Evaluation of the Completion Time and Catastrophic Failure Time of a Two State System

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Abstract

The distribution of the completion time vs catastrophic failure time of a two state system with a finite work requirement and with a down time constraint is derived in closed form in the Laplace transform domain. Various combinations of preemption policies are examined, and the numerical evaluation of the formulated problem is discussed

1 Introduction

A system alternates between an up state and a down state. The sojourn times in the up state are i.i.d. r.v. U_i with distribution $\hat{U}(x) = Pr\{U \le x\}$ and Laplace Stieltjes Transform (LST) $U^{\sim}(s)$. Similarly, the sojourn times in the down state are i.i.d. r.v. D_i with distribution $\hat{D}(x) = Pr\{D \le x\}$ and LST $D^{\sim}(s)$. A task, with an assigned constant work requirement a, is executed when the system is in the up state. The system reaches a catastrophic condition when the downtime exceeds a critical constant threshold b [3]. Let us denote $C(a, b)$ the completion time, i.e. the r.v. representing the completion of the task (with work requirement a) before the system reaches the catastrophic condition (with down time threshold b). Conversely, let us denote $F(a, b)$ the catastrophic failure time, i.e the r.v. representing the attainment of the catastrophic condition before the completion of the task.

In order to completely specify the problem, the accumulation process in both the up and the down state should be defined. Two alternatives are examined:

- the time spent in the up (down) state, is accumulated at each visit. We refer to this mechanism as *preemptive-resume (prs)* accumulation policy;
- the sojourn time accumulation starts from zero at each visit in the up (down) state, and the time previoulsy spent in the same state is lost. We refer to this mechanism as *preemptive-repeat (prd)* accumulation policy.

Accordingly, we can define 4 cases, namely UsDs, UsDd, UdDs and UdDd, where the capital letter refers to the states (U - up state, D - down state) and the small case letter refers to the accumulation policy (s - prs, d - prd). Limiting cases arise when the work requirement (a) or the down time constrain (b) are considered infinity. In the first case, only catastrophic failure is possible (cases Ds and Dd), and in the latter case only completion is possible (cases Us and Ud).

Let $\hat{C}(t, a, b)$ and $\hat{F}(t, a, b)$ be the Cdf of $C(a, b)$ and $F(a, b)$, respectively:

$$\hat{C}(t,a,b) = Pr\{C(a,b) \leq t\} \qquad \text{and} \qquad \hat{F}(t,a,b) = Pr\{F(a,b) \leq t\}$$

and let

$$C^{\sim}(s,a,b) \;=\; \int_0^{\infty} e^{-st}\, d\hat{C}(t,a,b) \qquad \text{and} \qquad F^{\sim}(s,a,b) \;=\; \int_0^{\infty} e^{-st}\, d\hat{F}(t,a,b)$$

be the Laplace Stieltjes transforms (LST). Finally, define:

$$U_a^{\sim}(s) = \int_0^{a} e^{-sh}\, d\hat{U}(h) \qquad \text{and} \qquad D_b^{\sim}(s) = \int_0^{b} e^{-sy}\, d\hat{D}(y)$$

If the system is Markovian (U and D are exponential r.v. of rates λ and μ, respectively), then

$$\hat{U}(x) = 1 - e^{-\lambda x} \;\;,\;\; U^{\sim}(s) = \frac{\lambda}{s+\lambda} \;\;,\;\; \text{and} \;\; U_a^{\sim}(s) = \frac{\lambda}{s+\lambda}[1 - e^{-(s+\lambda)a}]$$

$$\hat{D}(x) = 1 - e^{-\mu x} \;\;,\;\; D^{\sim}(s) = \frac{\mu}{s+\mu} \;\;,\;\; \text{and} \;\; D_b^{\sim}(s) = \frac{\mu}{s+\mu}[1 - e^{-(s+\mu)b}]$$

Let $u_{-1}(t)$ be the unit step function.

2 Transform domain analysis

The following analysis is mutuated from [3]. The LST expressions $C^{\sim}(s,a,b)$ (or $F^{\sim}(s,a,b)$) are derived by conditioning the initial up time to be $U = h = const$, and the subsequent down time to be $D = y = const$, and then unconditioning w.r.t. U, D. Furthermore, the Laplace transform (LT) of the resulting expression is evaluated w.r.t. the variable for which a *prs* accumulation policy is assumed; hence in the UsD· cases we take the LT w.r.t. a (denoting the transform variable by w), and in the U·Ds case we take the LT w.r.t. b (denoting the transform variable by v). Therefore, in the UsDs case a triple transformation is needed.

Here we show in details the derivation of the UsDd case, only.

UsDd case

Time domain conditioned description:

$$\hat{C}(t,a,b|U=h,D=y) = \begin{cases} u_{-1}(t-a) & h \geq a \\ 0 & h < a, y \geq b \\ \hat{C}(t-(h+y), a-h, b) & h < a, y < b \end{cases}$$

LST domain conditioned description:

$$C^{\sim}(s,a,b|U=h,D=y) = \begin{cases} e^{-sa} & h \geq a \\ 0 & h < a, y \geq b \\ e^{-s(h+y)} C^{\sim}(s, a-h, b) & h < a, y < b \end{cases}$$

Unconditioning on U results:

$$C^{\sim}(s,a,b|D=y) = \begin{cases} e^{-sa}[1-\hat{U}(a)] & \\ e^{-sy} \int_0^a e^{-sh} C^{\sim}(s,a-h,b) d\hat{U}(h) & y < b \end{cases}$$

Unconditioning on D results:

$$C^{\sim}(s,a,b) = e^{-sa}[1-\hat{U}(a)] + \int_0^b e^{-sy} d\hat{D}(y) \int_0^a e^{-sh} C^{\sim}(s,a-h,b) d\hat{U}(h)$$

Taking the Laplace transform with respect to $a \to w$, we have:

$$C^{\sim\star}(s,w,b) = \frac{1-U^{\sim}(s+w)}{s+w} + D_b^{\sim}(s)U^{\sim}(s+w)C^{\sim\star}(s,w,b)$$

From the above expression we get:

$$C^{\sim\star}(s,w,b) = \frac{1-U^{\sim}(s+w)}{(s+w)[1-D_b^{\sim}(s)U^{\sim}(s+w)]}$$

Under the Markovian assumption (U and D exponential):

$$C^{\sim\star}(s,w,b) = \frac{1}{s+w+\lambda-\lambda D_b^{\sim}(s)}$$

Taking the inverse Laplace transform ($w \to a$) of the above expression , we finally get:

$$C^{\sim}(s,a,b) = e^{-a(s+\lambda-\lambda D_b^{\sim}(s))}$$

The evaluation of the other cases follows the same pattern. Table I contains the closed form expressions for all the considered cases.

Case	Completion time	Catastrophic failure time
UsDd	$C^{\sim\star}(s,w,b) = \frac{1-U^{\sim}(s+w)}{(s+w)[1-D_b^{\sim}(s)U^{\sim}(s+w)]}$	$F^{\sim\star}(s,w,b) = \frac{e^{-sb}[1-\hat{D}(b)]U^{\sim}(s+w)}{w[1-D_b^{\sim}(s)U^{\sim}(s+w)]}$
UdDs	$C^{\sim\star}(s,a,v) = \frac{e^{-sa}[1-\hat{U}(a)]}{v[1-U_a^{\sim}(s)D^{\sim}(s+v)]}$	$F^{\sim\star}(s,a,v) = \frac{U_a^{\sim}(s)[1-D^{\sim}(s+v)]}{(s+v)[1-U_a^{\sim}(s)D^{\sim}(s+v)]}$
UdDd	$C^{\sim}(s,a,b) = \frac{e^{-as}[1-\hat{U}(a)]}{1-D_b^{\sim}(s)U_a^{\sim}(s)}$	$F^{\sim}(s,a,b) = \frac{e^{-bs}[1-\hat{D}(b)]U_a^{\sim}(s)}{1-D_b^{\sim}(s)U_a^{\sim}(s)}$
UsDs	$C^{\sim\star\star}(s,w,v) = \frac{1-U^{\sim}(s+w)}{v(s+w)[1-U^{\sim}(s+w)D^{\sim}(s+v)]}$	$F^{\sim\star\star}(s,w,v) = \frac{U^{\sim}(s+w)[1-D^{\sim}(s+v)]}{w(s+v)[1-U^{\sim}(s+w)D^{\sim}(s+v)]}$
Us	$C^{\sim\star}(s,w) = \frac{1-U^{\sim}(s+w)}{(s+w)[1-D^{\sim}(s)U^{\sim}(s+w)]}$	$F^{\sim\star}(s,w) = 0$
Ud	$C^{\sim}(s,a) = \frac{e^{-as}[1-\hat{U}(a)]}{1-D^{\sim}(s)U_a^{\sim}(s)}$	$F^{\sim}(s,a) = 0$
Ds	$C^{\sim\star}(s,v) = 0$	$F^{\sim\star}(s,v) = \frac{U^{\sim}(s)[1-D^{\sim}(s+v)]}{(s+v)[1-U^{\sim}(s)D^{\sim}(s+v)]}$
Dd	$C^{\sim}(s,b) = 0$	$F^{\sim}(s,b) = \frac{e^{-bs}[1-\hat{D}(b)]U^{\sim}(s)}{1-D_b^{\sim}(s)U^{\sim}(s)}$

Table I - Transform domain description of the completion and catastrophic failure time

We can easily particularize the formulas of Table I in the Markovian case, when U and D are exponential.

3 Considerations about the numerical solution

For the double LT expressions in Table I, a symbolical inversion is performed w.r.t. the LT variable (either w or v), and then the time domain solution is evaluated by applying to the

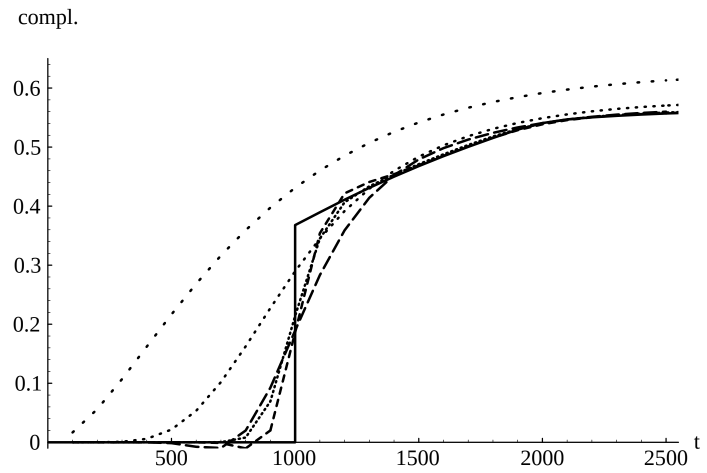

Figure 1 - Distribution of completion time for the UdDs system.

LST expression the Jagerman's inversion numerical technique [2].

For the UsDs case, some further considerations are necessary to reach a single transform description. We observe that the UsDs system has a finite processing time, in the sense that both the completion and the catastrophic failure time are upper bounded by $(a+b)$. Let us introduce:

$W(t)$: the r.v. representing the accumulated time spent in the up state during $(0,t)$, and

$V(t)$: the r.v. representing the accumulated time spent in the down state during $(0,t)$.

$W(t)+V(t)=t$ by definition and, if $t<a+b$ and $W(t)\geq a$, then $V(t)<b$.

For $t<a+b$, the following holds:

$$\begin{aligned}\hat{C}_{UsDs}(t,a,b) &= Pr\{C_{UsDs}(a,b)\leq t\} = Pr\{W(t)\geq a, V(t)<b\} = Pr\{W(t)\geq a\} \\ &= Pr\{C_{Us}(a)\leq t\} = \hat{C}_{Us}(t,a)\,.\end{aligned}$$

Similarly

$$\begin{aligned}\hat{F}_{UsDs}(t,a,b) &= Pr\{F_{UsDs}(a,b)\leq t\} = Pr\{V(t)\geq b, W(t)<a\} = Pr\{V(t)\geq b\} \\ &= Pr\{F_{Ds}(b)\leq t\} = \hat{F}_{Ds}(t,b)\,.\end{aligned}$$

Hence, for $t<a+b$, the numerical evaluation of the UsDs system is reduced to the evaluation of either the Us or Ds system, for which a single transform description is available in Table I.

The accuracy of the Jagerman's method is a function of the number of iterations, but its efficiency is reduced if the function to be transformed has a very steep behavior or presents discontinuities (steps).

If U has infinite positive distribution and $0<a,b<\infty$ then $\hat{C}(t,a,b)=0$ for any $t<a$ and $\hat{C}(a,a,b)>0$. Hence, $\hat{C}(t,a,b)$ is not continuous in $t=a$ but it has a step. On the other hand, $\hat{F}(t,a,b)=0$ for any $t<b$ but also $\hat{F}(b,a,b)=0$, and the function is continuous in $t=b$. Due to these properties (common to all the cases), the Jagerman's method provides better results if applied to the shifted functions $\hat{C}_a(t,a,b)=\hat{C}(t-a,a,b)$ and $\hat{F}_b(t,a,b)=\hat{F}(t-b,a,b)$ rather than for $\hat{C}(t,a,b)$ and $\hat{F}(t,a,b)$. In LST domain:

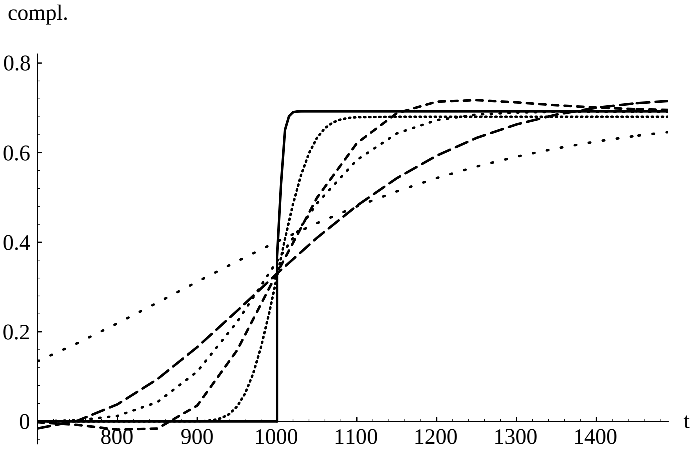

Figure 2 - Distribution of completion time for the UsDd system.

$$C^{\sim}(s,a,b) = e^{-as} C_a^{\sim}(s,a,b) \qquad \text{and} \qquad F^{\sim}(s,a,b) = e^{-bs} F_b^{\sim}(s,a,b) \, .$$

For the case of the UsDd Markovian system, the shifted functions become:

$$C_a^{\sim}(s,a,b) = e^{-a(\lambda - \lambda D_b^{\sim}(s))} \qquad \text{and} \qquad F_b^{\sim}(s,a,b) = \frac{\lambda e^{-b\mu}}{s + \lambda - \lambda D_b^{\sim}(s)} [1 - e^{-a(s+\lambda-\lambda D_b^{\sim}(s))}] \, .$$

The gain of numerically solving $C_a^{\sim}(s,a,b)$ depends on the step of the function at $t = a$ (i. e. $\hat{C}(a,a,b)$) and the rate of the transient time compared to a. The gain in the calculation of $F_b^{\sim}(s,a,b)$ depends only on the relation of the transient time to b.

4 Numerical examples

We compare three computational methods: namely, the numerical inverse transformation of $C^{\sim}(s,.)$ ($F^{\sim}(s,.)$ (IT), the numerical inverse transformation of the shifted function $C_a^{\sim}(s,.)$ ($F_b^{\sim}(s,.)$ (ITSF), and the PH approximation technique [1] (PH). In the latter case we approximate the constant work reqirement a and the downtime constraint b by an Erlang distribution of assigned order and with the same expected value (a or b, respectively).

Figure 1 depicts the defective distribution of the completion time for the UdDs system, where the up (down) time is exponentially distributed with paramater $\lambda = 10^{-3}$ ($\mu = 0.1$) and the up (down) time constraint is $a = 1000$ ($b = 30$). The solid line is obtained by the ITSF of order 50 while the dashed lines comes from the IT of order 10 (rear one) and order 50 (dense one). The dotted lines are obtained by the PH method of order 2×2 (rear dotted curve), 10×10 (middle dense dotted curve), and order 100×100 (dense dotted curve). Here the first (second) value is the order of the Erlang approximating the deterministic work requirement (down time constraint).

In this example, the distribution of the completion time shows a step at $t = a$ equal to $e^{-a\lambda}$ (i.e. the probability, that the system never fails before completing the work requirement). The only method which is able to capture the discontinuity is the ITSF. Both the IT and the PH methods provide continuous functions closer and closer to the exact one by increasing the order. However, we can recognize two main differences between them. The IT method provides an overshot and undershot around the discontinuity (which can produce values under 0, and occasionally above 1 as well), but reaches always the correct steady state result. The

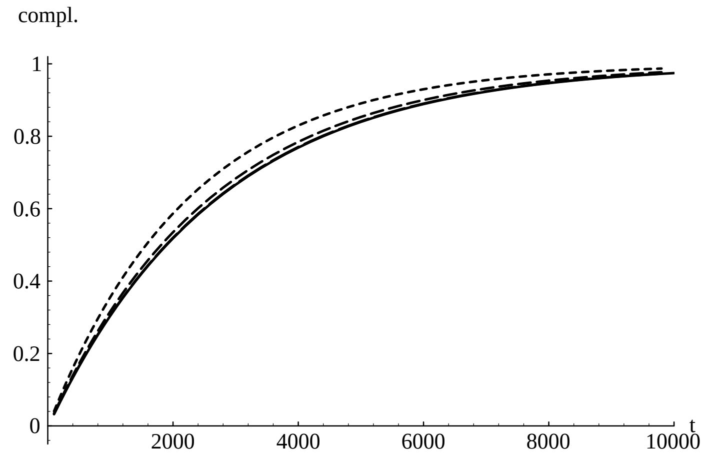

Figure 3 - Distribution of catastrophic failure time for the Dd system.

PH method provides always a correct Cdf function, but the steady state value depends on the order of the approximation.

Figure 2 shows the result of the UsDd case with the same parameter values. The *prs* accumulation of the useful work when the system stays in the up state increases the probability of completion and decreases the probability of the catastrophic failure before completion. At time $t = a$, the ditribution has the same step ($e^{-a\lambda}$), but after the discontinuity the system reaches the steady state very sharply. In Figure 2, the solid and the dashed lines represent the results of the ITSF and IT numerical method, respectively, of the same order as in Figure 1. The dotted lines are obtained by the PH method of order 10×10 (rear dotted curve), 100×100 (middle dens dotted curve), and order 1000×10 (dense dotted curve). Due to the sharp change of the function beyond $t = a$ the approximations of the same order as before provides less accurate results. The curve PH (1000×10) provides the closest approximation around $t = a$, but its steady state value is less accurate than the PH (100×100).

Figure 3 introduces a case (Dd) in which the catastrophic failure time distribution has no dicontinuities, and the behavior of the function is rather regular; hence, both the IT and the PH methods give quite accurate results.

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