Partial Loss in Reward Models

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1 Introduction

Stochastic reward models have been applied for performance and reliability analysis of computer systems for a long time [1, 2]. Previously applied models assumed no reward loss at all (referred to as *preemptive resume*) or a complete loss of reward accumulated so far (*preemptive repeat*). In this paper we consider the intermediate case when only a portion of the accumulated reward is lost at a state transition of the considered system. Two partial loss models are considered.

2 Partial loss of the incremental accumulated reward

Let $\{Z(t), t \geq 0\}$ be an ergodic semi-Markov process (SMP) over the state space $R = \{1, 2, ..., N\}$ with kernel $Q(\cdot) = [Q_{ij}(\cdot)]$. Throughout this paper we assume that $Q(t)$ is the canonical kernel of $Z(t)$, i.e., $Q_{ii} = 0, \forall i$ and the mean sojourn time is positive and finite in all states. Suppose whenever the SMP is in state i, reward is accumulated at rate r_i. When the SMP undergoes a state transition out from state i a fraction $(1 - \mathcal{A}_i)$ of the reward accumulated during the last sojourn in state i is lost. $\mathcal{A}_i$ is a r.v. over $(0, 1)$ with cdf $A_i(\cdot)$. Let $Y(t)$ denote the amount of accumulated reward at time t, and T_n the epoch of the n^{th} state transition of the SMP. The dynamics of the right continuous process $\{Y(t), t \geq 0\}$ is defined as follows:

$$\frac{dY(t)}{dt} = r_{Z(t)} \qquad \text{for} \quad T_n < t < T_{n+1} \qquad (1)$$

$$Y(T_n) = Y(T_{n-1}) + \mathcal{A}_{Z(T_n^-)}[Y(T_n^-) - Y(T_{n-1})] \qquad (2)$$

Limiting distribution of $Y(t)$: By this definition $Y(T_n)$ is a non-decreasing series. Supposing $Z(t)$ is an ergodic process with steady state distribution $\underline{\pi} = \{\pi_i\}$, where $\pi_i > 0, \forall i$. Both $lim_{n\to\infty} Y(T_n)$ and $lim_{t\to\infty} Y(t)$ go to infinity if $\exists i \in R$ such that $\pi_i r_i E[\mathcal{A}_i] > 0$.

2.1 Accumulated reward up to time t

Define

$$P_i(t, w) = Pr(Y(t) \leq w \mid Z(0) = i).$$

Theorem 1 *The double transform equation for $P_i(t, w)$ is given by:*

$$P_i^{*\sim}(s, v) = \frac{1 - Q_i^{\sim}(s + vr_i)}{s + vr_i} + \sum_{k \in R} \int_{\tau=0}^{\infty} e^{s\tau} A_i^{\sim}(vr_i\tau)\, dQ_{ik}(\tau) \cdot P_k^{*\sim}(s, v)$$

where $Q_i(t) = \sum_{j \in R} Q_{ij}(t)$.

Proof **1** *Conditioning on H, the sojourn time in state i, we have:*

$$P_i(t, w | H = \tau) = \begin{cases} U_w(w - r_i t) & \text{if} : \tau > t \\ \displaystyle\sum_{k \in R} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot \int_0^1 P_k(t - \tau, w - \alpha\tau r_i) dA_i(\alpha) & \text{if} : \tau < t \end{cases} \qquad (3)$$

Taking the Laplace-Stieltjes transform with respect to w ($\to v$) and unconditioning with respect to H (by $Q_i(t)$) results in:

$$P_i^{\sim}(t, v) = e^{-vr_i t}(1 - Q_i(t)) + \sum_{k \in R} \int_{\tau=0}^{t} A_i^{\sim}(vr_i\tau) \cdot P_k^{\sim}(t - \tau, v)\, dQ_{ik}(\tau)$$

A final Laplace transformation with respect to t ($\to s$) results in the theorem. □

Corollary 1: In case of (state dependent) deterministic loss ratio (i.e., $\mathcal{A}_i = \alpha_i$)

$$P_i^{*\sim}(s, v) = \frac{1 - Q_i^{\sim}(s + vr_i)}{s + vr_i} + \sum_{k \in R} Q_{ik}^{\sim}(s + vr_i\alpha_i) \cdot P_k^{*\sim}(s, v) \qquad (4)$$

Corollary 2: In a CTMC environment with generator $B = [b_{ij}]$ (using the notation $b_i = -b_{ii}$)

$$P_i^{*\sim}(s,v) = \frac{1}{s + vr_i + b_i} + \sum_{k\in R, k\neq i} \frac{b_{ik}}{s + vr_i\alpha_i + b_i} \cdot P_k^{*\sim}(s,v) \,. \qquad (5)$$

Whose solution, in matrix form, is:

$$P^{*\sim}(s,v) = (sI + vR_\alpha - B)^{-1} D_1(s,v) \qquad (6)$$

where the diagonal matrices are defined as $R_\alpha =$ diag$<r_i\alpha_i>$ and $D_1(s,v)=$diag$<\frac{s+vr_i\alpha_i+b_i}{s+vr_i+b_i}>$.

2.2 Completion time

The completion time ($\mathcal{C}(w)$) and its conditional distribution are defined as follow

$$\mathcal{C}(w) = \min[t : Y(t) \geq w],$$
$$C_i(t,w) = Pr(\mathcal{C}(w) \leq t | Z(0) = i).$$

Note that $Y(t)$ has a non-monotone trajectory, hence

$$Pr(Y(t) \leq w | Z(0) = i) \neq Pr(C(w) > t | Z(0) = i)$$

Theorem 2 *With (state dependent) deterministic loss ratio, $C_i(t,w)$ satisfies the following double transform domain equation:*

$$C_i^{\sim *}(s,v) = \frac{r_i\,[1 - Q_i^\sim(s + v\,r_i)]}{s + v\,r_i} + \sum_{k\in R} Q_{ik}^\sim(s + v\,r_i\,\alpha_i)\, C_k^{\sim *}(s,v) \qquad (7)$$

Proof **2** *Conditioning on the sojourn time in state i (H), let us define:*

$$C_i(t,w|\,H = h) = \begin{cases} U\left(t - \dfrac{w}{r_i}\right) & \text{if} : h\,r_i \geq w \\ \displaystyle\sum_{k\in R} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot C_k(t-h, w - hr_i\alpha_i) & \text{if} : h\,r_i < w \end{cases} \qquad (8)$$

Eq. (7) is obtained from (8) by unconditioning with respect to H, taking the Laplace-Stieltjes transform with respect to t and taking the Laplace transform with respect to w. □

Corollary 3: In a CTMC environment with generator $B = [b_{ij}]$ ($b_i = -b_{ii}$)

$$C_i^{\sim *}(s,v) = \frac{r_i}{s + vr_i + b_i} + \sum_{k\in R, k\neq i} \frac{b_{ik}}{s + vr_i\alpha_i + b_i} \cdot C_k^{\sim *}(s,v) \qquad (9)$$

From which

$$C^{*\sim}(s,v) = (sI + vR_\alpha - B)^{-1} D_2(s,v) \qquad (10)$$

where $D_2(s,v)=$diag$<\frac{r_i(s+vr_i\alpha_i+b_i)}{s+vr_i+b_i}>$.

3 Partial loss on the total accumulated reward

The same ergodic semi-Markov environment is considered but with a different accumulation process. Whenever the SMP is in state i reward is accumulated at rate r_i. When the SMP undergoes a transition out from state i the fraction $(1 - \mathcal{A}_i)$ of the total accumulated reward is lost and the fraction $\mathcal{A}_i$ of the total reward is resumed in the new state. In the sequel we assume that $\mathcal{A}_i = \alpha_i$ ($0 \leq \alpha_i \leq 1$) is deterministic[1]. The dynamics of the right continuous reward process $\{Y(t), t \geq 0\}$ is defined as follows:

$$\frac{dY(t)}{dt} = r_{Z(t)} \quad \text{for } T_n \leq t < T_{n+1} \qquad (11)$$
$$Y(T_n) = \mathcal{A}_{Z(T_n^-)} Y(T_n^-) \qquad (12)$$

3.1 Limiting distribution of $Y(t)$

In contrast with the partial incremental loss case, $Y(T_n)$ does not increase to infinity (if a state $i \in R$ exists such that $\alpha_i < 1$), since the average reward accumulated during a sojourn in any state is finite and independent of $Y(t)$, while the amount of reward lost at any state transition is proportional to $Y(t)$. A larger $Y(t)$ results in a larger reward loss which suggests that $Y(t)$ has a finite limiting behaviour.

We shall derive the limiting joint distribution of the $\{(Y(t), Z(t)), t \geq 0\}$ process. We first look at $\{(Y(T_n), Z(T_n)), n \geq 0\}$ process. Define

$$f_n(w,i)dw = Pr\{Y(T_n) \in (w, w+dw), Z(T_n) = i\}$$

$$f(w,i) = \lim_{n\to\infty} f_n(w,i)$$

$$f^*(v,i) = \lim_{n\to\infty} E(e^{-vY(T_n)}; Z(T_n) = i) = \int_{w=0}^{\infty} e^{-vw} f(w,i)\, dw.$$

Theorem 3 *The Laplace transform of the limiting joint distribution of the $\{(Y(T_n), Z(T_n))\}$ process is given by:*

$$f^*(v,j) = \sum_{i\in R} \alpha_i f^*(v\alpha_i, i) Q_{ij}^\sim(vr_i\alpha_i). \qquad (13)$$

[1] The case when $\mathcal{A}_i$ is a r.v. over $(0,1)$ with distribution $A_i(.)$ can be considered following the same approach, but it results in very complicated expressions.

Proof **3** *Suppose a transition to state j occurs after a sojourn time τ in state i, then*

$$f(w,j|i,\tau) = f(\frac{w}{\alpha_i} - r_i\tau, i).$$

Unconditioning with respect to the preceding state and the sojourn time gives

$$f(w,j) = \sum_{i\in R} \int_{\tau=0}^{\frac{w}{r_i\alpha_i}} f(\frac{w}{\alpha_i} - r_i\tau, i)\, dQ_{ij}(\tau),$$

finally the Laplace transformation with respect to $w(\to v)$ results in the theorem. □

Now we define the Laplace transform of the limiting joint distribution of the $\{(Y(t), Z(t))\}$ process as:

$$g_t(w,i)dw = Pr\{Y(t) \in (w, w+dw), Z(t) = i\}$$

$$g(w,i) = \lim_{t\to\infty} g_t(w,i)$$

$$g^*(v,i) = \lim_{t\to\infty} E(e^{-vY(t)}; Z(t) = i) = \int_{w=0}^{\infty} e^{-vw} g(w,i)\, dw.$$

and let $\gamma_i = \int_0^\infty t\; dQ_i(t)$, be the mean sojourn time in state i.

Theorem 4

$$g^*(v,i) = f^*(v,i)\frac{1 - Q_i^{\sim}(vr_i)}{vr_i\gamma_i}. \tag{14}$$

Proof **4** *Suppose the SMP is in steady state at time 0; then the age of the current soujourn is given by the equilibrium distribution of $Q_i(t)$ if the current state is i. Hence Y(0) equals the accumulated reward at the last state transition plus the reward accumulated since that.* □

Corollary 4: In the special case when $Z(t)$ is a CTMC with generator $B = [b_{ij}]$ and $b_i = -b_{ii}$

$$f^*(v,j) = \sum_{i\in R, i\neq j} \frac{\alpha_i b_{ij}}{vr_i\alpha_i + b_i} f^*(v\alpha_i, i) \tag{15}$$

$$g^*(v,i) = \frac{1}{\gamma_i(b_i + vr_i)} f^*(v,i). \tag{16}$$

3.2 Accumulated reward up to time t

To apply a regenerative approach similar to the one used in Theorem 1, a more complicated description has to be used. Indeed, in this case, it is not enough to consider the difference between the present and the target value of the reward accumulation process. Let us define:

$$V_i(t,w,\eta) = Pr(Y(t) \le w \mid Z(0) = i, Y(0) = \eta)$$

The regenerative description of the process evolution is the following:

$$V_i(t,w,\eta|\, H=\tau) = \begin{cases} U_w(w - \eta - r_i t) & \text{if} : \tau > t \\ \sum_{k\in R} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot V_k(t-\tau, w, (\eta + \tau r_i)\alpha_i) & \text{if} : \tau < t \end{cases} \tag{17}$$

Unfortunately Eq. (17) does not exhibit any closed form transform domain expression. A different analysis approach is followed when the environment is a CTMC.

Underlying CTMC: In the special case when $Z(t)$ is a CTMC with generator $B = [b_{ij}]$, assuming $Y(0) = 0$ let us define:

$$S_i(t,w) = Pr(Y(t) \le w \mid Z(t) = i).$$

Note that the condition applies for time t!

Theorem 5 *$S_i(t,w)$ satisfies the following double transform domain equation:*

$$S_i^{**}(s,v) = \frac{1}{v(s + r_i v + b_i)} + \sum_{k\in R, k\neq i} \alpha_k b_{ik}\; S_k^{**}(s, \alpha_k v) \tag{18}$$

Proof **5** *The forward argument describing the evolution of the process is:*

$$S_i(t+dt, w) = (1 - b_i dt) S_i(t, w - r_i dt) + \sum_{k\in R, k\neq i} b_{ik}\; dt\; S_k(t, \frac{w}{\alpha_k} + \sigma(dt))$$

Taking the limit $dt \to 0$, provides:

$$\frac{\partial S_i(t,w)}{\partial t} + r_i \frac{\partial S_i(t,w)}{\partial w} = -b_i S_i(t,w) + \sum_{k\in R, k\neq i} b_{ik}\; S_k(t, \frac{w}{\alpha_k}) \tag{19}$$

Taking the Laplace transform with respect to t ($\to s$)

$$sS_i^*(s,w) - S_i(0,w) + r_i \frac{\partial S_i^*(s,w)}{\partial w} = -b_i S_i^*(s,w) + \sum_{k\in R, k\neq i} b_{ik}\; S_k^*(s, \frac{w}{\alpha_k})$$

where $S_i(0,w) = 1$. Taking the Laplace transform with respect to w ($\to v$)

$$sS_i^{**}(s,v) - \frac{1}{v} + r_i v S_i^{**}(s,v) - S_i^*(s,0) = -b_i S_i^{**}(s,v) + \sum_{k\in R, k\neq i} b_{ik}\alpha_k\; S_k^{**}(s, \alpha_k v)$$

where $S_i^*(s,0) = 0$ (supposing $r_i > 0$), from which the theorem comes. □

3.3 Completion time

Define

$$F_i(t,w,x) = Pr(C(w) \le t \mid Z(0) = i, Y(0) = x) \ .$$

Considering an SMP environment the regenerative approach provides

$$F_i(t,w,\eta| H=\tau) = \begin{cases} U_t(t - \frac{w-x}{r_i}) & \text{if} : \tau > \frac{w-x}{r_i} \\ \sum_{k \in R} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot & \text{if} : \tau < \frac{w-x}{r_i} \\ \quad F_k(t-\tau, w, (\eta + \tau r_i)\alpha_i) & \end{cases} \tag{20}$$

Eq. (20) is also hard to evaluate.

Underlying CTMC:

Theorem 6 *When $Z(t)$ is a CTMC with generator $B = [b_{ij}]$, $F_i(t,w,x)$ satisfies the backward differential equation:*

$$\begin{aligned} & \textit{if } x < w: \\ & \frac{\partial F_i(t,w,x)}{\partial t} + r_i \frac{\partial F_i(t,w,x)}{\partial x} = \\ & - b_i F_i(t,w,x) + \sum_{k \in R, k \ne i} b_{ik} \ F_k(t,w,\alpha_i x) \\ & \textit{if } x \ge w: \\ & F_i(t,w,x) = 1 \end{aligned} \tag{21}$$

Proof **6** *The backward argument describing the evolution of the process is:*

$$F_i(t,w,x) = (1 - b_i dt) F_i(t-dt, w, x - r_i dt) + \sum_{k \in R, k \ne i} b_{ik} \ dt \ F_k(t-dt, w, \alpha_i x + \sigma(dt))$$

which proves the theorem. □

Equation (21) can be expressed in double transform domain as:

$$F_i^{**}(s,w,v) = \frac{r_i}{s + r_i v + b_i} F_i^*(s,w,0) + \sum_{k \in R, k \ne i} \frac{b_{ik}}{\alpha_i (s + r_i v + b_i)} \ F_k^{**}(s, w, \frac{v}{\alpha_i}) \ . \tag{22}$$

Unfortunately (22) can not be used for numerical analysis since $F_i^*(s,w,0)$ is not known in this case.

Comment: Note that all the above results can be easily specified for renewal and Poisson background processes. There are several application examples of renewal processes with partial reward loss in telecommunication [3].

4 Numerical analysis techniques

Different numerical techniques, to evaluate reward measures, can be applied to the different analytical descriptions provided above. Eq. (6) and (10) can be directly evaluated applying a numerical inverse transform method.

Expressions like (15) can be evaluated numerically using the iterative procedure $f_0^*(v,\cdot) = 1$ and $f_{n+1}^*(v,j) = \sum_{i \in R, i \ne j} \frac{\alpha_i b_{ij}}{v r_i \alpha_i + b_i} f_n^*(v\alpha_i, i)$. Alternatively, the same iterative approach can be applied for the "time domain" version of (15):

$$f(w,j) = \sum_{i \in R, i \ne j} \frac{b_{ij}}{r_i \ \alpha_i} \int_{\tau=0}^{w} e^{-\frac{b_i}{r_i \ \alpha_i}(w-\tau)} f(\frac{\tau}{\alpha_i}, i) d\tau. \tag{23}$$

In this case, a convolution integral has to be evaluated numerically at each iteration step.

The moments of reward measures can be obtained based on double transform expressions like Eq. (18). E.g., the mean of the accumulated reward at time t, defined as $E_i = \mathrm{E}(Y(t)|Z(0) = i)$, can be obtained by a symbolic inverse transform with respect to s, a symbolic derivation with respect to v, evaluating the limit $v \to 0$ and solving the obtained linear system. That is

$$S_i^{\sim}(t,v) = \ e^{-(r_i v + b_i)t} \ + \sum_{k \in R, k \ne i} \alpha_k b_{ik} \ S_k^{\sim}(t, \alpha_k v)$$

$$E_i(t) = r_i \ t \ e^{-b_i t} \ + \sum_{k \in R, k \ne i} \alpha_k^2 b_{ik} \ E_k(t)$$

Equations like (19) and (21) can be handled by the application of numerical differential equation solvers.

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