Preemptive Repeat Identical Transitions in Markov Regenerative Stochastic Petri Nets

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Abstract
The recent literature on Markov Regenerative Stochastic Petri Nets (MRSPN) assumes that the random firing time associated to each transition is resampled each time the transition fires or is disabled by the firing of a competitive transition. This modeling assumption does not cover the case of preemption mechanisms of repeat identical nature (pri). In this policy, an interrupted job must be repeated with an identical requirement so that its associated random variable must not be resampled. The paper investigates the implication of a pri policy into a MRSPN and describes an analytical procedure for the derivation of expressions for the transient probabilities.

Key words: Stochastic Petri Nets, Semi-Markov Reward Models, Markov regenerative processes, preemptive repeat identical policy.

1 Introduction
The analysis of stochastic systems with non-exponential timing is of increasing interest in the literature and requires the development of suitable modeling tools. Choi et al. have shown in [8] that the marking process underlying a Stochastic Petri Net (SPN), where at most one generally distributed transition is enabled in each marking, belongs to the class of Markov Regenerative Stochastic Processes (MRGPs). For this reason they referred to this new class of Petri nets as Markov Regenerative Stochastic Petri Net (MRSPN). Following the line opened in [8], different approaches have been proposed to deal with non-exponential systems [11, 6, 14, 9].

The analysis technique proposed for this class of models, consists in identifying a sequence of time points at which it is possible to forget the past history of the process. These points, indicated as regeneration points, are such that the future evolution of the stochastic process only depends on the state entered when a regeneration time point occurs. Based on the sequence of the regeneration time points, an analytical formulation of the process is available [10, 8].

The models discussed in the previous references require that the generally distributed (or deterministic) transitions are assigned a firing policy of enabling memory type [1]. Bobbio and Telek [15] have introduced the class of AgeMRSPN, in which a general transition can be associated an age memory policy. The enabled/disabled status of the transition is marked by a binary reward variable, so that the process subordinated to two consecutive regeneration points can be a semi-Markov reward process. AgeMRSPN have proved to be useful in representing situations in which an interrupted job is resumed without loss of the previous work. In [16], a computationally effective approach to the steady state analysis of AgeMRSPN with subordinated CTMC is proposed.
The technique is applied to the steady-state solution of a preemptive \( M/D/1/(n+m)/(n+m) \) queueing system with two classes of customers.

In all the previous literature on MRSPN [9, 3], based on the semantics exposed in [1], it has been implicitly assumed that, for an enabling memory transition, the associated random firing time is resampled each time the transition fires or is disabled by the firing of a competitive transition. This assumption cannot include the case of a \( pri \) policy [13] in which an interrupted job must be repeated with an identical requirement.

The paper is aimed at investigating the conditions under which a \( pri \) policy can be modeled by means of the class of MRSPN. Finally, an analytical procedure for the derivation of expressions for the transient and steady state probabilities are provided.

A practical situation, in which a \( pri \) policy is of value [7]: arises in connection with the execution of jobs whose duration is a random variable of known distribution. Selecting a job corresponds to picking up a sample from the job duration distribution. If the execution is suspended and restarted a processing time identical to the one of the interrupted job is required.

The paper is organized as follows. Section 2 introduces the concept of a \( pri \) policy in connection with MRSPN. Section 3 characterizes the process subordinated to a dominant \( pri \) transition. Section 4 presents the transient solution of a preemptive \( M/G/1/2/2 \) queueing system with two classes of customers, one of which behaves according to a \( pri \) mechanism.

### 2 pri memory policy

We adopt very standard notation. A marked Petri Net is a tuple \( PN = (P, T, I, O, H, M_0) \), where: \( P \) is the set of places, \( T \) the set of transitions. \( I, O \) and \( H \) are the input, the output and the inhibitor functions, respectively, and \( M_0 \) is the initial marking. The reachability set \( RS(M_0) \) is the set of all the markings that can be generated from the initial marking \( M_0 \). The marking process \( M(t) \) denotes the marking occupied by the PN at time \( t \).

**Proposition 1** A regeneration time point \( \tau_i^g \) in the marking process \( M(t) \) is the epoch of entrance in a marking \( M_{g,i} \) in which the Markov property holds.

To provide an analytical formulation of the stochastic process underlying a MRSPN, the following matrix valued functions (\( V(t) = [V_{ij}(t)] \), \( K(t) = [K_{ij}(t)] \)) and \( E(t) = [E_{ij}(t)] \) are defined on the reachability set \( RS(M_0) \) [10, 8]:

\[
V_{ij}(t) = \Pr[M(t) = j | M(\tau_i^g) = i] \\
K_{ij}(t) = \Pr[M(\tau_i^g) = j \cdot \tau_i^g \leq t | M(\tau_i^g) = i] \\
E_{ij}(t) = \Pr[M(\tau_i^g) = j \cdot \tau_i^g > t | M(\tau_i^g) = i]
\]

Matrix \( V(t) \) is the transition probability matrix and provides the probability that the stochastic process \( M(t) \) is in marking \( j \) at time \( t \) given it was in marking \( i \) at \( t = 0 \). The matrix \( K(t) \) is the global kernel of the MRSPN and provides the cdf of the event that the next regeneration marking is \( M_{1,j} \) given marking \( i \) at \( \tau_i^g = 0 \). Finally, the matrix \( E(t) \) is the local kernel since it describes the behavior of the marking process \( M(t) \) between two consecutive regeneration time points. The generic element \( E_{ij}(t) \) provides the probability that the process is found in state \( j \) at time \( t \) starting from \( i \) at \( \tau_i^g = 0 \) before the next regeneration time point.

The transient behavior of the MRSPN can be evaluated by solving the following generalized Markov renewal equation (in matrix form) [10, 8]:

\[
V(t) = E(t) + K \ast V(t)
\]

where \( K \ast V(t) \) is a matrix, whose \((i,j)\)-th entry is:

\[
[K \ast V(t)]_{ij} = \sum_k \int_0^1 dK_{ik}(y) V_{kj}(t-y)
\]

Equation (2) implies that the analysis of the whole process can be decomposed into the analysis of the marking process between any two successive regeneration points. The restriction of \( M(t) \) between two successive regeneration points is referred to as the subordinated process.

A transition \( tr_g \) is associated with a memory variable \( a_g \) [1]. \( a_g \) is a functional that depends on the time during which \( tr_g \) has been enabled and keeps track of the amount of the elapsed time. The functional dependence of the memory variable on the past enabling time of the transition is named the memory policy. The semantics of different memory policies has been discussed in [1] where three alternatives have been proposed referred respectively as Resampling memory, Enabling memory and Age memory policy.

**Resampling memory** and **Enabling memory** policies can be classified as repeat type policies since the age variable is reset when the transition is disabled. On the contrary, the **Age memory** policy is a resume type policy since the memory variable is reset only when the transition fires while its value is maintained if the transition is disabled and then enabled again. In order to track the enabling/disabling condition of a generally distributed transition \( tr_g \) in [5] a binary reward variable was introduced according to the following values:

- \( r_{i,1}^g = 1 \) if \( tr_g \) is enabled in marking \( k \);
- \( r_{i,2}^g = 0 \) if \( tr_g \) is not enabled in marking \( k \).

In this setting, the value of the memory variable can be computed as the accumulated reward and the firing of a transition can be formulated as a completion time problem [13, 2]. A transition fires when the elapsed time accumulated in the corresponding memory variable reaches a threshold \( \gamma_g \) equal to the value of the random firing time initially sampled from its cdf. Therefore, in order to completely define the firing conditions of a transition \( tr_g \) at a given time \( t \), two elements must be known: the value of the memory variable \( a_g \) at time \( t \), and the value of the threshold \( \gamma_g \).
Figure 1: Pictorial representation of different firing time sampling policies

All the previous literature on MRSPN [9, 3], was based on the semantics proposed in [1] where it was implicitly assumed that in connection with both repeat type policies (resampling and enabling), the threshold \( \tau_g \) is resampled each time the memory variable is reset (either because the corresponding transition fires or is disabled). The resulting MRSPN model cannot keep memory of the threshold value (firing time) of any transition beyond its current enabling period. According to this semantics, the repeat memory policies are suited to represent a preemptive repeat different (prd) execution mechanism in which the threshold value is sampled each time the transition fires or is disabled. The age memory policy is suited to represent a preemptive resume (prs) execution mechanism where the threshold value is sampled only when the transition fires, and, if the transition is disabled before the threshold is reached, then the value of the age variable is maintained.

This modeling framework was, however, inconsistent with the pri preemption policy. If a pri policy needs to be modeled, the threshold value must be maintained identical across successive enabling/disabling cycles, until the transition fires.

Figure 1 gives a pictorial description of the introduced firing time sampling strategies. In the picture, we indicate with \( E, D, F \) the enabling, disabling and firing time points of a transition, respectively. In the prd and prs cases the memory variable and the memory policy completely define the firing process of the transition. In the pri case, instead, the knowledge of these quantities is not enough, because, the value of the previously sampled threshold must be remembered after the transition is disabled (time point \( D \)).

In order to cover this case, the rule by which the threshold of a transition has to be sampled must be specified. The sampling policy is relevant only in connection with repeat type policies, and we consider the two classical alternatives [13]:

- **prd-sampling** - the threshold (firing time) is sampled each time the age variable is reset;
- **prs-sampling** - the threshold (firing time) is sampled only after the firing of the transition.

The combination of the memory policy with the sampling policy completely specifies the execution of the net. At the entrance in a new tangible marking, the completion time is computed for each enabled transition, given the memory variable and the sampled threshold. The transition with minimal completion time is the one which fires.

**Definition 1** - A transition is dormant in those markings in which the corresponding age variable is equal to zero and is active in those markings in which the age variable is greater than zero. The activity cycle of a transition is the period of time in which a transition is active between two dormant periods.

**Definition 2** - A transition is sampled if the threshold value of its random firing time has already been set up. The sampled cycle of a transition is the period of time during which the threshold value is not resampled.

For a prd-sampling policy, sampled and activity cycles are coincident. For a prs-sampling policy sampled and activity cycles are different: the sampled cycle is the interval of time between the first time the transition is enabled after it was fired, and the time instance in which it fires again. The sampled cycle strictly contains the activity cycle. A transition can be dormant but its threshold can be sampled, thus causing a conditional dependence in the underlying marking process.

In either cases, the definition of the sampling policy assumes that at the time point in which a threshold is resampled, the memory variable is zero. In the light of the previous discussion, we can particularize Proposition 1 by stating that a regeneration time point \( \tau^* \) is the epoch of entrance in a marking \( M_{tr} \) in which all the memory variables are zero and all the thresholds are not sampled.

The prs-sampling policy is the one implicitly assumed in the previous literature and its behavior has been completely characterized in [5]. The combination of the repeat type memory policies with the pri sampling policy is described in the following:

- **pri-Resampling Memory** - The activity cycle ends at the first firing of any transition (including \( tr_g \) itself), while the sampled cycle ends only when \( tr_g \) fires. Thus the same sampled value is maintained identical over successive activity cycles.

- **pri-Enabling Memory** - The activity cycle starts as soon as \( tr_g \) becomes enabled, and ends either when \( tr_g \) fires, or when it becomes disabled by the firing of a competitive transition. The sampled cycles ends only when \( tr_g \) fires.

The above features can be compared with prd and prs policies in the following table:
Let us suppose that a regeneration period starts at time $t = 0$ from marking $i$ and is dominated by a transition $\tau_{ij}$ with memory variable $a_j$ and random firing time $\gamma_j$. Transition $\tau_{ij}$ is priEnabling memory, so that the next regeneration point is the firing time of $\tau_{ij}$ itself. The process subordinated to the dominant transition is a semi-Markov process. Since the global and local kernels $K(t)$ and $E(t)$ can be evaluated row by row, given the above assumptions, we provide an analytical procedure for determining the non-zero entries of the $i$-th row.

To better understand the developed mathematical formalism, we summarize the notation:

- $\Omega$: reachability set $\mathcal{R}(M_0)$;
- $m$: cardinality of the reachability set;
- $\tau_{ij}$: vector grouping the reward rates associated to $\tau_{ij}$ during its sampled cycle;
- $R^i$: subset of $\Omega$ grouping the states reachable from state $i$ inside the sampled cycle of $\tau_{ij}$ in which $\tau_{ij}$ is enabled; for any $k \in R^i$, the reward rate is equal to 1 and $a_j$ is strictly increasing;
- $h$: cardinality of $R^i$;
- $R^c_i$: subset of $\Omega$ in which $\tau_{ij}$ is not enabled, but still sampled: for any $k \in R_c^i$, the reward rate is equal to 0 and $a_j$ is not increasing;
- $m$: cardinality of $R^c_i$;
- $R^c_i$: subset of $\Omega$ in which $\tau_{ij}$ is not enabled and not sampled: $\Omega = R^i + R^c_i + R^c_i$;
- $Z^i(t)$: right-continuous subordinated semi-Markov process defined over $R^i + R^c_i$;
- $Q^i(t) = \mathbb{Q}^i(t)$: kernel of the subordinated semi-Markov process;
- $w$: threshold value sampled from the firing time $r_{ij}$ associated with transition $\tau_{ij}$.

$Z^i(t)$ starts at time $t = 0$ in marking $M_0$ with probability 1, so that the initial probability vector is $\mathbb{P}_0 = [0, 0, \ldots, 1, \ldots, 0]$ (a vector with all the entries equal to 0 except entry $i$, which equals 1).

For notational convenience we renumber the states in $\Omega$ so that the states numbered $1, 2, \ldots, h$ belong to the subset $R^i$, in which the dominant $pri$ transition $\tau_{ij}$ is enabled and the states numbered $h + 1, h + 2, \ldots, m + h$ belong to $R^c_i$ in which $\tau_{ij}$ is disabled. By this ordering of states $Q^i(t)$ can be partitioned into the following submatrices $Q(t) = \begin{bmatrix} Q_1(t) & Q_2(t) \\ Q_3(t) & Q_4(t) \end{bmatrix}$ where $Q_1(t)$ describes the transitions inside $R^i$, $Q_2(t)$ from $R^i$ to $R^c_i$, $Q_3(t)$ from $R^c_i$ to $R^c_i$ and $Q_4(t)$ inside $R^c_i$. We denote by $H$ the
time duration until the first embedded time point in
$Z'(t)$ starting from state $k$ at time $t = 0$, and by $Q_k(t)$
the cdf of $H (Q_k(t) = \sum_{j=1}^{k} Q_{kj}(t))$.

Let us fix the value of the firing requirement $\gamma_g = w$, and let us define the following matrix functions $P^i(t, w), F^i(t, w)$ and $\Delta^i$ of dimension $n \times n$:

$$P_{ik}(t, w) = Pr\{Z_i'(t) = \ell, \tau_i^* > t \mid Z_i'(0) = k, \gamma_g = w\}$$

$$F_{ik}(t, w) = Pr\{Z_i'(\tau_i^*--) = \ell, \tau_i^* \leq t, tr_g\text{ fires(5)} \mid Z_i'(0) = k, \gamma_g = w\}$$

$$\Delta_{ik} = Pr\{\text{next tangible marking is } \ell \mid \text{current marking is } k, tr_g\text{ fires}\}$$

By the above definitions:

- $P_{ik}(t, w)$ is the probability of being in state $\ell$ at time $t$ before absorption at the barrier $w$, starting in state $k$ at $t = 0$.
- $F_{ik}(t, w)$ is the probability that $tr_g$ fires from state $\ell$ (hitting the absorbing barrier $w$ in $\ell$) before $t$, starting in state $k$ at $t = 0$. For $\ell \in R^c$, $F_{ik}(t, w) = 0$.
- $\Delta^i$ is the branching probability matrix and represents the successor tangible marking $\ell$ that is reached by firing $tr_g$ in state $k \in R^c$ (the firing of $tr_g$ in the subordinated process $M^i(t)$, can only occur in a state $k$ in which $r_i^j = 1$).

From (5) and (6), it follows that for any $t$:

$$\sum_{\ell \in R^c} \left[ P_{ik}(t, w) + F_{ik}(t, w) \right] = 1$$

Given that $G_p(w)$ is the cumulative distribution function of the firing time $v$, the elements of the $i$-th row of matrices $K(t)$ and $E(t)$ can be expressed as follows, as a function of the matrices $P(t, w)$ and $F(t, w)$:

$$K_{ij}(t) = \int_{w=0}^{\infty} \sum_{k \in R} F_{ik}(t, w) \Delta_{kj} dG_p(w)$$

$$E_{ij}(t) = \int_{w=0}^{\infty} P_{ij}(t, w) dG_p(w)$$

where the notation $K_{ij}(t)$ and $E_{ij}(t)$ refers to the modified numbering of the states.

Figure 2: A sampled path of the subordinated process $Z(t)$

In order to avoid unnecessarily cumbersome notation in the following derivation, we neglect the explicit dependence on the particular subordinated process $Z'(t)$, by eliminating the superscript $i$. It is however tacitly intended, that all the quantities $\tau, Q(t), P(t, w), F(t, w), \Delta, R$ and $R^c$ refer to the specific process subordinated to the regeneration period starting from state $i$.

### 3.1 Derivation of $P(t, w)$ and $F(t, w)$

A regeneration period of a prEnabling Memory type transition (tr$_g$) starts from a state ($k$) in which tr$_g$ is enabled ($k \in R$) and completes when tr$_g$ fires. The reward rate is equal to 1 for all the states inside $R$ and the age variable $a_g$ is continuously increasing. $a_g$ is reset to 0 when $Z(t)$ exits $R$ to enter $R^c$.

The subordinated process consists of a random number of unsuccessful activity cycles inside $R$, each one followed by period inside $R^c$ in which tr$_g$ is disabled, finally concluded by a successful activity cycle, at the end of which tr$_g$ fires. A subordinated process of this kind can be considered as a MRSPN with a dominant deterministic transition of prEnabling Memory type whose firing time equals the sampled work requirement of tr$_g$.

Figure 2 shows a possible realization of $Z(t)$, assuming that the sampled period starts at time $t = 0$ and tr$_g$ is enabled. The following cases can occur:

- the process is inside an active cycle $R$;
- tr$_g$ has already fired;
- tr$_g$ is disabled and $Z(t)$ enters $R^c$;
- $Z(t)$ is in $R^c$;
- $Z(t)$ leaves $R^c$ and enters $R$ (tr$_g$ is enabled again).

The subordinated MRGP can thus be described by the following matrix functions, where $T_1$ and $T_2$ denote the time point until $Z(t)$ visits $R$ and $T_2$ denotes the time point until $Z(t)$ visits $R^c$.

$$P_{1k}(t, w) = Pr\{Z(t) = \ell \mid \tau^*_i > t, T_1 > t, Z(0) = k \in R, \gamma_g = w\}$$

$$F_{1k}(t, w) = Pr\{Z(\tau^*_i--) = \ell \mid \tau^*_i \leq t, T_1 > \tau^*_i, Z(0) = k \in R, \gamma_g = w\}$$
By the above definitions it follows that:

- $P_{12k}(t, w)$ is the probability of being in state $t \in R$ at time $t$ before absorption at the barrier $w$ or leaving $R$, starting in state $k \in R$ at $t = 0$.
- $F_{12k}(t, w)$ is the probability that $t$ fires from state $t \in R$ before time $t$, supposing that the subdivided process never left $R$ up to $t$, starting in state $k \in R$ at $t = 0$.
- $P_{12s}(t, w)$ is the probability that the subdivided process left $R$ before time $t$ and before absorption at the barrier $w$ to reach state $t \in R^c$ (i.e. $t$ is the first visited state once entered $R^c$), assuming to start in state $k \in R$ at $t = 0$.
- $P_{21k}(t, w)$ is the probability of being in state $t \in R^c$ at time $t$ before leaving $R^c$, starting in state $k \in R^c$ at $t = 0$.
- $P_{21s}(t, w)$ is the probability that the subdivided process left $R^c$ before time $t$ and $t$ is the first visited state in $R$ starting in state $k \in R^c$ at $t = 0$.

According to the previous definitions, the following equalities hold:

$$P_{1k}(t, w) + P_{12k}(t, w) + F_{1k}(t, w) = 1$$
$$P_{2k}(t) + P_{21k}(t) = 1$$

**Theorem 1** - The LST transform of the matrix function $P(t, w)$ satisfies the following equation:

$$P^-(s, w) = [I - P^{-1}(s, w) P_{21}^{-1}(s)]^{-1} [P^{-1}(s, w) + P^{-1}(s, w) P^{-1}(s)]$$

where the matrix functions in the r.h.s are derived in the following lemmas.

**Lemma 1** - The double LST-LT transform of the probability function $P_{1k}(t, w)$ satisfies the following equation:

$$P_{12k}^-(s, w) = \delta_{kt} \left[1 - Q_{kt}^-(s + v)\right] + \sum_{w \in R^c} Q_{wt}^-(s + v) P_{21k}^-(s, w)$$

**Proof of Lemma 1** - The proof of the lemma follows the same procedure line developed in [13, 2] for the analysis of the distribution of the completion time in a semi-Markov reward process. Conditioning on $H = h$, and $\gamma_2 = w$ let us define:

$$P_{1k}(t, w \mid H = h) = \begin{cases} \delta_{kt} [U(t) - U(t - h)] & \text{if } h \geq w \\ \sum_{w \in R} \frac{dQ_{ku}(h)}{dQ_k(h)} P_{1u}(t - h, w - h) & \text{if } h < w \end{cases}$$

where $U(t)$ denotes the unit step function.

If $h < w$ then a transition occurs to state $w$ with probability $dQ_{ku}(h)/dQ_k(h)$ and the residual firing time $(w - h)$ should be accomplished starting from state $w$ at time $(t - h)$. Taking the LST transform with respect to $t$ (denoting the transform variable by $s$), the LT transform with respect to $w$ (denoting the transform variable by $v$) of (15) and unconditioning with respect to $H$, (15) becomes (15).

**Lemma 2** - The double LST-LT transform of the probability function $P_{12k}(t, w)$ satisfies the following equation:

$$P_{12k}^-(s, v) = \frac{1}{w} Q_{ku}^-(s + v) + \sum_{w \in R} Q_{ku}^-(s + v) P_{12k}^-(s, v)$$

**Proof of Lemma 2** - Conditioning on $H = h$, and $\gamma_2 = w$ let us define:

$$P_{12k}^-(s, v) = \begin{cases} 0 & \text{if } h \geq w \\ \frac{dQ_{ku}(h)}{dQ_k(h)} U(t - h) + \sum_{w \in R} \frac{dQ_{ku}(h)}{dQ_k(h)} P_{21u}(t - h, w - h) & \text{if } h < w \end{cases}$$

The derivation of $P_{2k}(t, w)$ based on (17) follows the same pattern [17] as for $P_{1k}(t, w)$ in Lemma 1.

**Lemma 3** - The double LST-LT transform of the probability function $P_{2k}(t, w)$ satisfies the following equation:

$$P_{2k}^-(s) = \delta_{kt} \left[1 - Q_{kt}^-(s)\right] + \sum_{w \in R} P_{21u}^-(s) Q_{ku}^-$$

**Proof of Lemma 3** - Lemma 3 can be directly derived from Lemma 1, by substituting subset $R$ with $R^c$ and assuming a threshold $w$ equal to infinity.

**Lemma 4** - The double LST-LT transform of the probability function $P_{21k}(t, w)$ satisfies the following equation:
resampling

Enabling of the transition has to be considered. Instead the arbitrary number of sampled cycles up to the indicated than the analysis of all the other cases, because the process is resident in $R$ at time $t$ before firing of $t_1$ can be decomposed into the mutually exclusive events that the process is resident in $R$ continuously from $t = 0$ or after 1, 2, ... passages through $R^c$ (see Figure 2); hence,

$$P(1)_{t}(s) = Q^c_{1}(s) + \sum_{u \in R^c} Q^c_{1u}(s) P(1)_{u}(s)$$

Proof of Lemma 4 - Lemma 4 can be directly derived from Lemma 2, by substituting subset $R$ with $R^c$ and assuming a threshold $u$ equal to infinity. □

Proof of Theorem 1 - The event that the process is resident in $R$ at time $t$ before firing of $t_1$ can be decomposed into the mutually exclusive events that the process is resident in $R$ continuously from $t = 0$ or after 1, 2, ... passages through $R^c$ (see Figure 2); hence,

$$P^c(s, w) = P^c_1(s, w) + P^c_2(s, w) P^c_2(s) + P^c_2(s, w) P^c_2(s) \cdot P^c_2(s, w) P^c_2(s) + P^c_2(s, w) P^c_2(s) + \ldots$$

The expression (21) is obtained by applying $\sum_{i \geq 0} M^i = [I - M]^{-1}$. □

Since only the $pri$-Resampling and $pri$-Enabling transitions can have more than one activity cycle in their sampled cycles, their analysis is more complicated than the analysis of all the other cases, because the arbitrary number of sampled cycles up to the firing of the transition has to be considered. Instead the analysis of the other general transitions requires only the evaluation of one sampled cycle, and only the matrix functions like $P(1, t)$ and $F(1, t)$ has to be evaluated.

Theorem 2 - The $LST$ transform of the firing probability matrix $F(t, w)$ satisfies the following equation:

$$F^c(s, w) = (I - P^c_1(s, w) P^c_2(s))^{-1} F^c(s, w)$$

Where $F(t, w)$ is derived in Lemma 5.

Lemma 5 - The double $LST-LT$ transform of the firing probability $F_{1,k}(t, w)$ satisfies the following equation:

$$F_{1,k}^c(s, v) = \delta_{k1} \frac{1 - Q^c_{1}(s + v)}{s + v} + \sum_{u \in R} Q^c_{1u}(s + v) F_{1,k}^c(s, v)$$

Proof of Lemma 5 - Conditioning on $H = h$ and $\gamma_2 = w$, let us define:

$$F_{1,k}^c(s, v) = \delta_{k1} \frac{1 - Q^c_{1}(s + v)}{s + v} + \sum_{u \in R} Q^c_{1u}(s + v) F_{1,k}^c(s, v)$$

From which the theorem comes. □

4 Preemptive $M/G/1/2/2$ queue with different customers

The $PN$ of Figure 3a models a $M/G/1/2/2$ queue in which the jobs submitted by customer 2 have higher priority and preempts the jobs submitted by customer 1. The server has a $pri$ Enabling Memory service discipline, which means, that the service of a preempted lower priority job starts from the beginning, when the server becomes available. Place $p_1$ ($p_4$) represents customer 1 (2) thinking, while place $p_2$ ($p_3$) represent job 1 (2) under service. Transitions $t_1$ and $t_3$ are EXP and represent the submission of a job of type 1 or 2, respectively. $t_2$ and $t_4$ are GEN transitions, and represent the completion of service of a job of type 1 or 2, respectively. A $pri$ service discipline is modeled by
assigning to \( tr_3 \) an \( \textit{pri Enabling Memory} \) type. The inhibitor arc from \( p_k \) to \( tr_3 \) models the described preemption mechanism: as soon as a type 2 job joins the queue the type 1 job under service (if any) is interrupted. The reachability graph of the PN of Figure 3a is in Figure 3b. Under a \( \textit{pri} \) service, after completion of the type 2 job, the service of the interrupted type 1 job is restarted, and the same job (with an identical work requirement) has to be completed. From Figure 3b, it is easily recognized that states \( s_1, s_2 \) and \( s_3 \) can all be regeneration states, while \( s_4 \) can never be a regeneration state.

Only EXP transitions are enabled in \( s_1 \) and the next regeneration states can be either \( s_2 \) or \( s_3 \) depending whether \( tr_1 \) or \( tr_2 \) fires first. From \( s_2 \) the next regeneration state can be only \( s_1 \), but multiple cycles \( (s_2 - s_3) \) can occur depending whether type 2 jobs arrive to interrupt the execution of the type 1 job. The dominant transition is \( tr_2 \) with \( \textit{Pri Enabling Memory} \) and the subordinated process is a reward \( S\text{MP} (tr_4 \text{ is GEN}); hence the results of the previous section can be applied for the evaluation of the relevant row of the \( E(t) \) and \( K(t) \) matrices.

Let us de/\( \text{猛地} \)fine the \( a \)

The rew/\( \text{….} \)ard vector is \( \mathbf{P} \) = \( [1, 0] \), and the firing of \( tr_2 \) leads to \( s_4 \) with probability \( 1 \). Applying Lemma 1 and 5 we obtain the following results for the nonzero entries in the double transform domain:

\[
P_{11}^a(s, w) = \frac{s}{(s + v + \lambda)}; \quad F_{11}^a(s, v) = \frac{1}{s + v + \lambda}
\]

after inverse Laplace transforming with respect to \( v \) we have:

\[
P_{11}^a(s, w) = \frac{s a}{s + \lambda}; \quad F_{11}^a(s, w) = 1 - a
\]

where \( a = 1 - e^{-(s + \lambda) v} \). Applying Lemma 2:

\[
P_{12}^a(s, v) = \frac{\lambda}{(s + v + \lambda)}; \quad P_{12}^a(s, w) = \frac{\lambda a}{s + \lambda}
\]

Let us define the matrix functions to describe the subordinated process:

\[
\mathbf{P}_i^a(s, w) = \begin{pmatrix}
s a & 0 & 0 \\
0 & s + \lambda & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\mathbf{P}_2^a(s, w) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 - G_z(s) & 0 \\
0 & G_z(s) & 0 \\
\end{pmatrix}
\]

\[
\mathbf{F}_1^a(s, w) = \begin{pmatrix}
1 - a & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

Due to Theorems 1 and 2, the relevant entries (1st row) of the \( \mathbf{P}^a(s, w) \) and the \( \mathbf{F}^a(s, w) \) matrices are given by:

\[
P_{11}^a(s, w) = \frac{s a}{s + \lambda}; \quad P_{12}^a(s, w) = \frac{(1 - G_z(s)) \lambda a}{s + \lambda}
\]

\[
F_{11}^a(s, w) = \frac{1 - G_z(s) \lambda a}{s + \lambda}
\]

With respect to the original state space numbering, the nonzero entries in the 2nd row of the \( LST \)
matrix functions $K^{-}(s)$ and $E^{-}(s)$ have the following expression:

$$E^{-}_{s_1,s_2}(s) = \int_{0}^{\infty} P_{11}^{-}(s, w) dG_2(w)$$

$$E^{-}_{s_2,s_1}(s) = \int_{0}^{\infty} P_{11}^{-}(s, w) dG_2(w)$$

$$K^{-}_{s_2,s_1}(s) = \int_{w=0}^{\infty} F_{11}^{-}(s, w) dG_2(w)$$

(29)

The time domain probabilities are calculated by first deriving $V^{-}(s)$ from (2) using a standard package for symbolic analysis (e.g. MATLAB), and then numerically inverting the resulting LST expressions resorting to the Jagerman’s method [12]. The plot of the state probabilities versus time together with the steady state results (dotted line) is reported in Figure 4.

- submitting rate $\lambda = 0.5$;
- higher priority customer (transition $fr_4$) with an exponentially distributed service time with parameter $\mu = 1$ ($G_2(s) = \frac{1}{1+\lambda s}$);
- lower priority customer (transition $fr_2$) with a service time uniformly distributed between $\alpha = 0.5$ and $\beta = 1.5$ ($G_2(t) = \frac{\beta-\alpha}{\beta-\alpha} (U(t-\alpha)-U(t-\beta))$).

5 Conclusion

The semantics of the SPN model with generally distributed transitions discussed in [1], provides a natural environment for the definition of a class of analytically tractable MRSPN’s. However, the proposed semantics, for repeat type memory policies, assumes a coincidence between the action of resetting the memory variable of a transition and the action of resampling its firing time threshold. Therefore, only prd policies can be modeled. This paper has shown that a pri policy can be considered if the instant at which the firing time is resampled is evaluated separately from the instant at which the memory variable is reset. The sampled period of a pri transition may contain several activity periods, and is the crucial factor in determining the sequence of the regeneration time points in the marking process. An analytical solution is given when the dominant transition is classified as priEnabling Memory type with subordinated reward semi-Markov process.

6 Acknowledgements

The authors are thankful to Sachin Garg for the fruitful discussions. A. Bobbio and K. S. Trivedi would like to thank NATO for grant No. CRG.940308. A. Puliafito would like to thank Italian CNR for grant No. 203.15.5. This research was carried out while M. Telek was visiting the Department of Operations Research of the University of North Carolina at Chapel Hill sponsored by the Hungarian OTKA (Grant No. W 015850).

Figure 4: Transient behavior of the state probabilities for the preemptive M/G/1/2/2 system with different customers.

References


