+(\lambda_6+s)v]} \;\right]$$

$$F^{s_1\sim *}(s,v) \;=\; \frac{s+\lambda_6}{s^2+(\lambda_5+\lambda_6)s+(\lambda_6+s)v}$$

An inverse Laplace transformation provides:

$$P^{s_1\sim}(s,w) \;=\; \left[\; \frac{s+\lambda_6}{\lambda_5+\lambda_6+s}\left(1-e^{-c_1 w}\right) \;,\; \frac{\lambda_5}{\lambda_5+\lambda_6+s}\left(1-e^{-c_1 w}\right) \;\right]$$

$$F^{s_1\sim}(s,w) \;=\; e^{-c_1 w}$$

where $c_1 = \dfrac{s(\lambda_5+\lambda_6+s)}{\lambda_6+s}$. Note that condition (13), which is also valid in LST domain, holds. The next step, which is the application of (14) and (15), results in the 2nd row of the kernel matrices. To go further in the analytical derivation, let assume that the service time of customer 2 on processor 1 is uniformly distributed on the interval (x_1, x_2). The cumulative distribution function $F(x)$, its derivative $f(x)$ and the Laplace transform $F^{\sim}(s)$ are:

$$F(x) = \begin{cases} 0 & 0 \le x < x_1 \\ \frac{x-x_1}{x_2-x_1} & x_1 \le x \le x_2 \\ 1 & x > x_2 \end{cases} \quad , \quad f(x) = \frac{1}{x_2-x_1} \quad , \quad F^{\sim}(s) = \frac{e^{-x_1 s}-e^{-x_2 s}}{s(x_2-x_1)}$$

The following expressions hold:

$$E^{\sim}_{s_1 s_1}(s) = \frac{s+\lambda_6}{\lambda_5+\lambda_6+s}\left[1-\frac{e^{-c_1 x_1}-e^{-c_1 x_2}}{c_1(x_2-x_1)}\right]$$

$$E^{\sim}_{s_1 s_4}(s) = \frac{\lambda_5}{\lambda_5+\lambda_6+s}\left[1-\frac{e^{-c_1 x_1}-e^{-c_1 x_2}}{c_1(x_2-x_1)}\right]$$

$$K^{\sim}_{s_1 s_0}(s) = \frac{e^{-c_1 x_1}-e^{-c_1 x_2}}{c_1(x_2-x_1)}$$

To evaluate $\alpha_{s_1 s_1}$ and $\alpha_{s_1 s_4}$, Equation (55) is used:

$$\alpha_{s_1 s_1} = \int_0^\infty w \; dG_2(w) = \frac{x_1+x_2}{2}$$

$$\alpha_{s_1 s_4} = \int_0^\infty \frac{\lambda_5}{\lambda_6} \; w \; dG_2(w) = \frac{\lambda_5(x_1+x_2)}{2\lambda_6}$$

While the 2nd row of the ϕ matrix is know from the fact that next regeneration period starts always from s_1.

6.2 The analysis of the subordinated process starting from s_3

Following the same approach, one can observe that the subordinated processes starting from marking s_1 and s_3 are the same two-state $CTMC$ with rates λ_5 and λ_6, i.e. $\mathbf{A}^{s_1} = \mathbf{A}^{s_3}$. Hence, also the partitioned state space related measures (35) - (39) are the same. The difference between the relevant rows of the kernel matrices comes from the different preemption policies of t_4 (pri) with respect to t_2 (prs).

In order to analyze the subordinated process of a pri dominant transition we also need (from Equation (40) - (42)):

$$P\mathcal{E}^{s_3\sim}(s,w) \;=\; s\int_0^w e^{-(s+\lambda_5)z}dz \;=\; \frac{s}{s+\lambda_5}\left[1-e^{-(s+\lambda_5)w}\right]$$

$$F\mathcal{E}^{s_3\sim}(s,w) \;=\; e^{-(s+\lambda_5)w}$$

$$P\mathcal{ED}^{s_3\sim}(s,w) \;=\; \lambda_5\int_0^w e^{-(s+\lambda_5)z}dz \;=\; \frac{\lambda_5}{s+\lambda_5}\left[1-e^{-(s+\lambda_5)w}\right]$$

from which Theorem 3 gives:

$$P^{s_3\sim}(s,w) \;=\; \left[\;\frac{s(s+\lambda_6)(1-c_2)}{(s+\lambda_5)(s+\lambda_6)-\lambda_5\lambda_6(1-c_2)}\;,\;\;\frac{s\lambda_5(1-c_2)}{(s+\lambda_5)(s+\lambda_6)-\lambda_5\lambda_6(1-c_2)}\;\right]$$

$$F^{s_3\sim}(s,w) \;=\; \frac{(s+\lambda_5)(s+\lambda_6)c_2}{(s+\lambda_5)(s+\lambda_6)-\lambda_5\lambda_6(1-c_2)}$$

where $c_2 = e^{-(s+\lambda_5)w}$. Once again, condition (13) holds.

From the above expressions, the 4th row of the kernel matrices $E^{\sim}_{s_3s_3}(s)$, $E^{\sim}_{s_3s_5}(s)$, $K^{\sim}_{s_3s_0}(s)$ can be obtained by unconditioning with respect to the firing time distribution of t_4.

Considering again the special case when the service time of customer 2 on processor 2 is uniformly distributed on the interval (x_3, x_4), we obtain:

$$E^{\sim}_{s_3s_3}(s) = \frac{s+\lambda_6}{s+\lambda_5+\lambda_6}\left[\frac{(s+\lambda_6)c_3}{(x_4-x_3)\lambda_5\lambda_6}+1\right]$$

$$E^{\sim}_{s_3s_5}(s) = \frac{\lambda_5}{s+\lambda_5+\lambda_6}\left[\frac{(s+\lambda_6)c_3}{(x_4-x_3)\lambda_5\lambda_6}+1\right]$$

$$K^{\sim}_{s_3s_0}(s) = -\frac{(s+\lambda_6)c_3}{(x_4-x_3)\lambda_5\lambda_6}$$

where $c_3 = ln[s(s+\lambda_5+\lambda_6)+\lambda_5\lambda_6e^{-(s+\lambda_5)x_4}] - ln[s(s+\lambda_5+\lambda_6)+\lambda_5\lambda_6e^{-(s+\lambda_5)x_3}]$.

To compute $\alpha_{s_3s_3}$ and $\alpha_{s_3s_5}$ (49) is used, while $\phi_{s_3s_0} = 1$.

$$\alpha_{s_3s_3} = \int_0^\infty \frac{1}{\lambda_5}\,(e^{\lambda_5 w}-1)\;dG_4(w) = \frac{1}{\lambda_5}\left[\frac{e^{\lambda_5x_4}-e^{\lambda_5x_3}}{\lambda_5(x_4-x_3)}-1\right]$$

$$\alpha_{s_3s_5} = \int_0^\infty \frac{1}{\lambda_6}\,(e^{\lambda_5 w}-1)\;dG_4(w) = \frac{1}{\lambda_6}\left[\frac{e^{\lambda_5x_4}-e^{\lambda_5x_3}}{\lambda_5(x_4-x_3)}-1\right]$$

6.3 Numerical results

The transient analysis can be performed by applying Equation (4), where the complete kernel matrices are given by:

$$\mathbf{K}^{\sim}(s) = \begin{bmatrix} 0 & \frac{c\,\lambda}{s+\lambda+\lambda_5} & \frac{\lambda_5}{s+\lambda+\lambda_5} & \frac{(1-c)\,\lambda}{s+\lambda+\lambda_5} & 0 & 0 \\ K^{\sim}_{s_1 s_0}(s) & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda_6}{s+\lambda+\lambda_6} & 0 & 0 & 0 & \frac{c\,\lambda}{s+\lambda+\lambda_6} & \frac{(1-c)\,\lambda}{s+\lambda+\lambda_6} \\ K^{\sim}_{s_3 s_0}(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda_6}{s+\lambda_6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_6}{s+\lambda_6} & 0 & 0 \end{bmatrix}$$

$$\mathbf{E}^{\sim}(s) = \begin{bmatrix} \frac{s}{s+\lambda+\lambda_5} & 0 & 0 & 0 & 0 & 0 \\ 0 & E^{\sim}_{s_1 s_1}(s) & 0 & 0 & E^{\sim}_{s_1 s_4}(s) & 0 \\ 0 & 0 & \frac{s}{s+\lambda+\lambda_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & E^{\sim}_{s_3 s_3}(s) & 0 & E^{\sim}_{s_3 s_5}(s) \\ 0 & 0 & 0 & 0 & \frac{s}{s+\lambda_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{s}{s+\lambda_6} \end{bmatrix}$$

The steady state analysis requires the computation of the following matrices:

$$\boldsymbol{\phi} = \begin{bmatrix} 0 & \frac{c\,\lambda}{\lambda+\lambda_5} & \frac{\lambda_5}{\lambda+\lambda_5} & \frac{(1-c)\,\lambda}{\lambda+\lambda_5} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda_6}{\lambda+\lambda_6} & 0 & 0 & 0 & \frac{c\,\lambda}{\lambda+\lambda_6} & \frac{(1-c)\,\lambda}{\lambda+\lambda_6} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{1}{\lambda+\lambda_5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{s_1 s_1} & 0 & 0 & \alpha_{s_1 s_4} & 0 \\ 0 & 0 & \frac{1}{\lambda+\lambda_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{s_3 s_3} & 0 & \alpha_{s_3 s_5} \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_6} \end{bmatrix}$$

Suppose that the two processors are of different classes, and the average computation time on processor 1 is less than on processor 2 (i.e.: $x_2 - x_1 > x_4 - x_3$). The considered design problem consists in optimizing the switching probability c in order to optimize the performance characteristics of the system.

The steady state probabilities (v_j) can be evaluated based on $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ by Equation (8) and (9). With a given traffic pattern the dependence of the system performance on

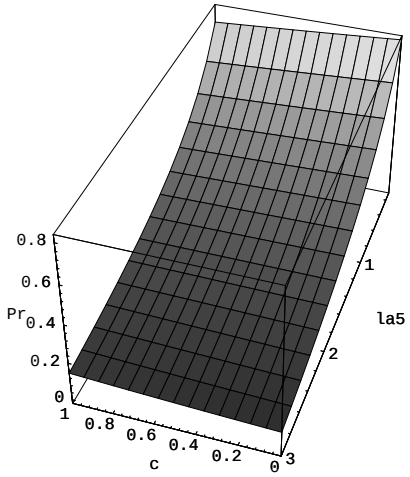

Figure 5: Probability of state s_0 (idle state) versus c and λ_5

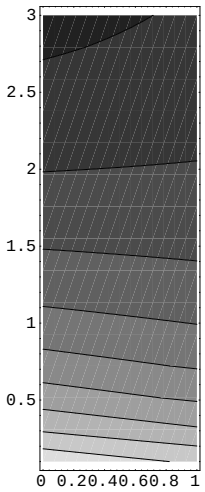

Figure 6: Surface plot of Figure 5

c can be measured by the steady state probability of the idle state (s_0). The better the system performance is, the higher is the steady state probability of s_0.

In Figure 5, the steady state probability of s_0 is depicted as a function of the switching probability (c) and the submission rate of customer 1 (λ_5). Figure 5 contains the 3D-view of the function. In order to emphasize the dependence on c Figure 6 shows the surface plot of it. The parameters of the model are set as follows:

- the submission rate of customer 1 (λ_5) is varying from 0.1 to 3,
- the service rate of customer 1 (λ_6) is 1,
- the submission rate of customer 2 (λ) is 0.2,
- the service time of customer 2 is assumed to be uniformly distributed. On the slower processor (processor 1) the service time (firing time of t_2) is uniformly distributed on $(0, 2)$ (i.e.: $x_1 = 0, x_2 = 2$), while on the faster processor (processor 2) the service time (firing time of t_4) is uniformly distributed on $(0, 1)$ (i.e.: $x_3 = 0, x_4 = 1$),

- the switching probability (c) is varying between 0 and 1.

As can be observed from the figures, the optimal value of c depends on λ_5 as well. For the case when $\lambda_5 > 1.78$ (frequent preemption of the low priority job), $c = 1$ (*prs* policy) results in the best performance, and if $\lambda_5 < 1.78$ (rare preemption of the low priority job), $c = 0$ (*pri* policy) results in the best performance. No probabilistic mixture of the two preemption policies ($0 < c < 1$) results in a better performance than the one of the two extreme cases.

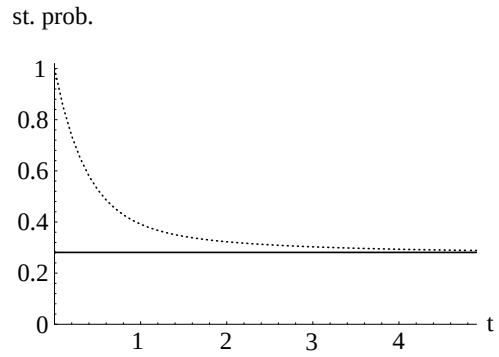

Figure 7: Transient probability of s_0

The transient behavior of the system idle probability ($Pr(\mathcal{M}(t) = s_0)$) is depicted on Figure 7, when the submission rate of customer 1 is $\lambda_5 = 1.5.$ and $c = 0.5$. The horizontal line indicates the steady state probability of s_0 (0.281063) obtained by the steady state analysis method.

7 Computational complexity

Let us briefly summarize the elementary computational steps for the evaluation of the transient and steady-state solution in the proposed approach.

7.1 Transient analysis

Let us first suppose that all the MEM transitions are deterministic. The computational method can be divided in the following steps:

1. generation of all the subordinated process according to the operational procedure given in Theorem 1;
2. for each subordinated process, symbolic derivation of the non-zero entries of the corresponding row in the kernels $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ (in the LST domain);
3. symbolic matrix inversion and matrix multiplication in order to obtain the $\mathbf{V}^{\sim}(s)$ matrix (Equation 4) in the LST domain;

4. time domain solution obtained by a numerical inversion of the entries of the $\mathbf{V}^{\sim}(s)$.

If the dominant MEM transition is not deterministic, Step 2) in the above list should be replaced by:

2′ symbolic derivation of the entries of the $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ matrices in LST domain as in the deterministic case;

2″ unconditioning of the entries of the $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ matrices, according to the Cdf of the firing time of the MEM transitions (Equations (14) and (15)).

Any standard algorithm generating reachability graph from standard PN description can be used for Step 1); during the generation of the state space of a given subordinated process, matrices $\boldsymbol{\Delta}$ and $\mathbf{S}$ must be filled in. In our case Step 2) is done manually, only because of the intrinsic limitations of the mathematical packages we used in manipulating symbolic algebraic expressions. The complexity of this step depends on the non-zero entries of the involved matrices, and on the complexity of the process subordinated to the dominant MEM transitions.

The computational complexity of Step 3) depends on the dimension of the matrices (i.e. the number of tangible markings) and the complexity of the elements of the kernels (the difficulty of Step 3 is related to the difficulty of Step 2). We performed these symbolic computations using MATHEMATICA. The complexity of the numerical inversion at Step 4) also depends on two factors; the complexity of the function to invert, and the prescribed accuracy. We used the Jagerman's method [21] implemented in MATHEMATICA language.

7.2 Steady-state analysis

In the general case the steady-state analysis follows the same procedure described for the transient one. However, a very efficient technique has been extensively discussed in Section 5 in the particular case where all the subordinated processes are $CTMC$s. The main difference is that the introduced method is composed by numerical analysis steps only, hence it does not require any symbolic and transform domain manipulation as it is in the case of transient analysis.

If the transition dominating the considered regeneration period is deterministic, the steady-state analysis requires for each subordinated process the transient solution for the instantaneous and integral probabilities of a $CTMC$ up to the firing time w of the dominant transition. Any standard algorithm can be used at this purpose, and it is known that the computation of the instantaneous probabilities and of the integral probabilities can be performed at the same time with a very little overhead [28, 29]. When the transition dominating the considered regeneration period is not deterministic, numerical integration according to the distribution of the firing time has to be performed [26].

8 Conclusion

The paper has provided a detailed discussion and an analytical procedure to deal with the class of *MRSPN* with non-overlapping dominant transitions. The analytical description is based on the Markov renewal theory and allows to include different preemption policies associated to different dominant transitions into a single *MRSPN* model. Complete analytical descriptions have been provided for the transient and steady state cases, when all the subordinated processes are restricted to be *CTMC* or *SMP*.

The complexity of the proposed approach has been discussed and, for the steady-state analysis with subordinated *CTMC*s, a procedure ready to be implemented into an automatic solution tool has been provided.

The transient analysis still requires some symbolic manipulation. However, we believe that all the necessary elements for its automatic computation have been fully identified.

Our approach extends previously available results and offer the possibility to adopt different preemption policies inside the same model, thus widening the field of applicability of stochastic Petri nets.

Future extensions are related with the introduction of interlaced, or state dependent, memory policies, where the memory of a transition can be modified by the occurrence of some condition on the net.

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Appendix A Proof of Theorem 3

The event that the process is resident in $\mathcal{R}^i = \mathcal{E}^i \cup \mathcal{D}^i$ at time t before the firing of the dominant transition tr_g can be decomposed into the mutually exclusive events according to the number of cycles between $\mathcal{D}$ and $\mathcal{E}$. Hence:

$$\begin{aligned}
\mathbf{P}^{\sim}(s,w) &= [\,\mathbf{P}\mathcal{E}^{\sim}(s,w),\ \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}^{\sim}(s)\,] \\
&\quad + \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}\mathcal{E}^{\sim}(s)\,[\mathbf{P}\mathcal{E}^{\sim}(s,w),\ \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}^{\sim}(s)] \\
&\quad + [\mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}\mathcal{E}^{\sim}(s)]^2\ [\mathbf{P}\mathcal{E}^{\sim}(s,w),\ \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}^{\sim}(s)] \\
&\quad + \ \ldots \qquad (57) \\
&= \sum_{u=0}^{\infty} [\mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}\mathcal{E}^{\sim}(s)]^u\ [\mathbf{P}\mathcal{E}^{\sim}(s,w),\ \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}^{\sim}(s)]
\end{aligned}$$

Expression (23) is obtained from (57) by applying the relation $\sum_{u=0}^{\infty}\mathbf{M}^u = [\mathbf{I}-\mathbf{M}]^{-1}$.

Similarly, for matrix $\mathbf{F}^{\sim}(s,w)$:

$$\begin{aligned}
\mathbf{F}^{\sim}(s,w) &= \mathbf{F}\mathcal{E}^{\sim}(s,w) + \mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}\mathcal{E}^{\sim}(s)\mathbf{F}\mathcal{E}^{\sim}(s,w) + \\
&\quad [\mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}\mathcal{E}^{\sim}(s)]^2\,\mathbf{F}\mathcal{E}^{\sim}(s,w) + \ \ldots \qquad (58) \\
&= \sum_{u=0}^{\infty} [\mathbf{P}\mathcal{E}\mathcal{D}^{\sim}(s,w)\,\mathbf{P}\mathcal{D}\mathcal{E}^{\sim}(s)]^u\ \mathbf{F}\mathcal{E}^{\sim}(s,w)
\end{aligned}$$

which proves the theorem.

An alternative proof of this theorem can be given using the regenerative property at the first return to $\mathcal{E}$.

Appendix B Proof of Theorem 4

To evaluate $F_{ij}(t,w)$ we should consider two different cases for the firing of the dominant MEM transition. The first one is when the transition fires before leaving $\mathcal{E}$ and the second one is when the subordinated process leaves $\mathcal{E}$ for state $k \in \mathcal{D}$ at time t' accumulating w' firing time units, and it stays in $\mathcal{D}$ up to $t"$ when it enters $\ell \in \mathcal{E}$. For the quantitative analysis of these cases we introduce:

$$\overline{P\mathcal{E}\mathcal{D}}_{ij}(t) = Pr\{Z(T_1) = j \in \mathcal{D}\,,\, T_1 < t \,|\, Z(0) = i \in \mathcal{E}\} \qquad (59)$$

and

$$\widehat{P\mathcal{E}\mathcal{D}}_{ij}(t,w) = Pr\{Z(T_1) = j \in \mathcal{D}\,,\, \tau_1^* > T_1 \,|\, Z(0) = i \in \mathcal{E}\,,\, T_1 = t\,,\, \gamma_g = w\} \qquad (60)$$

$F_{ij}(t,w)$ can then be expressed as:

$$\begin{aligned}
&F_{ij}(t,w) = F\mathcal{E}_{ij}(t,w) + \\
&\sum_{k\in\mathcal{D}}\sum_{\ell\in\mathcal{E}} \int_{t'=0}^{t}\int_{t"=t'}^{t}\int_{w'=0}^{w} F_{\ell j}(t-t",w-w')\ d\widehat{P\mathcal{E}\mathcal{D}}_{ik}(t',w')\ d\overline{P\mathcal{E}\mathcal{D}}_{ik}(t')\ dP\mathcal{D}\mathcal{E}_{k\ell}(t")
\end{aligned} \qquad (61)$$

The LST with respect to w results in:

$$F_{ij}^{\sim}(t,v) = F\mathcal{E}_{ij}^{\sim}(t,v) + \sum_{k\in\mathcal{D}}\sum_{\ell\in\mathcal{E}}\int_{t'=0}^{t}\int_{t"=t'}^{t} F_{\ell j}^{\sim}(t-t",v)\ \widehat{P\mathcal{E}\mathcal{D}}_{ik}^{\sim}(t',v)\ d\overline{P\mathcal{E}\mathcal{D}}_{ik}(t')\ dP\mathcal{D}\mathcal{E}_{k\ell}(t") \tag{62}$$

By the definition of $\widehat{P\mathcal{E}\mathcal{D}}_{ij}(t,w)$ and $\overline{P\mathcal{E}\mathcal{D}}_{ij}(t)$ we can write:

$$F_{ij}^{\sim}(t,v) = F\mathcal{E}_{ij}^{\sim}(t,v) + \sum_{k\in\mathcal{D}}\sum_{\ell\in\mathcal{E}}\int_{t'=0}^{t}\int_{t"=t'}^{t} F_{\ell j}^{\sim}(t-t",v)\ dP\mathcal{E}\mathcal{D}_{ik}^{\sim}(t',v)\ dP\mathcal{D}\mathcal{E}_{k\ell}(t") \tag{63}$$

Which is a triple convolution according to the time variable t, whose LST is:

$$F_{ij}^{\sim\sim}(s,v) = F\mathcal{E}_{ij}^{\sim\sim}(s,v) + \sum_{k\in\mathcal{D}}\sum_{\ell\in\mathcal{E}} F_{\ell j}^{\sim\sim}(s,v)\ P\mathcal{E}\mathcal{D}_{ik}^{\sim\sim}(s,v)\ P\mathcal{D}\mathcal{E}_{k\ell}(s) \tag{64}$$

Expression (25) is obtained from (64), by remembering the relation between the LST and the LT of a given function:

$$A^{\sim}(v) \;=\; v\,A^{*}(v)\ .$$

The derivation of $\mathbf{P}^{\sim *}(s,v)$ follows the same pattern.

Appendix C Proof of Theorem 5

Conditioning on the sojourn time $H = h$ in state $k \in \mathcal{E}$, we have:

$$F\mathcal{E}_{k\ell}(t,w\,|\,H=h) \;=\; \begin{cases} \delta_{k\ell}\,U\,(t\;-\;w) & \text{if}: h \geq w \\ \displaystyle\sum_{u\in\mathcal{E}} \frac{dQ_{ku}(h)}{dQ_k(h)}\ \cdot\ F\mathcal{E}_{u\ell}(t-h,w-h) & \text{if}: h < w \end{cases} \tag{65}$$

where $U(t)$ is the unit step function. In (65), two mutually exclusive events are identified. If $h \geq w$, a sojourn time equals to w is accumulated before leaving state k, so that the firing of the dominant MEM transition occures at $t = w$. If $h < w$ then a transition occurs to state u with probability $dQ_{ku}(h)/dQ_k(h)$ and the residual firing time $(w - h)$ should be accumulated starting from state u at time h. Taking the LST with respect to t (denoting the transform variable by s), the LT with respect to w (denoting the transform variable by v) of (65) and then unconditioning with respect to H proves the Theorem.

Similarly conditioning on $H = h$ the probability of sojourn in $\mathcal{E}$ is:

$$P\mathcal{E}_{k\ell}(t,w\,|\,H=h) \;=\; \begin{cases} \delta_{k\ell}\ [U(t)\;-\;U(t-w)] & \text{if}: h\ \geq\ w \\ \delta_{k\ell}\ [U(t) - U(t-h)]\ +\ \displaystyle\sum_{u\in\mathcal{E}} \frac{dQ_{ku}(h)}{dQ_k(h)}\, P\mathcal{E}_{u\ell}(t-h,w-h) & \text{if}: h\ <\ w \end{cases} \tag{66}$$

The derivation of the matrix function $\mathbf{P}\mathcal{E}(t,w)$ based on (66) follows the same pattern as for the function $\mathbf{F}\mathcal{E}(t,w)$.

Finally conditioning on $H = h$, $P\mathcal{E}\mathcal{D}_{k\ell}(t,w)$ can be defined as:

$$P\mathcal{E}\mathcal{D}_{k\ell}(t,w \mid H=h) = \begin{cases} 0 & \text{if} : h \geq w \\ \dfrac{dQ_{k\ell}(h)}{dQ_k(h)}\, U(t-h) + \displaystyle\sum_{u \in R} \dfrac{dQ_{ku}(h)}{dQ_k(h)}\, P\mathcal{E}\mathcal{D}_{u\ell}(t-h, w-h) & \\ & \text{if} : h < w \end{cases} \tag{67}$$

The derivation of the matrix function $\mathbf{P}\mathcal{E}\mathcal{D}(t,w)$ based on (67) follows the same pattern as for the function $\mathbf{F}\mathcal{E}(t,w)$.

The matrix functions $\mathbf{P}\mathcal{D}(t)$ and $\mathbf{P}\mathcal{D}\mathcal{E}(t)$ can be obtained from $\mathbf{P}\mathcal{E}(t,w)$ and $\mathbf{P}\mathcal{E}\mathcal{D}(t,w)$ by interchanging $\mathcal{E}$ and $\mathcal{D}$ and by letting $w \to \infty$.