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MATCHING THREE MOMENTS WITH MINIMAL ACYCLIC PHASE TYPE DISTRIBUTIONS

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□ *A number of approximate analysis techniques are based on matching moments of continuous time phase type (PH) distributions. This paper presents an explicit method to compose minimal order continuous time acyclic phase type (APH) distributions with a given first three moments. To this end we also evaluate the bounds for the first three moments of order n APH distributions (APH(n)). The investigations of these properties are based on a basic transformation, which extends the APH($n-1$) class with an additional phase in order to describe the APH(n) class.*

Keywords Acyclic phase type distributions; Moment bounds; Moment fitting.

Mathematics Subject Classification Primary 60J99; Secondary 60J10.

1. INTRODUCTION

The problem of finding a member of a distribution family with given moments arises in many applications. The use of the PH distribution family is very popular because stochastic models with PH distributions allow the use of effective Markov chain techniques.

Various moment matching methods were proposed by Johnson and Taaffe[3–6]. These methods were based on special PH structures and applied optimization techniques to minimize the difference of the moments of the PH distribution from the required ones.

One of the main difficulties in moment matching with PH distributions is that the PH family is limited with respect to its moments. The most well-known limit is the lower bound of the coefficient of variation of order

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n PH distributions. The bounds of the higher moments are not known. Hence, it is not known if the problem of moment matching with order n PH distributions is solvable or not. That is why the problem of moment matching requires the characterization of moment bounds.

The need for simple, quick and easy to implement moment fitting methods initiated research for explicit moment fitting procedures. The bounds of the first three moments of the APH(2) class and an explicit method to compose an APH(2) with given three moments (inside the bounds) is presented in Telek[10].

In the subsequent work (Osogami[8]), the problem of capturing three moments of all positive distributions (i.e., also the ones out of the APH(2) bounds) is considered.

To provide a general moment fitting method one should characterize the borders of the moment set of APH(n) distributions. Osogami and Harchol-Balter provided an outer and an inner moment bounds in Osogami[9] and they proposed an almost minimal fitting procedure using these bounds (Osogami[8]).

An important technical outcome of their investigation is the introduction of the so called *normalized moments* which transforms the problem of three moment fitting into the problem of two normalized moments fitting. By this transformation the moment bounds can be visualized on the plane of the normalized second and third moments. Also, the authors of the present work heavily utilized this graphical representation of the problem.

In Osogami[8], an extended set of the APH(n) class is used for fitting in which probability mass is allowed at 0. In this work we refer to the class of order n APH distributions with possible mass at 0 as APHM(n). A significant drawback of the APHM(n) class with respect to the APH(n) class is that a PH distribution with mass at 0 destroys the skip-free property when it is plugged into a matrix geometric solver (e.q., an APH/APH/1 queue forms a quasi birth death process, while an M/APHM/1 queue results in an G/M/1 type and an APHM/M/1 queue in an M/G/1 type model).

In the present work, we extend the results presented in Osogami[8,9] and Telek[10] in the following directions:

- We provide the exact normalized moment bounds of the APH(n) and the APHM(n) class for the first three moments.
- We present an APH structure with a minimal number of free parameters which exhibits the whole flexibility of the APH(n) class with respect to the first three moments. This minimal structure is composed of an Erlang($n-1$) distribution and an additional exponential phase, as opposed to the slightly more redundant EC class proposed in Osogami[8] which is composed of an Erlang($n-2$) and two additional phases (i.e. an APH(2) distribution).

- Based on the exact bounds and the proposed structure we present an explicit method to construct an APH(n) and an APHM(n) with given first three moments using minimal number of phases and minimal number of parameters.

The reason for considering the class of acyclic PH distributions is that the characterization of the realizable moments is tractable for this class, whereas it does not appear to be so for the entire class.

The rest of the paper is organized as follows: section 2 presents the basic notation used in the paper. The main results, the moment bounds and the fitting methods are provided in sections 3 and 4, respectively. Examples demonstrate the properties of the fitting procedure in section 5. The involved proofs of the main theorems are shifted to the appendix.

2. NOTATIONS

The definition and basic properties of PH distributions are provided in various text books, e.g., in Latouche[7]. In this paper we consider APH distributions whose properties are investigated in Cumani[2].

By m_i we denote the ith non-central moment, i.e. $m_i = E(X^i)$. The second and third normalized moments are defined as

$$n_2 = \frac{m_2}{m_1^2} \quad \text{and} \quad n_3 = \frac{m_3}{m_1 m_2}. \tag{1}$$

The ith moment of a continuous time PH distribution with initial probability vector α and generator A is $m_i = i!\alpha(-A)^{-i}\mathbb{1}$, where $\mathbb{1}$ is the column vector of ones. Multiplying A with a positive constant c results in the PH distribution with representation (α, cA) and moments $m_i = c^{-i} i!\alpha(-A)^{-i}\mathbb{1}$. The second and third normalized moments are independent of c. Utilizing this property, in the rest of the paper, we need to focus on the fitting of the second and third normalized moments only and do not have to consider the fitting of the first moment. Having set n_2 and n_3, the desired first moment can be achieved by a simple multiplication.

The Erlang(n) distribution (also referred to as $E(n)$), which is the sum of n independent exponentially distributed random variables with parameter μ, plays a special role in the sequel. Its first three non-normalized moments are

$$m_1 = \frac{n}{\mu}, \quad m_2 = \frac{n(n+1)}{\mu^2}, \quad m_3 = \frac{n(n+1)(n+2)}{\mu^3}$$

and its normalized moments are

$$n_2 = \frac{n+1}{n} \quad \text{and} \quad n_3 = \frac{n+2}{n}.$$

3. MOMENT BOUNDS

Before presenting the moment bounds of the APH(n) and APHM(n) classes we present some important properties of APH distributions.

Corollary 3.1. *Any* APH(n) *distribution can be realized as the composition of an* APH($n-1$) *distribution with representation* (α, A) *and an additional phase with transition rate* λ *according to Figure* 1 *(gray circles represent the absorbing states). The initial distribution of the composed* PH *distribution is defined by the initial distribution of the* APH($n-1$) *distribution* α *and an additional parameter* $0 \leq p \leq 1$.

Proof. The corollary is an easy consequence of the classical result of Cumani[2] according to which any APH(n) distribution can be transformed into the canonical form depicted in Figure 2.

Corollary 3.2. *Let* m_1, n_2 *and* n_3 *be the first moment and the second and third normalized moments of an* APH($n-1$) *distribution, respectively. The extension of this* APH($n-1$) *distribution with an additional phase with transition rate* λ *and parameter* $0 \leq p \leq 1$ *according to Figure* 1 *results in an* APH(n) *distribution with the following normalized moments*

$$\text{a)}\ \ \hat{n}_2 = \frac{2(1+pa) + a^2pn_2}{(1+pa)^2}, \quad \text{b)}\ \ \hat{n}_3 = \frac{6(1+pa) + a^2pn_2(n_3a+3)}{2(1+pa)^2 + a^2pn_2(1+pa)}, \tag{2}$$

where parameter a *is the ratio of the mean of the* APH($n-1$) *distribution and that of the additional phase, i.e.* $a = \frac{m_1}{1/\lambda}$.

Proof. Let m_1, m_2 and m_3 denote the first three moments of the APH($n-1$) distribution. The first three moments of the APH(n) distribution resulting from the composition can be written as

$$\hat{m}_1 = p\left(m_1 + \frac{1}{\lambda}\right) + (1-p)\frac{1}{\lambda}, \quad \hat{m}_2 = p\left(m_2 + \frac{2m_1}{\lambda} + \frac{2}{\lambda^2}\right) + (1-p)\frac{2}{\lambda^2},$$

$$\hat{m}_3 = p\left(m_3 + \frac{3m_2}{\lambda} + \frac{6m_1}{\lambda^2} + \frac{6}{\lambda^3}\right) + (1-p)\frac{6}{\lambda^3}.$$

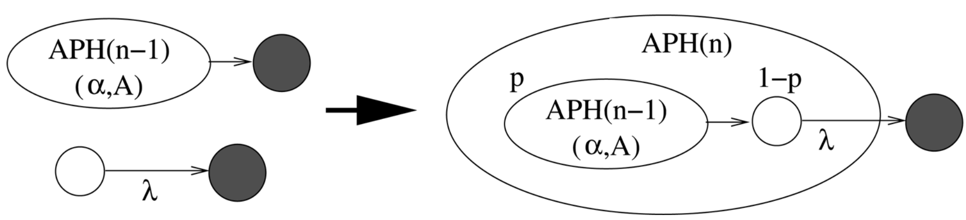

FIGURE 1 Realization of an APH(n) distribution by the composition of an APH($n-1$) distribution and an additional phase.

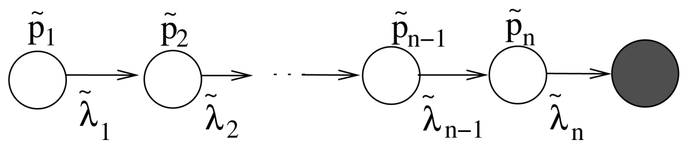

FIGURE 2 Canonical form of APH(n) distributions.

Introducing the n_2, n_3, $\hat{n}_2$, $\hat{n}_3$ normalized moments according to (1) and $a = \frac{m_1}{1/\lambda}$ results in the corollary.

We will use the basic transformation provided in Corollary 3.2 to connect the bounds of the APH($n-1$) class to the bounds of the APH($n-1$) class. Corollary 3.1 shows that this transformation is sufficient to describe the whole APH(n) class. The second and third normalized moments resulting from a given transformation will be denoted as $\{\hat{n}_2, \hat{n}_3\} = \mathcal{T}(\{n_2, n_3\}, a, p)$.

The main results about the moment bounds of the APH(n) and APHM(n) class are as follows:

Theorem 3.1. *n_2 and n_3 can be the second and third normalized moments of an* APH(n) *distribution iff they are subject to the following constraints: $n_2 \geq \frac{n+1}{n}$ and*

$$
\begin{aligned}
&n_3 \textit{ lower bound: } \begin{cases} l_n \leq n_3 & \textit{if } \dfrac{n+1}{n} \leq n_2 \leq \dfrac{n+4}{n+1}, \quad \text{a)} \\[2ex] \dfrac{n+1}{n} n_2 < n_3 & \textit{if } \dfrac{n+4}{n+1} \leq n_2, \quad \text{b)} \end{cases} \\
&n_3 \textit{ upper bound: } \begin{cases} n_3 \leq u_n & \textit{if } \dfrac{n+1}{n} \leq n_2 \leq \dfrac{n}{n-1}, \quad \text{c)} \\[2ex] n_3 < \infty & \textit{if } \dfrac{n}{n-1} < n_2, \quad \text{d)} \end{cases}
\end{aligned}
\tag{3}
$$

where l_n and u_n are defined as follows:

$$l_n = \frac{(3+a_n)(n-1)+2a_n}{(n-1)(1+a_n p_n)} - \frac{2a_n(n+1)}{2(n-1)+a_n p_n(n a_n + 2n - 2)}, \tag{4}$$

$$u_n = \frac{1}{n^2 n_2}\left(2(n-2)(n n_2 - n - 1)\sqrt{1+\frac{n(n_2-2)}{n-1}} + (n+2)(3nn_2 - 2n - 2)\right) \tag{5}$$

where

$$p_n = \frac{(n+1)(n_2-2)}{3n_2(n-1)}\left(\frac{-2\sqrt{n+1}}{\sqrt{4(n+1)-3nn_2}} - 1\right),$$

$$a_n = \frac{n_2-2}{p_n(1-n_2)+\sqrt{p_n^2+\frac{p_n n(n_2-2)}{n-1}}}.$$

The proof of Theorem 3.1 is provided in Appendix A.

The numerical values of the bounds of the APH(n) class are illustrated in Figure 3. The darkest shaded area contains those normalized moments pairs that can be realized with two phases. Points in the second darkest shaded area can be realized with three phases. The dashed segment of the lower bound corresponds to case a) of (3) and it is not linear. The linear segment drawn with solid line corresponds to case b). The upper bound (case c)) is represented with dotted line. The dashed dotted line corresponds to $n_3 = n_2$ which is the lower limit of non-negative random variables.

The structure of the bounds are presented in Figure 5. The figure indicates the $n_3 = 2n_2 - 1$ line as well because it plays a special role. The second and third normalized moments of the Erlang(n) ($n \geq 1$) distributions lie on this line. We refer to this line as the Erlang line. The set of normalized moments pairs $\{n_2, n_3\}$ that satisfy the inequalities of Theorem 3.1 for a given n will be denoted by $\mathcal{APH}(n)$.

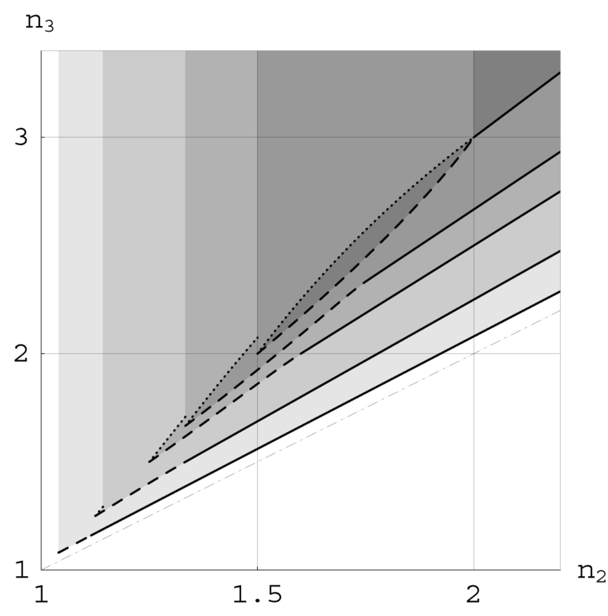

FIGURE 3 Bounds of normalized moments of the APH(n) class ($n = 2, 3, 4, 8, 25$).

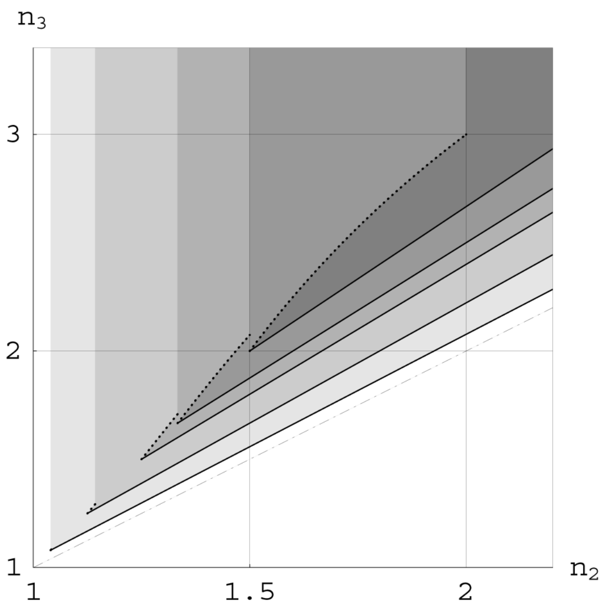

FIGURE 4 Bounds of normalized moments of the APHM(n) class ($n = 2, 3, 4, 8, 25$).

The following disjoint subsets of $\mathscr{APH}(n)\backslash\mathscr{APH}(n-1)$ are used in the sequel (see Figure 6):

- Subset 1) $\frac{n+1}{n} \leq n_2 < \frac{n}{n-1}$ and $l_n \leq n_3 \leq u_n$;
- Subset 2) $\frac{n}{n-1} \leq n_2 < \frac{n+4}{n+1}$ and $l_n \leq n_3 \leq l_{n-1}$;
- Subset 3) $\frac{n}{n-1} \leq n_2 < \frac{n-1}{n-2}$ and $u_{n-1} \leq n_3$;
- Subset 4) $\frac{n+4}{n+1} \leq n_2 < \frac{n+3}{n}$ and $\frac{n+1}{n} n_2 < n_3 \leq l_{n-1}$,
- Subset 5) $\frac{n+3}{n} \leq n_2$ and $\frac{n+1}{n} n_2 < n_3 \leq \frac{n}{n-1} n_2$.

Theorem 3.2. *n_2 and n_3 can be the second and third normalized moments of an* APHM(n) *distribution iff they are subject to the following constraints: $n_2 \geq \frac{n+1}{n}$ and*

$$
\begin{aligned}
&n_3 \textit{ lower bound:} \quad \frac{n+2}{n+1} n_2 \leq n_3 \\
&n_3 \textit{ upper bound:} \quad \begin{cases} n_3 \leq u_n & \textit{if } \dfrac{n+1}{n} \leq n_2 \leq \dfrac{n}{n-1}, \\ n_3 < \infty & \textit{if } \dfrac{n}{n-1} < n_2 \end{cases}
\end{aligned} \tag{6}
$$

where u_n is the same as in Theorem 3.1.

The proof of Theorem 3.2 is provided in Appendix B.

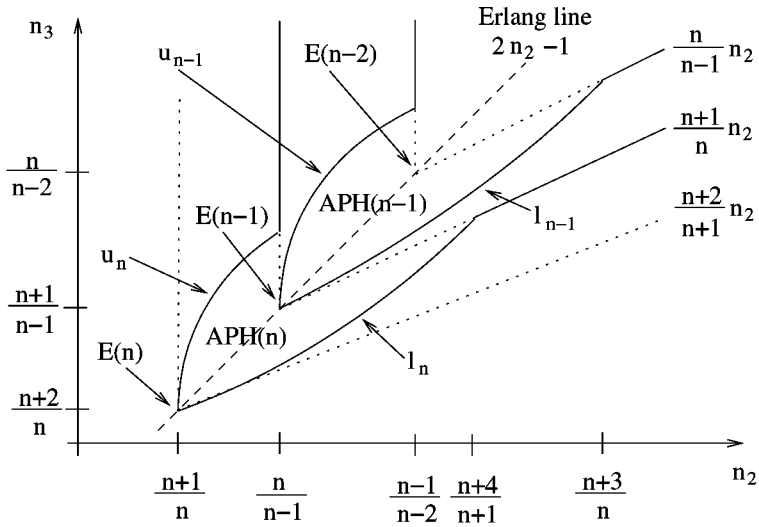

FIGURE 5 Structure of the APH(n) moment bounds for $n \geq 3$.

As in case of the APH(n) class, we illustrate both numerical values (Figure 4) and the structure (Figure 7) of the moment bounds of the APHM(n) class. The set of normalized moments pairs $\{n_2, n_3\}$ that satisfy the inequalities of Theorem 3.2 is denoted by $\mathscr{APHM}(n)$.

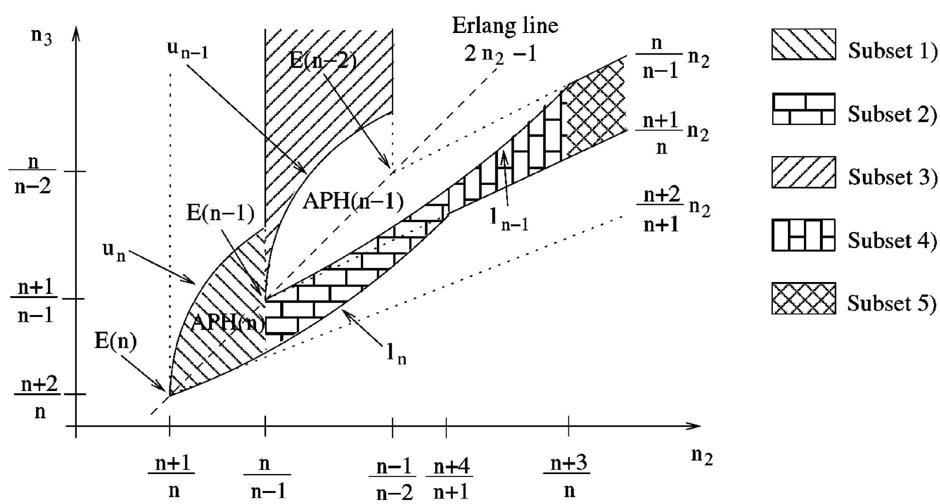

FIGURE 6 The disjoint subsets of $\mathscr{APH}(n)\backslash\mathscr{APH}(n-1)$ used in the proof of Theorem 3.1.

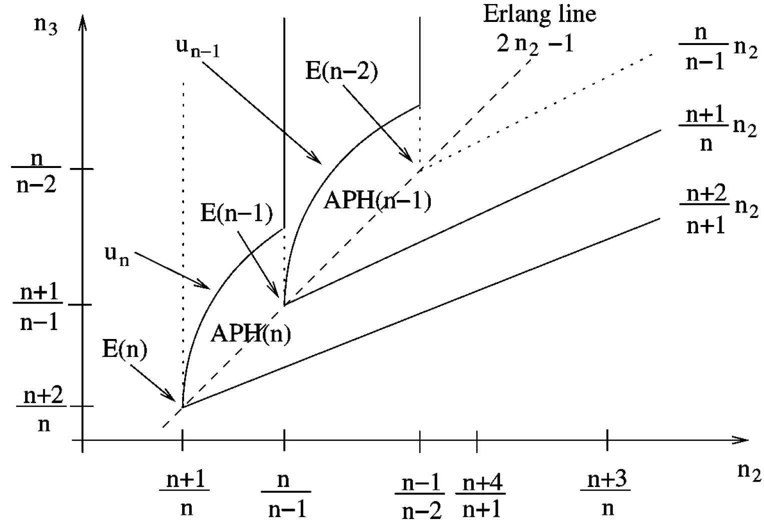

FIGURE 7 Structure of the APHM(n) moment bounds for $n \geq 3$.

4. EXPLICIT MOMENT MATCHING WITH MINIMAL NUMBER OF PARAMETERS

In this section we present moment matching procedures which result in minimal order APH(n) and APHM(n) distributions with minimal number of parameters (two and three, respectively) to set. The provided explicit methods are applicable over the whole $\mathcal{APH}(n)\backslash\mathcal{APH}(n-1)$ and $\mathcal{APHM}(n)\backslash\mathcal{APHM}(n-1)$ sets.

Theorem 4.1. *Given the normalized moments n_2 and n_3, such that $\{n_2, n_3\} \in \mathcal{APH}(n)\backslash\mathcal{APH}(n-1)$, we have the following two cases for an* APH(n) *distribution which matches the second and third normalized moments:*

i) *if $n_2 \leq \frac{n}{n-1}$ or $n_3 \leq 2n_2 - 1$, (i.e., the point is in subset 1), 2), 4) or 5)) then the Erlang-Exp distribution, presented in Figure 8, with parameters*

$$a = \frac{(b\, n_2 - 2)(n-1)\, b}{(b-1)\, n}, \quad p = \frac{b-1}{a}, \tag{7}$$

and

$$b = \frac{2(4 - n(3n_2 - 4))}{n_2(4 + n - nn_3) + \sqrt{nn_2} \times\sqrt{12{n_2}^2(n+1) + 16n_3(n+1) + n_2(n(n_3 - 15)(n_3+1) - 8(n_3+3))}}$$

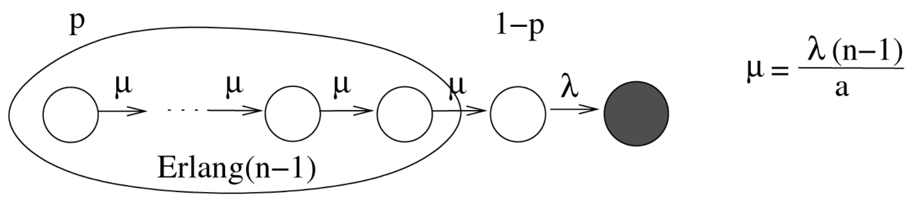

FIGURE 8 Erlang-Exp structure with parameters (a, p, λ, n).

ii) *if* $n_2 > \frac{n}{n-1}$ *and* $n_3 > u_{n-1}$, (*i.e., the point is in subset* 3)) *the Exp-Erlang distribution, presented in Figure* 9, *with parameters*

$$a = \frac{2(f-1)(n-1)}{(n-1)(n_2 f^2 - 2f + 2) - n}, \quad p = (f-1)a.$$

Parameter f is defined in the following steps.

$$K_1 = n-1, \quad K_2 = n-2, \quad K_3 = 3n_2 - 2n_3, \quad K_4 = n_3 - 3, \quad K_5 = n - n_2,$$

$$K_6 = 1 + n_2 - n_3, \quad K_7 = n + n_2 - nn_2, \quad K_8 = 3 + 3n_2^2 + n_3 - 3n_2 n_3,$$

$$K_9 = 108K_1^2\Big(4K_2^2 K_3 n^2 n_2 + K_1^2 K_2 K_4^2 nn_2^2 + 4K_1 K_5(K_5^2 - 3K_2 K_6 nn_2) + \sqrt{-16K_1^2 K_7^6 + (4K_1 K_5^3 + K_1^2 K_2 K_4^2 nn_2^2 + 4K_2 nn_2(K_4 n^2 - 3K_6 n_2 + K_8 n))^2}\Big),$$

$$K_{10} = \frac{K_4^2}{4K_3^2} - \frac{K_5}{K_1 K_3 n_2}, \quad K_{11} = \frac{2^{1/3}(3K_5^2 + K_2(K_3 + 2K_4)nn_2)}{K_3 K_9^{1/3} n_2},$$

$$K_{12} = \frac{K_9^{1/3}}{3\, 2^{7/3} K_1^2 K_3 n_2}, \quad K_{13} = \sqrt{K_{10} + K_{11} + K_{12}},$$

$$K_{14} = \frac{6K_1 K_3 K_4 K_5 + 4K_2 K_3^2 n - K_1^2 K_4^3 n_2}{4K_1^2 K_3^3 K_{13} n_2}, \quad K_{15} = -\frac{K_4}{2\, K_3}$$

$$K_{16} = \sqrt{2K_{10} - K_{11} - K_{12} - K_{14}}, \quad K_{17} = \sqrt{2K_{10} - K_{11} - K_{12} + K_{14}},$$

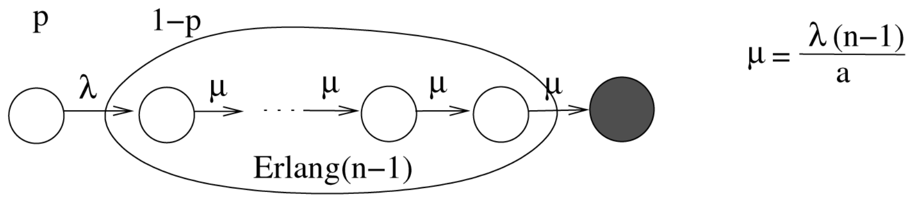

FIGURE 9 Exp-Erlang structure with parameters (a, p, λ, n).

$$K_{18} = 36\,K_5^3 + 36\,K_2K_4K_5nn_2 + 9\,K_1K_2K_4^2nn_2^2 - \sqrt{81\left(4K_5^3 + 4K_2K_4K_5nn_2 + K_1K_2K_4^2nn_2^2\right)^2 - 48\left(3K_5^2 + 2K_2K_4nn_2\right)^3},$$

$$K_{19} = -\frac{K_5}{K_1\,K_4\,n_2} - \frac{2^{2/3}(3K_5^2 + 2K_2K_4nn_2)}{3^{1/3}K_1K_4n_2K_{18}^{1/3}} - \frac{K_{18}^{1/3}}{6^{2/3}K_1K_4n_2},$$

$$K_{20} = 6K_1K_3K_4K_5 + 4K_2K_3^2n - K_1^2K_4^3n_2, \quad K_{21} = K_{11} + K_{12} + \frac{K_5}{2nK_1K_3},$$

$$K_{22} = \sqrt{\frac{3\;K_4^2}{4K_3^2} - \frac{3K_5}{K_1\;K_3n_2} + \sqrt{4K_{21}^2 - \frac{nK_2}{n_2K_1^2K_3}}},$$

$$f = \begin{cases} K_{13} + K_{15} - K_{17} & \textit{if } u_{n-1} < n_3 < 3n_2/2, \\ K_{19} & \textit{if } n_3 = 3n_2/2, \\ -K_{13} + K_{15} + K_{16} & \textit{if } n_3 > 3n_2/2 \textit{ and } K_{20} > 0 \\ K_{15} + K_{22} & \textit{if } K_{20} = 0 \\ K_{13} + K_{15} + K_{17} & \textit{if } K_{20} < 0 \end{cases}$$

Comments on the theorem:

- Similar to Corollary 3.2, only a and p affect the second and third normalized moments while λ does not. With λ the required first moment can be set.
- For a given n_2, f is a continuous function of n_3 as it is demonstrated in Figure 10. In the figure $n_2 = 1.7$, $n_3 > 2.5$ $(n = 3)$ and the turning points are $n_3 = 2.55$ $(n_3 = 3n_2/2)$ and $n_3 = (45 + 3\sqrt{55})/10 \sim 6.7248595$ $(K_{20} = 0)$.

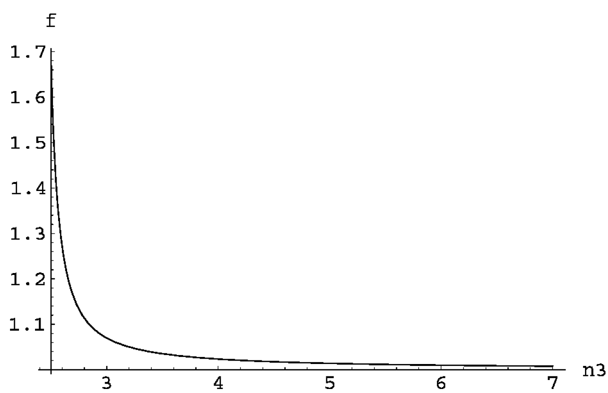

FIGURE 10 f is a continuous function of n_3 $(n = 4,\ n_2 = 1.7)$.

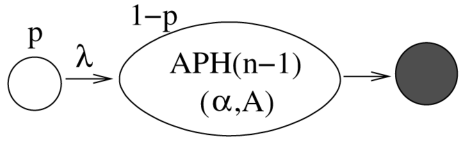

FIGURE 11 Extension of an APH($n-1$) with an additional phase in the front

In order to prove Theorem 4.1 we need the following corollary.

Corollary 4.1. *Let n_2 and n_3 be the second and third normalized moments of an* APH($n-1$) *distribution. The extension of this* APH($n-1$) *distribution with an additional phase in the front according to Figure* 11 *results in an* APH(n) *distribution with the following normalized moments*

$$\text{a)}\ \hat{n}_2 = \frac{a^2 n_2 + 2p(1+a)}{p^2 + a^2 + 2ap}, \quad \text{b)}\ \hat{n}_3 = \frac{a^3 n_3 n_2 + 3p(n_2 a^2 + 2(a+1))}{(p+a)(a^2 n_2 + 2p(a+1))}. \quad (8)$$

Proof. Let m_1, m_2 and m_3 denote the first three moments of the APH ($n-1$) distribution. The first three moments of the APH(n) distribution of the given structure are:

$$\hat{m}_1 = p\left(m_1 + \frac{1}{\lambda}\right) + (1-p)m_1, \quad \hat{m}_2 = p\left(m_2 + \frac{2m_1}{\lambda} + \frac{2}{\lambda^2}\right) + (1-p)m_2,$$
$$\hat{m}_3 = p\left(m_3 + \frac{3m_2}{\lambda} + \frac{6m_1}{\lambda^2} + \frac{6}{\lambda^3}\right) + (1-p)m_3.$$

Introducing the n_2, n_3 normalized moments of the APH($n-1$) distribution from (1) results in the corollary.

Proof. Part i) of Theorem 4.1 is based on Corollary 3.2 where the APH($n-1$) distribution is a special one, an Erlang($n-1$) distribution with normalized moments $n_2 = \frac{n}{n-1}$ and $n_3 = \frac{n+1}{n-1}$. Substituting these normalized moments into (2) results in implicit equations for p and a. Solving the equation for p and a results in (7).

Part ii) of the theorem can be proven similarly applying Corollary 4.1. Once again the APH($n-1$) distribution is an Erlang($n-1$) distribution. Substituting its normalized moments into (8) results in implicit equations for p and a. This case leads to the fourth order equation $c_4 f^4 + c_3 f^3 + c_2 f^2 + c_1 f + c_0 = 0$, where

$$c_4 = n_2(3n_2 - 2n_3)(n-1)^2, \quad c_3 = 2n_2(n_3 - 3)(n-1)^2,$$
$$c_2 = 6(n-1)(n-n_2), \quad c_1 = 4n(2-n), \quad c_0 = n(n-2),$$

and parameter f is the solution which results in a proper APH(n) distribution.

Theorem 4.1 uses an Erlang-Exp distribution for the normalized moments pairs in subset 1), but it is also possible to find an Exp-Erlang distribution for those moments pairs according to the method presented in Part ii) of the theorem. The moments pairs of subset 3) are not reachable with order n Erlang-Exp distribution and the moments pairs of subsets 2), 4) and 5) are not reachable with order n Exp-Erlang distribution.

Theorem 4.2. *Given the normalized moments n_2 and n_3, such that $\{n_2, n_3\} \in \mathscr{APHM}(n)\backslash\mathscr{APHM}(n-1)$, we have the following two cases for an* APHM(n) *distribution, which matches the second and third normalized moments:*

i) *if $\{n_2, n_3\} \in \mathscr{APH}(n)$ and $n_3 \neq \frac{3}{2}n_2$ then the* APH(n) *distribution provided in Theorem* 4.1,
ii) *if $\{n_2, n_3\} \in \mathscr{APHM}(n)\backslash\mathscr{APH}(n)$ then the* APHM(n) *distribution presented in Figure* 12 *with parameter $q = \frac{1}{2n_2 - n_3}$ and* APH(n) *distribution provided by Theorem* 4.1 *for $n_2^* = \frac{n_2}{2n_2-n_3}$ and $n_3^* = \frac{n_3}{2n_2-n_3}$.*

Proof. Part i) of the theorem is a consequence of Theorem 4.1, while part ii) is a consequence of Corollary 4.2.

Corollary 4.2. *Consider an* APH(n) *distribution with normalized moments n_2 and n_3. The* APHM(n) *distribution constructed according to Figure 12 with the considered* APH(n) *distribution and parameter q has the following normalized moments:*

$$\text{a)}\ \hat{n}_2 = \frac{n_2}{q}, \quad \text{b)}\ \hat{n}_3 = \frac{n_3}{q}. \tag{9}$$

Proof. The non-normalized moments of the APHM(n) distribution are related to the non-normalized moments of the APH(n) distribution as $\hat{m}_i = qm_i$ ($i \geq 1$). From this relation the corollary is obtained simply by the definition of the normalized moments.

The methods presented in Osogami[8,9], which papers consider only APHM(n) distributions, make use of the transformation described in Corollary 4.2.

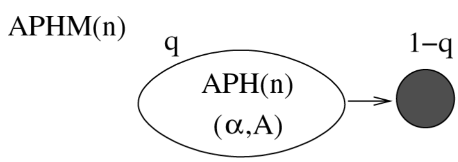

FIGURE 12 Composition of APHM(n) distribution.

Part ii) of Theorem 4.2 generates the APHM(n) distribution with the required normalized moments $\{n_2, n_3\}$ based on Corollary 4.2. The proposed solution is such that the normalized moments of the APH(n) distribution lie exactly on the Erlang line ($n_3 = 2n_2 - 1$). This is not the unique possibility. One could select other points of $\mathscr{APH}(n)\backslash\mathscr{APH}(n-1)$ which then can be transformed into an APHM(n) distribution with normalized moments $\{n_2, n_3\}$.

The transformation of Corollary 4.2 is such that any point of $\mathscr{APHM}(n)\backslash\mathscr{APH}(n)$ is reachable from the $\mathscr{APH}(n)\backslash\mathscr{APH}(n-1)$ segment of the Erlang line.

5. EXAMPLES

In this section we compare the presented moment matching methods against the one proposed in Osogami[8]. The considered aspects are: number of phases, presence of mass in the distributions, visual aspect of the resulting probability density functions (pdfs) and higher order moments. For what concerns the concrete numerical values, m_1 is set to 1 and $\{n_2, n_3\} \in \mathscr{APH}(5)\backslash\mathscr{APH}(4)$ in all the cases. In order to carry out the comparison, we have to consider different subsets of $\mathscr{APH}(n)\backslash\mathscr{APH}(n-1)$. Four cases have to be distinguished. In the first three cases, the two methods presented in this paper (the one that does not make use of mass at 0, Theorem 4.1, and the one that does, Theorem 4.2) result in the same distribution because the presence of mass does not influence the required number of phases. For the fourth case we apply both of the methods. The cases are as follows.

1. The point $\{n_2, n_3\}$ is in subset 1) (see Figure 6) above the Erlang line: application of Osogami[8] does not introduce mass at 0 but results in an

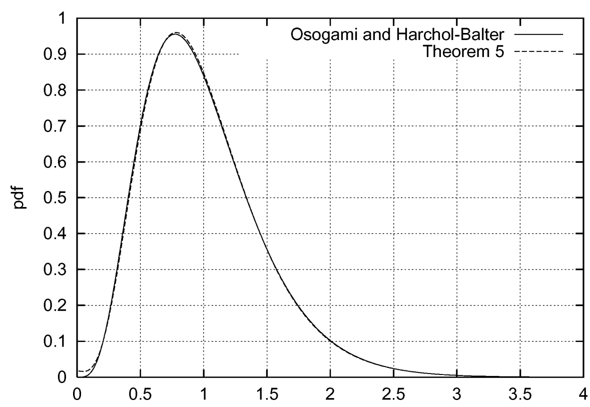

FIGURE 13 Pdfs in case 1: a point in subset 1) above the Erlang line.

TABLE 1 Behaviour of higher order non-normalized moments

Case	Method	4th	5th	6th	7th	8th
1	Paper Osogami[8]	2.95e + 00	5.64e + 00	1.21e + 01	2.90e + 01	7.67e + 01
	Without mass (Theorem 4.1)	2.96e + 00	5.67e + 00	1.23e + 01	2.99e + 01	8.04e + 01
2	Paper Osogami[8]	2.86e + 00	5.30e + 00	1.09e + 01	2.48e + 01	6.17e + 01
	Without mass (Theorem 4.1)	2.86e + 00	5.29e + 00	1.09e + 01	2.48e + 01	6.16e + 01
3	Paper Osogami[8]	2.50e + 01	9.47e + 02	4.98e + 04	3.08e + 06	2.17e + 08
	Without mass (Theorem 4.1)	1.42e + 01	2.05e + 02	4.29e + 03	1.07e + 05	3.08e + 06
4	Paper Osogami[8]	9.60e + 00	2.84e + 01	9.52e + 01	3.59e + 02	1.50e + 03
	Without mass (Theorem 4.1)	9.38e + 00	2.65e + 01	8.36e + 01	2.92e + 02	1.12e + 03
	With mass (Theorem 4.2)	9.65e + 00	2.89e + 01	9.94e + 01	3.88e + 02	1.70e + 03

APH with 6 phases instead of 5. Resulting pdfs for the pair $\{1.22, 1.445\}$ are depicted in Figure 13. Higher order moments have very similar behaviour. Table 1 gives the numerical values of higher order moments for all the four cases. (Note that the first three non-normalized moments are identical and can be easily calculated based on the normalized moments.)

2. The point $\{n_2, n_3\}$ is in subset 1) (see Figure 6) below the Erlang line: application of Osogami[8] results in an APH with 5 phases as the method presented in this paper but with mass at 0. Pdfs of this case for $\{1.22, 1.43\}$ are presented in Figure 14. Higher order moments have very similar behaviour.
3. When the point $\{n_2, n_3\}$ is in subset 3) both methods use 5 phases and there is no mass in the APH constructed by the method of Osogami[8]. As illustrated by Figure 15, the two methods result in similar pdfs for $\{1.4, 2\}$. In this case the higher order moments resulting from the method of Osogami and Harchol-Balter are much greater. This can be

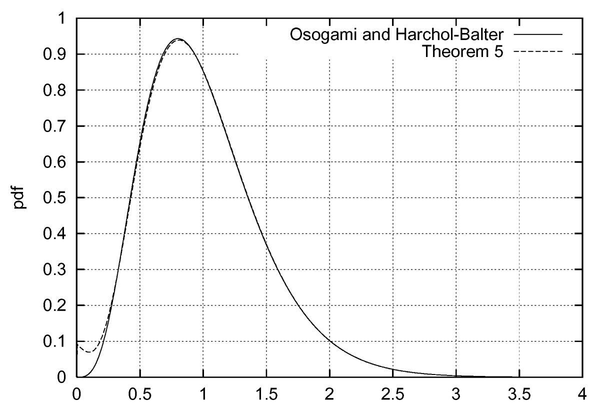

FIGURE 14 Pdfs in case 2: a point in subset 1) below the Erlang line.

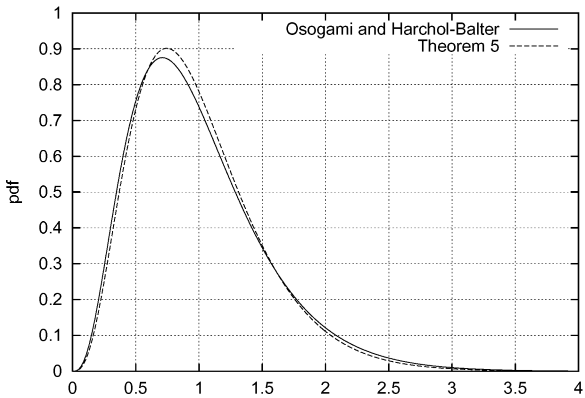

FIGURE 15 Pdfs in case 3: a point in subset 3).

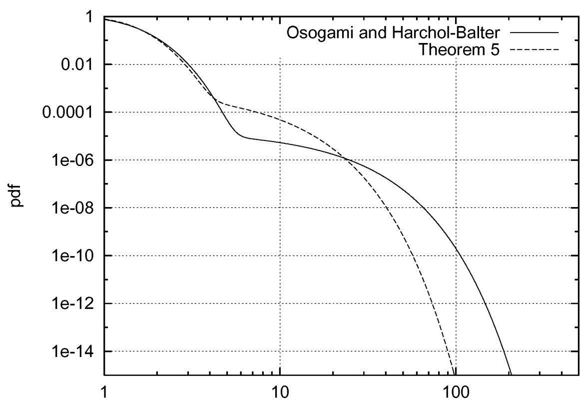

FIGURE 16 Tail behaviour of the pdfs in case 3: a point in subset 3).

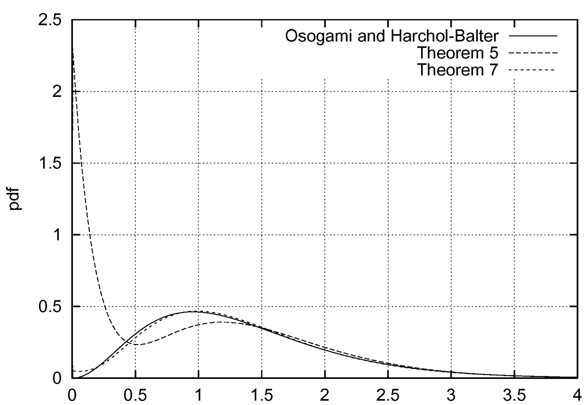

FIGURE 17 Pdfs in case 4: a point in subset 2), 4) or 5).

explained looking at the tail behaviour of the distributions depicted in Figure 16.

4. In case of a point $\{n_2, n_3\}$ in subset 2), 4) or 5) the method of Osogami[8] uses 4 phases with mass, the methods of this paper use 5 phases without mass (Theorem 4.1) and 4 phases with mass (Theorem 4.2). Pdfs for $\{1.75, 2.15\}$ are depicted in Figure 17 (note that the mass at 0 is not visualised in the figures). The behaviour of the higher moments is similar in all three cases.

APPENDIX A: PROOF OF THEOREM 3.1

We prove Theorem 3.1 by induction. For $n = 2$ the theorem is proved in Telek[10]. We assume that the theorem holds for $n - 1$ $(n \geq 3)$, i.e.,

a1) for any APH$(n-1)$ distribution the second and third normalized moments are in $\mathcal{APH}(n-1)$ and

a2) for any $\{n_2, n_3\} \in \mathcal{APH}(n-1)$ there is an APH$(n-1)$ distribution whose second and third normalized moments are $\{n_2, n_3\}$,

and based on this assumption we show that

s1) for any $\{n_2, n_3\} \in \mathcal{APH}(n-1)$, $0 < a < \infty$, $0 < p < 1$ the transformation of Corollary 3.2 results in $\mathcal{T}(\{n_2, n_3\}, a, p) \in \mathcal{APH}(n)$ and

s2) for any $\{n_2, n_3\} \in \mathcal{APH}(n) \backslash \mathcal{APH}(n-1)$ there is an APH(n) distribution whose second and third normalized moments are $\{n_2, n_3\}$.

We start with showing s1). First we prove that $\hat{n}_2 \geq \frac{n+1}{n}$. (Note that this inequality is shown in Aldous[1] as well. Here we give a different proof.)

According to (2a) $\hat{n}_2$ is a monotone increasing function of n_2. Based on assumption a1) the second normalized moment of an APH$(n-1)$ distribution $n_2 \geq \frac{n}{n-1}$. The derivative of the right-hand side (rhs) of (2a) with respect to a is $\frac{2p(an_2 - ap - 1)}{(1+ap)^3}$. This derivative becomes 0 at $a = \frac{1}{n_2 - p}$, where the second derivative is non-negative. Substituting this a value we have

$$\hat{n}_2 \geq 2 - \frac{p}{n_2} \geq 2 - \frac{1}{n_2}, \tag{10}$$

which is minimal at $p = 1$. Hence, the minimum of $\hat{n}_2$ is obtained at $n_2 = \frac{n}{n-1}$, $p = 1$, $a = n - 1$ and it is $\hat{n}_2 = \frac{n+1}{n}$. (This minimum is provided by the Erlang(n) distribution.)

In the following we treat the four inequalities of (3) separately where the last one, (3d), is evident.

Inequality (3a): We rewrite (2) using $g = 1 + pa$:

$$\text{a)}\ \ \hat{n}_2 = \frac{2g + a(g-1)n_2}{g^2}, \quad \text{b)}\ \ \hat{n}_3 = \frac{6g + a(g-1)n_2(n_3 a + 3)}{2g^2 + a(g-1)gn_2}. \tag{11}$$

By its definition $1 \le g < \infty$. From (11a) we have $a = \frac{(\hat{n}_2 g - 2)g}{n_2(g-1)}$. Substituting it into (11b) gives

$$\hat{n}_3 = \frac{3}{g} + \frac{(\hat{n}_2 g - 2)^2}{(g-1)g\hat{n}_2}\frac{n_3}{n_2}. \tag{12}$$

According to (12) $\hat{n}_3$ is a monotone increasing function of n_3/n_2, hence, the minimal $\hat{n}_3$ value of the $\mathcal{APH}(n)$ class at $\hat{n}_2$ is obtained when n_3/n_2 takes the minimal value of the $\mathcal{APH}(n-1)$ class. The minimal n_3/n_2 value of the $\mathcal{APH}(n-1)$ class is

$$\min_{\mathcal{APH}(n-1)} n_3/n_2 = \frac{n+1}{n}, \tag{13}$$

since at $n_2 = n/(n-1)$ the lower bound of n_3 is $l_{n-1} = (n+1)/(n-1)$ and $dl_{n-1}/dn_2 \ge (n+1)/n$. Note that this minimal n_3/n_2 ratio is provided by the Erlang$(n-1)$ distribution. Hence,

$$\hat{n}_3 \ge \hat{n}_3^{min} = \frac{3}{g} + \frac{(\hat{n}_2 g - 2)^2}{(g-1)g\hat{n}_2}\frac{n+1}{n}. \tag{14}$$

According to (14) $\hat{n}_3^{min}$ has the following limits in the $g \in (1, \infty)$ range, $\lim_{g\to 1^+} \hat{n}_3^{min} = \infty$ and $\lim_{g\to\infty} \hat{n}_3^{min} = \frac{n+1}{n}\ \hat{n}_2$. This way the minimum of $\hat{n}_3^{min}$ is obtained at $g \to \infty$ when the derivative of $\hat{n}_3^{min}$ with respect to g has no root in the $(1, \infty)$ interval, and it is obtained at the root of the derivative of $\hat{n}_3^{min}$ when it has one root in the $(1, \infty)$ interval. The case of having two roots in the $(1, \infty)$ interval cannot occur as is shown below.

The derivative of $\hat{n}_3^{min}$ with respect to g equals to zero at

$$g_1 = \frac{r + \sqrt{rs}}{r - s}, \quad g_2 = \frac{r - \sqrt{rs}}{r - s} = \frac{\sqrt{r}(\sqrt{r} - \sqrt{s})}{r - s}$$

where $r = 4(n+1) - 3n\hat{n}_2$, $s = (n+1)(\hat{n}_2 - 2)^2$. By their definitions $s \ge 0$ and $r > 0$ if $\hat{n}_2 < 4(n+1)/3n$, and $r - s > 0$ if $\hat{n}_2 < (n+4)/(n+1)$. The following cases need to be considered:

- if $\frac{n+1}{n} < \hat{n}_2 < \frac{n+4}{n+1}$ then $g_1 \in (1, \infty)$, $g_2 \in (0, 1)$,
- if $\frac{n+4}{n+1} < \hat{n}_2 < \frac{4n+4}{3n}$ then $g_1 \in (-\infty, 0)$, $g_2 \in (0, 1)$, and
- if $\frac{4n+4}{3n} < \hat{n}_2$ then $r < 0$ and there is no real root.

The second root $g_2 \in (0,1)$ because $\sqrt{r} - \sqrt{s}$ and $r - s$ has the same sign when $\hat{n}_2 < 4(n+1)/3n$ and

$$\begin{array}{cc} \text{if } s < r \text{ then} & \text{if } s > r \text{ then} \\ s < \sqrt{sr} & s > \sqrt{sr} \\ r - s > r - \sqrt{sr} & r - s < r - \sqrt{sr} \\ 1 > \dfrac{r - \sqrt{sr}}{r - s} & 1 > \dfrac{r - \sqrt{sr}}{r - s}. \end{array}$$

Figure 18 demonstrates the behaviour of the roots when $n = 4$. The figure contains a horizontal line at $g = 1$ and a vertical line at $\hat{n}_2 = \frac{n+4}{n+1}$. The fact that the optimal g value tends to infinity as $\hat{n}_2$ tends to $\frac{n+4}{n+1}$ indicates that the lower bound is continuous at $\hat{n}_2 = \frac{n+4}{n+1}$.

Considering inequality (3a) we restrict our attention to the $\hat{n}_2 \in \left(\frac{n+1}{n}, \frac{n+4}{n+1}\right)$ interval. In this interval g_1 falls into $(1, \infty)$. Substituting g_1 into (14) results in inequality (3a).

Inequality (3b): The derivative of $\hat{n}_3^{min}$ with respect to g has no root in the $(1, \infty)$ interval when $\hat{n}_2 \in \left(\frac{n+4}{n+1}, \infty\right)$. Hence, $\hat{n}_3^{min}$ is a monotone decreasing function of g with $\lim_{g \to \infty} \hat{n}_3^{min} = \frac{n+1}{n} \hat{n}_2$.

Inequality (3c): If $\frac{n+1}{n} \leq \hat{n}_2 < \frac{n}{n-1}$ then $\frac{n}{n-1} \leq n_2 < \frac{n-1}{n-2}$ according to (10).

From (2a)

$$a = \frac{\hat{n}_2 - 1 \pm \frac{\sqrt{n_2(\hat{n}_2 -) + p}}{\sqrt{p}}}{n_2 - \hat{n}_2 p}. \tag{15}$$

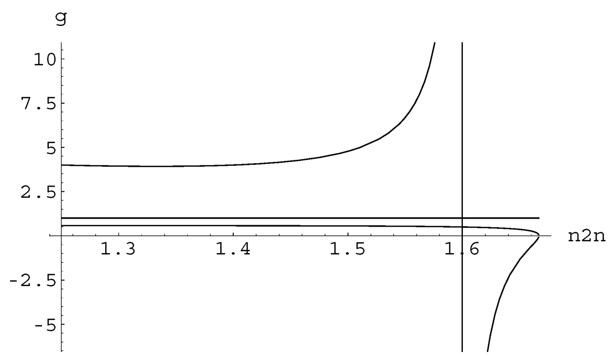

FIGURE 18 The roots of the derivative of (14) when $n = 4$.

For a given n_2, $\hat{n}_2$ pair $p \in \{n_2(2-\hat{n}_2), 1\}$, because a is complex when $p < n_2(2-\hat{n}_2)$. Substituting (15) into (2b) results in

$$\begin{aligned}\hat{n}_3 = & \frac{3n_2^2\hat{n}_2 + 2n_2(\hat{n}_2-2)n_3}{n_2^2\hat{n}_2} + \frac{n_2(\hat{n}_2-2)n_3 + (4n_3 - 3n_2\hat{n}_2)p}{n_2^2\hat{n}_2} \\ & \times \left(1 \pm \sqrt{\frac{n_2(\hat{n}_2-2)}{p} + 1}\right) \\ \leq & \frac{3n_2^2\hat{n}_2 + 2n_2(\hat{n}_2-2)n_3}{n_2^2\hat{n}_2} + \frac{n_2(\hat{n}_2-2)n_3 + (4n_3 - 3n_2\hat{n}_2)p}{n_2^2\hat{n}_2} \\ & \times \left(1 + \sqrt{\frac{n_2(\hat{n}_2-2)}{p} + 1}\right)\end{aligned} \tag{16}$$

In the $p \in \{n_2(2-\hat{n}_2), 1\}$ interval the maximum of $\hat{n}_3$ is reached when $p = 1$, since $\hat{n}_2 - 2 < 0$ and $4n_3 - 3n_2\hat{n}_2 > 0$ (according to (13)) when $\frac{n}{n-1} \leq n_2 < \frac{n-1}{n-2}$ and $\frac{n+1}{n} \leq \hat{n}_2 < \frac{n}{n-1}$. Furthermore, according to (16) $\hat{n}_3$ is a monotone function of n_3, hence the upper bound of $\hat{n}_3$ is obtained at the upper bound of n_3. Substituting the upper bound of $\hat{n}_3$ into (16) results in (3c).

The proof of s2) is divided into parts according to the five disjoint subsets of $\mathcal{APH}(n)\backslash\mathcal{APH}(n-1)$ (see Figure 6).

Subset 1): The following property holds for subset 1). Given an integer $n \geq 3$ and the pair $\{n_2, n_3\}$ such that

$$\frac{n+1}{n} \leq n_2 < \frac{n}{n-1} \quad \text{and} \quad l_n \leq n_3 \leq u_n$$

there is an APH(n) distribution of Erlang-Exp structure whose second and third normalized moments are n_2 and n_3, respectively.

The proof of this property of subset 1) can be carried out by the following steps.

1. Given an integer n and a real n_2 such that $\frac{n+1}{n} \leq n_2 < \frac{n}{n-1}$ the APH(n) distribution of Erlang-Exp structure with parameters

$$a = \frac{(b\,n_2 - 2)(n-1)\,b}{(b-1)\,n} \quad \text{and} \quad p = \frac{b-1}{a}$$

where

$$b = \frac{4 + n - n\,l_n}{2\,(n_2 + n\,n_2 - n\,l_n)}$$

has second and third normalized moments n_2 and l_n, respectively. This step of the proof is a direct consequence of Corollary 3.2.

2. Given an integer n and a real n_2 such that $\frac{n+1}{n} \leq n_2 < \frac{n}{n-1}$ the APH(n) distribution of Erlang-Exp structure with parameters

$$a = b - 1 \quad \text{and} \quad p = 1$$

where

$$b = \frac{n}{\sqrt{(n-1)(n(n_2-1)-1)}+1}$$

has second and third normalized moments n_2 and u_n, respectively. This step of the proof follows simply from Corollary 3.2.

3. Given an integer n and a real n_2 such that $\frac{n+1}{n} \leq n_2 < \frac{n}{n-1}$ the APH(n) distribution of Erlang-Exp structure with parameters

$$a = \frac{(b\, n_2 - 2)(n-1)\, b}{(b-1)\, n} \quad \text{and} \quad p = \frac{b-1}{a}$$

where

$$b \in \left[\frac{n}{\sqrt{(n-1)(n(n_2-1)-1)}+1}, \frac{4+n-n\, l_n}{2\,(n_2+n\, n_2-n\, l_n)}\right]$$

has second normalized moment n_2 and its third normalized moment is a continuous, monotone decreasing function of b, which can be seen from Corollary 3.2 and elementary properties of the emerging expressions.

Subset 2): Exactly the same proof applies for subset 2) as for subset 1), because $u_n > l_{n-1}$ in the $n_2 \in \left(\frac{n}{n-1}, \frac{n+4}{n+1}\right)$ range. (Note that u_n, defined in (5), is not the upper bound in the $n_2 \in \left(\frac{n}{n-1}, \frac{n+4}{n+1}\right)$ range.)

Subset 3): An APH(n) distribution with normalized moments $\{n_2, n_3\}$, where

$$\frac{n}{n-1} \leq n_2 < \frac{n-1}{n-2} \quad \text{and} \quad u_{n-1} \leq n_3,$$

can be composed of an APH$(n-1)$ distribution and an exponential phase according to Corollary 3.2 where the normalized moments pair of the APH$(n-1)$ distribution and the parameters of the transformation are

$$\left\{\frac{1}{2-n_2}, \frac{n_2(3-3n_2+n_3)}{(n_2-2)^2}\right\}, \quad p = 1,\ a = \frac{2-n_2}{n_2-1}.$$

The normalized moments pair $\left\{\frac{1}{2-n_2}, \frac{n_2(3-3n_2+n_3)}{(n_2-2)^2}\right\}$ is in subset 3) of the APH$(n-1)$ class.

Subset 5): The proof for subset 5) is similar to the one for subset 1). Given an integer $n \geq 3$ and the pair $\{n_2, n_3\}$ such that

$$\frac{n+3}{n} \leq n_2 \quad \text{and} \quad \frac{n+1}{n} n_2 < n_3 \leq \frac{n}{n-1} n_2$$

there is an APH(n) distribution of Erlang-Exp structure whose second and third normalized moments are n_2 and n_3, respectively.

We show this property of subset 5) in the following steps.

1. Given an integer n and a real n_2 such that $\frac{n+3}{n} \leq n_2$ the second and third normalized moments of the APH(n) distribution of Erlang-Exp structure whose parameters tends to

$$b \to \infty, \; a = \frac{(b\, n_2 - 2)(n-1)\, b}{(b-1)\, n} \to \infty \quad \text{and} \quad p = \frac{b-1}{a} \to \frac{n}{(n-1)n_2}$$

   tends to n_2 and $\frac{n+1}{n} n_2$, respectively.
2. Given an integer n and a real n_2 such that $\frac{n+3}{n} \leq n_2$ the APH(n) distribution of Erlang-Exp structure with parameters

$$a = \frac{(bn_2 - 2)(n-1)b}{(b-1)n} \quad \text{and} \quad p = \frac{b-1}{a}$$

   where

$$b = \frac{4 - 3n + n^2(n_2 - 1) + \sqrt{17n^2 + n^4(n_2-1)^2 - 6n^3(n_2-1) - 4n^2 n_2 + 12n(n_2-2)}}{2n_2}$$

   has second and third normalized moments n_2 and $\frac{n}{n-1} n_2$, respectively.
3. Given an integer n and a real n_2 such that $\frac{n+3}{n} \leq n_2$ the APH(n) distribution of Erlang-Exp structure with parameters

$$a = \frac{(bn_2 - 2)(n-1)\, b}{(b-1)n} \quad \text{and} \quad p = \frac{b-1}{a}$$

   where

$$b \in \left[\frac{4 - 3n + n^2(n_2 - 1) + \sqrt{17n^2 + n^4(n_2-1)^2 - 6n^3(n_2-1) - 4n^2 n_2 + 12n(n_2-2)}}{2n_2}, \infty \right]$$

has second normalized moment n_2 and its third normalized moment is a continuous, monotone decreasing function of b.

Subset 4): The same proof applies for subset 4) as for subset 5), because $\frac{n}{n-1}n_2 > l_{n-1}$ in the $n_2 \in (\frac{n+4}{n+1}, \frac{n+3}{n})$ range.

APPENDIX B: PROOF OF THEOREM 3.2

The proof is composed of two parts. Based on the fact that all APHM(n) distributions are obtained from an APH(n) distribution according to transformation of Corollary 4.2 we show that

s1) the second and third normalized moments of an APHM(n) distribution fulfills (6) and
s2) for any $\{n_2, n_3\} \in \mathcal{APHM}(n)$ there is an APHM(n) distribution whose second and third normalized moments are n_2 and n_3.

s1) The moment bounds of the APH(n) distributions are known from Theorem 3.1. The transformation of Corollary 4.2 allows to project an $\{n_2, n_3\} \in \mathcal{APH}(n)$ point along the $n_3^* = \frac{n_3}{n_2}n_2^*$ line to any $n_2^* \geq n_2$ point. Since the derivative of the upper bound of $\mathcal{APH}(n)$ is greater than the associated n_3/n_2 ratio (i.e., $du_n/dn_2 > u_n/n_2$) all over the $n_2 \in \left(\frac{n+1}{n}, \frac{n}{n-1}\right)$ interval the upper limit of the APHM(n) class is identical with the upper limit of the APH(n) class.

The derivative of the lower bound of $\mathcal{APH}(n)$ is also greater than the associated n_3/n_2 ratio (i.e., $dl_n/dn_2 > l_n/n_2$) all over the $n_2 \in \left(\frac{n+1}{n}, \frac{n+4}{n+1}\right)$ interval, but in this case it means that the transformation of Corollary 4.2 can project the points of the lower bound out of $\mathcal{APH}(n)$. The lower bound of $\mathcal{APHM}(n)$ is provided by the projection of the $\{n_2, n_3\} \in \mathcal{APH}(n)$ point with minimal n_3/n_2 ratio. This point is the $\left\{\frac{n+1}{n}, \frac{n+2}{n}\right\}$ point, and the lower bound is the $n_3 = \frac{n+2}{n+1}n_2$ line.

s2) For any point $\{n_2, n_3\} \in \mathcal{APH}(n)$ there is an APH(n) distribution with the given second and third normalized moments according to Theorem 3.1. For any point $\{n_2, n_3\} \in \mathcal{APHM}(n)\backslash\mathcal{APH}(n)$ the procedure provided in Theorem 4.2 results in an APHM(n) distribution with the given normalized moments.

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