Stochastic Petri Nets with Matrix Exponentially Distributed Firing Times

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Abstract

There are widely used analysis techniques for stochastic Petri nets with phase type (PH) distributed firing times. These techniques are based on the generation of an expanded Markov chain that describes the behaviour of the marking process and additionally the phase processes of the involved PH distributions.

In this paper we investigate the use of a similar approach when the firing time distribution does not allow a stochastic interpretation. We show that the same expansion based technique is applicable in case of matrix exponentially (ME) distributed firing times, but in this case the system does not allow any stochastic interpretation and the applied solution methods should not be restricted to non-negative numbers. Nevertheless, by defining an equivalence between PH and ME distributions it can be shown that marking probabilities remain the same when PH distributions are substituted by equivalent ME distributions.

Key words: Stochastic Petri Nets, Phase Type Distribution, Matrix Exponential Distribution, Numerical Analysis.

1 Introduction

Matrix exponential (ME) distributions [12] have some advantages in comparison to phase type (PH) distributions [15], but they are not as widely used in practice as PH distributions due to the fact that they do not allow a stochastic interpretation based on the theory of Markov processes as PH distributions do. One of the most important advantages of ME distributions is that an ME

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distribution of a given order can often approximate an empirical distribution much better than a PH of the same order. E.g., an ME distribution of order 5 can approximate an empirical distribution with coefficient of variation 0.082, while a PH distribution of order 12 is needed to capture such a low coefficient of variation [7].

To investigate the applicability of ME distributions in complex stochastic models we consider the analysis of stochastic Petri nets (SPNs) with ME distributed firing times. Following the general results in [12] it is very likely that in a stochastic model ME distributions may be used instead of PH distributions and several results will carry over. However, to the best of our knowledge, not much has been published about solution techniques for this more general model class. In particular, it is not easy to prove results in the general setting because probabilistic arguments do no longer hold. In [4] it has recently been shown that matrix geometric methods can be applied for quasi birth death processes (QBDs) with general arrival processes, an extension of ME distributions which allows one to incorporate correlations. To prove that the matrix geometric relations hold, the authors of [4] use an interpretation of ME distributions that has been proposed in [2]. However, the resulting proofs are non-trivial and the scope is limited to QBDs. In this paper we prove a similar result in the more general setting of SPNs with generally distributed firing times. Furthermore, we establish an equivalence relation between PH and ME distributions at an algebraic level which becomes the basis of our proofs.

The rest of the paper is organized as follows. Section 2 presents the basic definitions of ME and PH distributions and some important results about their representations. Section 3 defines the SPNs with ME distributed firing time and the associated initial vector and transition matrix. Section 4 presents the main result that SPNs with ME distributed firing time and with equivalent PH distributed firing time provide identical performance measures at the Petri net level. In Section 5 we introduce a small example to show possible applications of the approach. Finally, Section 6 concludes the paper.

2   Matrix Exponential and Phase Type Distributions

We first define ME distributions and fix some notations. Afterwards, an equivalence relation between different distributions is introduced.

Definition 1 *Let X be a random variable with cumulative distribution function (cdf)*

$$F_X(x) = Pr(X < x) = 1 - \alpha e^{Ax} \mathbb{1},$$

where α is an initial row vector of size n, $\mathbf{A}$ is a square matrix of size $n \times n$ and $\mathbb{1}$ is the column vector of ones of size n. In this case, we say that X is matrix exponentially distributed with representation $\alpha, \mathbf{A}$, or shortly, $ME(\alpha, \mathbf{A})$ distributed.

Definition 2 *If X is an $ME(\alpha, \mathbf{A})$ distributed random variable, where α and $\mathbf{A}$ have the following properties:*

- *$\alpha_i \geq 0$, $\alpha \mathbb{1} = 1$ (there is no probability mass at $t = 0$),*
- *$A_{ii} < 0$, $A_{ij} \geq 0$ for $i \neq j$, $\mathbf{A}\mathbb{1} \leq 0$,*
- *$\mathbf{A}$ is non-singular,*

then we say that X is phase type distributed with representation $\alpha, \mathbf{A}$, or shortly, $PH(\alpha, \mathbf{A})$ distributed.

The probability density function (pdf), the Laplace transform and the moments of X are

$$f_X(x) = \alpha e^{\mathbf{A}x}(-\mathbf{A})\mathbb{1}, \tag{1}$$

$$f_X^*(s) = E(e^{-sX}) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}(-\mathbf{A})\mathbb{1}, \tag{2}$$

$$\mu_n = E(X^n) = n!\alpha(-\mathbf{A})^{-n}\mathbb{1}. \tag{3}$$

Definition 3 *$ME(\alpha, \mathbf{A})$ ($PH(\alpha, \mathbf{A})$) is* non-redundant *if its cardinality equals to its algebraic order, where the* cardinality *of $ME(\alpha, \mathbf{A})$ ($PH(\alpha, \mathbf{A})$) is the size of vector α and square matrix $\mathbf{A}$ and the* algebraic order *of $ME(\alpha, \mathbf{A})$ ($PH(\alpha, \mathbf{A})$) is the degree of the denominator of $f_X^*(s)$ (which is a rational function of s).*

The initial vector and generator matrix representation of $F_X(x)$ is not unique. There are different initial vector and generator matrix pairs resulting in the same distribution. In the following we present results about the relations of different representations with identical size and different sizes.

Theorem 1 *[16] Let $ME(\alpha, \mathbf{A})$ and $ME(\gamma, \mathbf{G})$ be two non-redundant ME distributions with cdf $F_X(x)$ and $F_Y(x)$, respectively. $F_X(x) \equiv F_Y(x)$ if and only*

if there exists a non-singular matrix $\mathbf{B}$ *such that* $\gamma = \alpha\mathbf{B}$, $\mathbf{G} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ *and* $\mathbf{B}\mathbb{1} = \mathbb{1}$.

We denote two distributions with cdf $F_X(x)$ and $F_Y(x)$ as equivalent if $F_X(x) \equiv F_Y(x)$. The theorem implies that one of the two distributions can be a PH distribution whereas the other representation may not observe the conditions of Definition 2. Since $\mathbf{B}$ is a square matrix, both distributions have the same number of states. We now consider the case with a different number of states. To emphasize the size of different representations we indicate the length of the column vector of ones in the sequel, e.g., $\mathbb{1}_n$ denotes the vector of size n. Let $\mathbf{I}_{n,m}$ be the matrix of cardinality $n \times m$, whose element i, j equals to 1 when $i = j$ and to 0 otherwise.

In CTMC analysis lumpability is often used to define aggregated models with an identical behavior but a smaller state space [11,5]. Here we briefly introduce lumpability of ME distributions which is afterwards generalized to define an equivalence between PH and ME distributions. Let $(\gamma, \mathbf{G})$ be a PH distribution of cardinality m and let $\mathcal{P}$ be a partition of the state space $\{1, \ldots, m\}$ with $n \; (\leq m)$ partition groups. The partition can be defined by a $m \times n$ matrix $\mathbf{V}$ with elements from $\{0, 1\}$ such that $\mathbf{V}(i, j) = 1$ if state i belongs to the jth partition group and 0 otherwise. Since a state belongs to exactly one partition group and to each partition group belongs at least one state, each row of $\mathbf{V}$ contains exactly one element equal to 1 and each column contains at least one element equal to 1. Define a $n \times m$ matrix $\mathbf{W} = \left(diag(\mathbf{V}^T\mathbb{1}_n)\right)^{-1}(\mathbf{V}^T)$ where $diag(\mathbf{a})$ is a diagonal matrix with $\mathbf{a}(i)$ in position i. Observe that $\mathbf{W}$ results from the transposed matrix $\mathbf{V}$ by normalizing each row sum to 1. According to [11] matrix $\mathbf{V}$ describes a lumpable partition if and only if

$$\mathbf{G}\mathbf{V} = \mathbf{V}\mathbf{W}\mathbf{G}\mathbf{V} = \mathbf{V}\mathbf{A} \tag{4}$$

where $\mathbf{A} = \mathbf{W}\mathbf{G}\mathbf{V}$ and $\alpha = \gamma\mathbf{V}$ such that $(\alpha, \mathbf{A})$ is an equivalent PH distribution with n states. The equivalence of both distributions follows from Theorem 2 below and $(\alpha, \mathbf{A})$ is a PH distribution which follows from [11]. The following theorem generalizes (4) by allowing $\mathbf{V}$ to have a more general structure. It shows that we still obtain an equivalent distribution. However, the resulting distribution is usually not a PH distribution.

Theorem 2 *Let ME($\alpha, \mathbf{A}$) of cardinality n and ME($\gamma, \mathbf{G}$) of cardinality m, be two ME distributions with cdf $F_X(x)$ and $F_Y(x)$, respectively. If there exists a matrix $\mathbf{V}$ of cardinality $m \times n$, such that $\alpha = \gamma\mathbf{V}$, $\mathbf{V}\mathbf{A} = \mathbf{G}\mathbf{V}$, $\mathbf{V}\mathbb{1}_n = \mathbb{1}_m$ then $F_X(x) \equiv F_Y(x)$.*

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Proof. If $\alpha = \gamma \mathbf{V}$, $\mathbf{VA} = \mathbf{GV}$, $\mathbf{V}\mathbb{1}_n = \mathbb{1}_m$ then

$$
\begin{aligned}
F_X(x) &= 1 - \alpha e^{\mathbf{A}x}\mathbb{1}_n = 1 - \alpha \sum_{i=0}^{\infty} \mathbf{A}^i \frac{x^i}{i!}\mathbb{1}_n = 1 - \gamma \mathbf{V}\sum_{i=0}^{\infty}\mathbf{A}^i\frac{x^i}{i!}\mathbb{1}_n = \\
&= 1 - \gamma\sum_{i=0}^{\infty}\mathbf{G}^i\frac{x^i}{i!}\mathbf{V}\mathbb{1}_n = 1 - \gamma\sum_{i=0}^{\infty}\mathbf{G}^i\frac{x^i}{i!}\mathbb{1}_m = 1 - \gamma e^{\mathbf{G}x}\mathbb{1}_m = F_Y(x)
\end{aligned}
\tag{5}
$$

∎

Theorem 2 defines a generalization of lumpability. A second and alternative condition for equivalence is given in the following theorem.

Theorem 3 *Let ME(α, $\mathbf{A}$) of cardinality n and ME(γ, $\mathbf{G}$) of cardinality m, be two ME distributions with cdf $F_X(x)$ and $F_Y(x)$, respectively. If there exists a matrix $\mathbf{W}$ of cardinality $n \times m$, such that $\alpha\mathbf{W} = \gamma$, $\mathbf{AW} = \mathbf{WG}$, $\mathbf{W}\mathbb{1}_m = \mathbb{1}_n$ then $F_X(x) \equiv F_Y(x)$.*

Proof. If $\alpha\mathbf{W} = \gamma$, $\mathbf{AW} = \mathbf{WG}$, $\mathbf{W}\mathbb{1}_m = \mathbb{1}_n$ then

$$
\begin{aligned}
F_X(x) &= 1 - \alpha e^{\mathbf{A}x}\mathbb{1}_n = 1 - \alpha\sum_{i=0}^{\infty}\mathbf{A}^i\frac{x^i}{i!}\mathbb{1}_n = 1 - \alpha\sum_{i=0}^{\infty}\mathbf{A}^i\frac{x^i}{i!}\mathbf{W}\mathbb{1}_m = \\
&= 1 - \alpha\mathbf{W}\sum_{i=0}^{\infty}\mathbf{G}^i\frac{x^i}{i!}\mathbb{1}_m = 1 - \gamma\sum_{i=0}^{\infty}\mathbf{G}^i\frac{x^i}{i!}\mathbb{1}_m = 1 - \gamma e^{\mathbf{G}x}\mathbb{1}_m = F_Y(x)
\end{aligned}
\tag{6}
$$

∎

Theorem 3 is an extension of weak lumpability [11] for PH distributions where $\mathbf{W}$ is allowed to contain only non-negative entries and PH distributions are only related to PH distributions.

The following theorem introduces a necessary and sufficient condition for two ME distributions to be equivalent. To simplify the discussion we assume that the redundant Jordan blocks are eliminated from the considered representation, otherwise the transformation matrix converting one representation into the other would be far more complex.

Theorem 4 *Let ME(α, $\mathbf{A}$) of cardinality n and ME(γ, $\mathbf{G}$) of cardinality m ($m > n$), be two ME distributions with cdf $F_X(x)$ and $F_Y(x)$, respectively, such that ME(α, $\mathbf{A}$) is non-redundant and the Jordan decomposition of $\mathbf{G}$ does not contain multiple Jordan blocks or sub-blocks. $F_X(x) \equiv F_Y(x)$ if and only if there exist a pair of matrices $\mathbf{V}$ and $\mathbf{W}$ of cardinality $m \times r$ and $n \times r$ ($n \le r \le m$), respectively, and an ME(η, $\mathbf{H}$) representation of size r such that $\eta = \gamma\mathbf{V}$, $\mathbf{VH} = \mathbf{GV}$, $\mathbf{V}\mathbb{1}_r = \mathbb{1}_m$ and $\alpha\mathbf{W} = \eta$, $\mathbf{AW} = \mathbf{WH}$, $\mathbf{W}\mathbb{1}_r = \mathbb{1}_n$.*

Proof. According to Theorem 2 and 3 if $\eta = \gamma\mathbf{V}$, $\mathbf{VH} = \mathbf{GV}$, $\mathbf{V}\mathbb{1}_r = \mathbb{1}_m$ and $\alpha\mathbf{W} = \eta$, $\mathbf{AW} = \mathbf{WH}$, $\mathbf{W}\mathbb{1}_r = \mathbb{1}_n$ then $F_X(x) \equiv F_Y(x)$.

To show the opposite direction we construct such matrices $\mathbf{V}$ and $\mathbf{W}$.

From $F_X(x) \equiv F_Y(x)$ for $i \geq 1$ we have

$$(-1)^i \frac{d^i}{dx^i} F_X(x)|_{x=0} = \alpha\mathbf{A}^i\mathbb{1}_n = (-1)^i \frac{d^i}{dx^i} F_Y(x)|_{x=0} = \gamma\mathbf{G}^i\mathbb{1}_m, \qquad (7)$$

which means that the non-vanishing eigenvalues of $\mathbf{G}$ equal the eigenvalues of $\mathbf{A}$.

Let $\mathbf{A} = \mathbf{\Delta}^{-1}\mathbf{\Lambda}_A\mathbf{\Delta}$ and $\mathbf{G} = \mathbf{\Gamma}^{-1}\mathbf{\Lambda}_G\mathbf{\Gamma}$ be the Jordan decomposition of $\mathbf{A}$ and $\mathbf{G}$, where the Jordan blocks (sub-blocks) are ordered such that the Jordan blocks (sub-blocks) of $\mathbf{G}$ which are common with the one of $\mathbf{A}$ are listed in the same order as in $\mathbf{\Lambda}_A$, i.e.,

$$\mathbf{\Lambda}_A = \mathbf{I}_{n,m}\mathbf{\Lambda}_G\mathbf{I}_{m,n}. \qquad (8)$$

Furthermore, let r be the number of non-zero elements of $\mathbf{\Gamma}\mathbb{1}_m$ and without loss of generality we assume that the first r elements of $\mathbf{\Gamma}\mathbb{1}_m$ are non-zero. $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ are normalized such that $\mathbf{I}_{r,m}\mathbf{\Gamma}\mathbb{1}_m = \mathbb{1}_r$ and $\mathbf{\Delta}\mathbb{1}_n = \mathbb{1}_n$. Due to this normalization $\mathbf{\Gamma}\mathbb{1}_m = \mathbf{I}_{m,r}\mathbb{1}_r$, because the first r elements $\mathbf{\Gamma}\mathbb{1}_m$ equal to one and the last $m - r$ elements equal to zero. Note that, $\mathbf{\Delta}\mathbb{1}_n = \mathbb{1}_n$ and $\mathbf{\Gamma}\mathbb{1}_m = \mathbf{I}_{m,r}\mathbb{1}_r$ imply $\mathbf{\Delta}^{-1}\mathbb{1}_n = \mathbb{1}_n$ and $\mathbb{1}_m = \mathbf{\Gamma}^{-1}\mathbf{I}_{m,r}\mathbb{1}_r$.

If $F_X(x) \equiv F_Y(x)$ then the algebraic order of $\text{ME}(\gamma, \mathbf{G})$ and the algebraic order of $\text{ME}(\alpha, \mathbf{A})$ equal to n, since $\text{ME}(\alpha, \mathbf{A})$ is non-redundant.

From the fact that the contributing Jordan blocks (sub-blocks) of $\mathbf{\Lambda}_G$ are listed first it follows that $\gamma\mathbf{\Gamma}^{-1}$ and $\mathbf{\Gamma}\mathbb{1}_m$ are such that the first n elements of vector $\gamma\mathbf{\Gamma}^{-1}\circ(\mathbf{\Gamma}\mathbb{1}_m)^T$ (where $\circ$ denotes the element-wise vector multiplication) are non-zero and the last $m - n$ elements equal to 0. This structure ensures that the order of $\text{ME}(\gamma, \mathbf{G})$ is n and the remaining Jordan blocks (sub-blocks) vanish. Based on this property for $i \geq 1$ we have

$$\begin{aligned}
\gamma\mathbf{G}^i\mathbb{1}_m &= \gamma\mathbf{\Gamma}^{-1}\mathbf{\Lambda}_G^i\mathbf{\Gamma}\mathbb{1}_m \\
&= \gamma\mathbf{\Gamma}^{-1}\mathbf{I}_{m,n}\mathbf{I}_{n,m}\mathbf{\Lambda}_G^i\mathbf{I}_{m,n}\mathbf{I}_{n,m}\mathbf{\Gamma}\mathbb{1}_m \\
&= \gamma\mathbf{\Gamma}^{-1}\mathbf{I}_{m,n}\left(\mathbf{I}_{n,m}\mathbf{\Lambda}_G\mathbf{I}_{m,n}\right)^i\mathbf{I}_{n,m}\mathbf{\Gamma}\mathbb{1}_m \\
&= \gamma\mathbf{\Gamma}^{-1}\mathbf{I}_{m,n}\mathbf{\Lambda}_A^i\mathbb{1}_n,
\end{aligned} \qquad (9)$$

where we first utilized that the last $m - n$ eigenvalues of $\mathbf{G}$ vanish then the block diagonal structure of $\mathbf{\Lambda}_G$ and finally (8) and the normalization of $\mathbf{\Gamma}$.

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Substituting (9) and the Jordan decomposition of $\mathbf{A}$ into (7) gives

$$\gamma\boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,n}\boldsymbol{\Lambda}_A^i\mathbb{1}_n = \alpha\boldsymbol{\Delta}^{-1}\boldsymbol{\Lambda}_A^i\mathbb{1}_n, \tag{10}$$

from which

$$\gamma\boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,n} = \alpha\boldsymbol{\Delta}^{-1}, \tag{11}$$

because the rank of $\boldsymbol{\Lambda}_A$ is n.

Let $\eta = \gamma\boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,r}$, $\mathbf{H} = \mathbf{I}_{r,m}\boldsymbol{\Lambda}_G\mathbf{I}_{m,r}$ and $\mathbf{V} = \boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,r}$, then $\eta = \gamma\mathbf{V}, \mathbf{VH} = \mathbf{GV}, \mathbf{V}\mathbb{1}_r = \mathbb{1}_m$ follows. The first equality comes by definition, the last equality is a consequence of the normalizing condition and

$$\underbrace{\boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,r}}_{\mathbf{V}}\underbrace{\mathbf{I}_{r,m}\boldsymbol{\Lambda}_G\mathbf{I}_{m,r}}_{\mathbf{H}} = \underbrace{\boldsymbol{\Gamma}^{-1}\boldsymbol{\Lambda}_G\boldsymbol{\Gamma}}_{\mathbf{G}}\underbrace{\boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,r}}_{\mathbf{V}}$$

equals due to the diagonal block structure of $\boldsymbol{\Lambda}_G$.

Now, let $\mathbf{W} = \boldsymbol{\Delta}^{-1}\mathbf{I}_{n,r}$, then $\alpha\mathbf{W} = \eta$, $\mathbf{AW} = \mathbf{WH}$, $\mathbf{W}\mathbb{1}_r = \mathbb{1}_n$ follows. The second equality comes similarly due to the diagonal block structure of $\boldsymbol{\Lambda}_G$, the last equality is a consequence of the normalizing condition and

$$\alpha\underbrace{\boldsymbol{\Delta}^{-1}\mathbf{I}_{n,r}}_{\mathbf{W}} = \underbrace{\gamma\boldsymbol{\Gamma}^{-1}\mathbf{I}_{m,r}}_{\eta}$$

comes from (11) and the fact that the last $r - n$ elements of η equal to zero because the algebraic order of $\mathrm{ME}(\eta, \mathbf{H})$ is n. ∎

The result of Theorem 4 can be summarized as follows. If there is a non-redundant and a redundant representation of an ME (or PH) distribution, then a transformation with matrix $\mathbf{V}$ according to Theorem 2 (to eliminate the redundancy due to the closing vector) and a transformation with matrix $\mathbf{W}$ according to Theorem 3 (to eliminate the redundancy due to the initial vector) can transform one representation into the other.

We are going to use these results together with the one of Asmussen and O'Cinneide [3] which states that any ME distribution with strictly positive density in $(0, \infty)$ has a PH representation of possible higher order. The numerical procedure presented in [13] and implemented in the MoMI tool of S. Mocanu generates such a PH representation.

We use these results according to the following steps:

- we replace the ME representations of our models with the equivalent PH representations (e.g., computed with the approach from [13]),

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- we generate the generator matrix of the expanded Markov chain which describes the model based on the equivalent PH representations,
- at the same time we generate an expanded matrix of the model in a similar way as the expanded Markov chain using the ME representations.
- with the use of the transformations with matrix $\mathbf{V}$ and $\mathbf{W}$ we show that the aggregate model parameters obtained from the two descriptions are identical.

If the pdf of a ME distribution touches the x axis it has no finite dimensional PH representation and the mentioned proof is not applicable. We believe that the expansion based analysis method remains valid also for this case, but our proof does not cover this case.

In the following section we perform these steps in the context of SPNs. However, before we do so, an example for equivalent ME distributions is shown.

2.3 An Example

We use as an example the PH distribution $(\alpha, \mathbf{A})$ with

$$
\alpha = \left(\begin{matrix} 0.2 \ 0.0 \ 0.0 \ 0.1 \ 0.7 \end{matrix} \right) \ , \quad \mathbf{A} = \begin{pmatrix} -3 & 0 & 0 & 1 & 1/2 \\ 10/3 & -8 & 4/3 & 2/3 & 2/3 \\ 0 & 15/2 & -11 & 1/2 & 3/4 \\ 1 & 0 & 0 & -4 & 2 \\ 1 & 0 & 0 & 1 & -5 \end{pmatrix}
$$

and the PH distribution $(\eta, \mathbf{H})$ with

$$
\eta = \left(\begin{matrix} 0.1 \ 0.1 \ 0.1 \ 0.7 \end{matrix} \right) \ , \quad \mathbf{H} = \begin{pmatrix} -4 & 2 & 2 & 0 \\ 1 & -5 & 0 & 1 \\ 1/2 & 1/2 & -4 & 2 \\ 1/2 & 1/2 & 1 & -5 \end{pmatrix} .
$$

Both PH distributions are equivalent since $\mathbf{AV} = \mathbf{VH}$ and $\alpha \mathbf{V} = \eta$ for

$$\mathbf{V} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Observe that the equivalence between the two representations does not result from lumpability since $\mathbf{V}$ contains elements that differ from 0 or 1.

Furthermore, both distributions are equivalent to $(\gamma, \mathbf{G})$ with

$$\gamma = \begin{pmatrix} 0 & 0 & 1.0 \end{pmatrix} , \quad \mathbf{G} = \begin{pmatrix} -4.39428798 & 1.83425863 & 0.230777986 \\ 0.216420106 & -3.89971738 & 3.29773452 \\ 0 & 1.20599464 & -3.70599464 \end{pmatrix} .$$

since $\mathbf{W}\gamma = \eta$ and $\mathbf{WH} = \mathbf{GW}$ with

$$\mathbf{W} = \begin{pmatrix} 0.605741202 & 0.605741202 & -0.573237485 & 0.361755080 \\ 0.390216878 & 0.390216878 & 0.721893310 & -0.502327066 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{pmatrix} .$$

Although $\mathbf{W}$ contains negative elements, $(\gamma, \mathbf{G})$ is still a PH distribution.

Now consider the PH distribution $(\alpha', \mathbf{A})$ with $\alpha' = (1, 0, 0, 0, 0)$. This PH distribution is equivalent to $(\eta', \mathbf{H})$ with $\eta' = (0.5, 0.5, 0, 0)$ which can be seen using the same matrix $\mathbf{V}$ as above. Both PH distributions are equivalent to the ME distribution $(\gamma, \mathbf{G}')$ with

$$\gamma = \begin{pmatrix} 0 & 0 & 1.0 \end{pmatrix} , \quad \mathbf{G}' = \begin{pmatrix} -6 & -0.618033989 & 3.55278640 \\ -1 & -5.61803399 & 5.34164079 \\ 0 & -1.11803399 & -3.81966011 \end{pmatrix} .$$

since $\mathbf{W}'\gamma = \eta'$ and $\mathbf{W}'\mathbf{H} = \mathbf{G}'\mathbf{W}'$ with

$$\mathbf{W}' = \begin{pmatrix} 0.276393202 & 0.276393202 & -0.447213595 & 0.894427191 \\ 1.17082039 & 1.17082039 & -0.894427191 & -0.447213595 \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix} .$$

3 Stochastic Petri Nets with ME Distributed Firing Times

In this section we introduce Stochastic Petri Nets where the transition firing times are ME distributed. We consider first Petri Nets and their reachability graph. Afterwards the reachability graph is expanded by considering detailed states including the phases of the different distributions.

3.1 Basic Notations and Definitions

We briefly present some basic definitions and results for Petri nets, further results and details can be found in the literature [14].

Definition 4 *A Petri Net is a five tuple $PN = (P, T, I, O, M_0)$ where*

- P *is a set of places,*
- T *is a set of transitions such that $P \cap T = \emptyset$,*
- $I : P \times T \to \mathbf{N}$ *is the input function,*
- $O : T \times P \to \mathbf{N}$ *is the output function, and*
- $M_0 : P \to \mathbf{N}$ *is the initial marking.*

We assume that the transitions of the net are ordered such that for $t, t' \in T$ with $t \neq t'$ either $t < t'$ or $t > t'$ holds. Denote by $\bullet t = \{p | p \in P \wedge I(p, t) > 0\}$ and $t\bullet = \{p | p \in P \wedge O(t, p) > 0\}$ the input and output bag of transition t, respectively. Marking M_0 is a specific marking, i.e., can be interpreted as a vector of length $|P|$ including the token population of each place. Transition t is enabled in marking M if and only if $M(p) \geq I(p, t)$ for all $p \in \bullet t$. If t is enabled in marking M and fires, then a new marking M' with $M'(p) = M(p) - I(p, t) + O(t, p)$ is generated. We use the notation $M \xrightarrow{t} M'$. To keep the later notation simpler we assume that $M \xrightarrow{t} M'$ implies $M \neq M'$. The set of markings available from M_0 with repeated application of relation $\xrightarrow{t}$ defines the reachability set $\mathcal{RS}$ of the net. The reachability graph $\mathcal{RG}$ is a directed and labeled graph with vertex set $\mathcal{RS}$ and an arc labeled with t between $M, M' \in \mathcal{RS}$ if and only if $M \xrightarrow{t} M'$. Further assumptions about $\mathcal{RS}$ and $\mathcal{RG}$, like finiteness or strong connectivity will be made later when necessary.

Let $Ena(M) = \{t | t \in T$ and for all $p \in P : \ M(p) \geq I(p, t)\}$ be the set of enabled transitions in marking M. The concept underlying our definition of newly enabled transitions is denoted as enabling memory in [1]. The approach is applicable for age memory policy as well. Only the structure of set $\mathcal{S}$ in (12) and the definition of $\mathbf{R}_{t',t}(M', M)$ in (16) changes in that case, but we do not present this case here to avoid further complication of notations and discussions. Furthermore, we assume single server semantics for all transitions.

We now assume that a distribution $\mathrm{ME}(\alpha_t, \mathbf{A}_t)$ is associated with transition t. Let $\mathcal{S}_t = \{1, \ldots, n_t\}$ be the state space of the ME distribution associated with transition t, where in case of ME distributions we use the term state in the same sense as for PH distributions.

If all distributions associated to transitions are PH distributions, then the underlying stochastic process is a Markov chain [1]. However, if we allow ME distributions that do not belong to the PH class, then the process has no probabilistic interpretation. Consequently, we give a purely algebraic description of the resulting state space and matrix. In the next section we present an interpretation in terms of the stationary and transient distribution. In the following we use the convention that empty sums and Kronecker sums are 0, empty product or Kronecker products are 1.

We denote by $\mathcal{S}$ the state space of the process which considers the marking process and the states or phases of all enabled distributions. Thus, $\mathcal{S}$ can be decomposed into subsets $\mathcal{S}(M)$ for each $M \in \mathcal{RS}$, such that

$$\mathcal{S} = \bigcup_{M \in \mathcal{RS}} \mathcal{S}(M) = \bigcup_{M \in \mathcal{RS}} M \underset{t \in Ena(M)}{\times} \mathcal{S}_t \ . \tag{12}$$

Let n be the number of states of the complete SPN and $n(M)$ be the number of states for marking M.

$$n = \sum_{M \in \mathcal{RS}} n(M) = \sum_{M \in \mathcal{RS}} \prod_{t \in Ena(M)} n_t \tag{13}$$

Let $\pi(0)$ be the initial distribution vector for SPN. $\pi(0)$ is a row vector of size n, composed by blocks of size $n(M)$, $\pi(0, M)$, representing the initial values associated with marking M. When M_0 is the initial marking $\pi(0, M) = \underline{0}$ for $\forall M \neq M_0$ and

$$\pi(0, M_0) = \bigotimes_{t \in Ena(M_0)} \alpha_t. \tag{14}$$

The transition matrix $\mathbb{A}$ of the SPN is block structured and submatrix $\mathbb{A}(M, M')$ describes all transition starting in marking M and ending in marking M'. We begin with the case $M = M'$. Since every transition of the SPN changes the marking we have

$$\mathbb{A}(M, M) = \bigoplus_{t \in Ena(M)} \mathbf{A}_t \ . \tag{15}$$

Non-diagonal block $\mathbb{A}(M', M)$ equals

$$\mathbb{A}(M', M) = \sum_{t: M' \overset{t}{\to} M} \mathbb{A}_t(M', M)$$

$$\text{where} \quad \mathbb{A}_t(M', M) = \bigotimes_{t' \in T} \mathbf{R}^A_{t', t}(M', M) \quad \text{and}$$

$$\mathbf{R}^A_{t', t}(M', M) = \begin{cases} \mathbf{I}_{n_{t'}, n_{t'}} & \text{if } t' \neq t \text{ and } t' \in Ena(M') \cap Ena(M) \\ \alpha_{t'} & \text{if } t' \neq t, \ t' \notin Ena(M') \text{ and } t' \in Ena(M) \\ \mathbb{1}_{n_{t'}} & \text{if } t' \neq t, \ t' \in Ena(M') \text{ and } t' \notin Ena(M) \\ -\mathbf{A}_t \, \mathbb{1}_{n_t} & \text{if } t' = t, \ t \in Ena(M') \text{ and } t \notin Ena(M) \\ -\mathbf{A}_t \, \mathbb{1}_{n_t} \alpha_t & \text{if } t' = t \text{ and } t' \in Ena(M') \cap Ena(M) \\ 1 & \text{otherwise} \end{cases}$$

$$(16)$$

The superscript in $\mathbf{R}^A_{t', t}(M', M)$ identifies the generator of the firing time distributions. The structure of the matrices and submatrices follows from the hierarchical Kronecker representation of SPNs and related models (see e.g., [6] for details). During the above described generation of the state space and the transition matrix we used simple algebraic steps without utilizing any special property of the contributing matrices. We apply the same procedure to generate the state space and the transition matrix for the ME representation.

We now assume that distribution $ME(\alpha_t, \mathbf{A}_t)$ of size n_t has a PH representation $PH(\gamma_t, \mathbf{G}_t)$ of size m_t, for all transitions. Let m be the number of states of the obtained SPN, and $m(M)$ be the number of states for marking M.

$$m = \sum_{M \in \mathcal{RS}} m(M) = \sum_{M \in \mathcal{RS}} \prod_{t \in Ena(M)} m_t. \tag{17}$$

The composition of the associated initial vector $\nu(0)$ and transition matrix $\mathbb{G}$ follows the same pattern.

In this case, when all distributions are PH distributions, $\nu(0)$ is a probability vector with only non-negative entries and $\mathbb{G}$ is a generator matrix of a Markov chain which has non-negative elements everywhere except in the diagonal, which is composed by strictly negative entries. Instead if ME distributions are included, then $\pi(0)$ is a general vector which may contain negative entries and $\mathbb{A}$ is a general transition matrix which may contain positive and negative entries at every position.

4 Analysis of the SPN based on $\mathbb{A}$ and $\mathbb{G}$

We consider two SPNs, namely $SPN-ME$ and $SPN-PH$. In $SPN-ME$ all distributions are non-redundant ME distributions. $ME(\alpha_t, \mathbf{A}_t)$ of order n_t is associated with transition t. $SPN-PH$ results from $SPN-ME$ by associating PH representation $PH(\gamma_t, \mathbf{G}_t)$ of order m_t ($\geq n_t$) with transition t such that $F_{ME(\alpha_t, \mathbf{A}_t)}(x) \equiv F_{PH(\gamma_t, \mathbf{G}_t)}(x)$.

To simplify the discussion, instead of the two step procedure presented in Theorem 4, we discuss the result of the transformations of Theorem 2 and 3 separately. I.e., we first investigate the behaviour of $SPN-ME$ and $SPN-PH$ assuming that $\gamma_t \mathbf{V}_t = \alpha_t$, $\mathbf{G}_t \mathbf{V}_t = \mathbf{V}_t \mathbf{A}_t$ and $\mathbf{V}_t \mathbb{1}_{n_t} = \mathbb{1}_{m_t}$, then we analyze them assuming that $\gamma_t = \alpha_t \mathbf{W}_t$, $\mathbf{W}_t \mathbf{G}_t = \mathbf{A}_t \mathbf{W}_t$ and $\mathbb{1}_{n_t} = \mathbf{W}_t \mathbb{1}_{m_t}$. This is not a restriction since we may transform any $SPN-ME$ to $SPN-PH$ in two steps by first building $SPN-ME_1$ from $SPN-PH$ using Theorem 3 and then $SPN-ME$ from $SPN-ME_1$ using Theorem 2 as it is in Theorem 4.

Let $\nu(0)$ and $\mathbb{G}$ be the initial vector and the transition matrix of $SPN-PH$ which are an initial probability vector and a generator matrix of a Markov chain and let $\pi(0)$ and $\mathbb{A}$ be the initial vector and transition matrix for $SPN-ME$. In the following subsections we show how transient and stationary vectors are computed from $\nu(0)$, $\mathbb{G}$ and $\pi(0)$, $\mathbb{A}$. Then we show how the vectors for both nets are related and that both vectors include the same aggregated probabilities for the markings of the SPN and also result in the same results at the Petri net level (i.e., same token distributions at the places and same transition throughputs).

These results imply that we can use analytical techniques also for SPNs with ME distributed firing times. As long as the distribution function remains identical, PH and ME distributions give the same results.

4.1 Transient Distribution

Let $\nu(\tau)$ and $\pi(\tau)$ be the vector representing the state of $SPN-PH$ and $SPN-ME$ at time τ, respectively, which are given by

$$\nu(\tau) = \nu(0)e^{\mathbb{G}\tau} = \nu(0)\sum_{k=0}^{\infty}\frac{(\mathbb{G}\tau)^k}{k!} \quad \text{and} \quad \pi(\tau) = \pi(0)e^{\mathbb{A}\tau} = \pi(0)\sum_{k=0}^{\infty}\frac{(\mathbb{A}\tau)^k}{k!} \, . \tag{18}$$

Define

$$\mathbf{g}^{(k)} = \nu(0)\mathbb{G}^k \quad \text{and} \quad \mathbf{a}^{(k)} = \pi(0)\mathbb{A}^k$$

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then
$$\nu(\tau) = \sum_{k=0}^{\infty} \mathbf{g}^{(k)} \frac{\tau^k}{k!} \quad \text{and} \quad \pi(\tau) = \sum_{k=0}^{\infty} \mathbf{a}^{(k)} \frac{\tau^k}{k!} \ . \tag{19}$$

The different vectors can be decomposed into subvectors according to the state space partition defined by the marking of the net. E.g., $\nu(\tau, M)$ is the vector of state probabilities at time τ for all states of $SPN{-}PH$ that belong to marking M. Let $Prob(\tau, M)$ be the probability of marking M at time τ which is given by

$$Prob(\tau, M) = \nu(\tau, M) \, \mathbb{1}_{m(M)} \ . \tag{20}$$

Similarly the throughput of transition t at time τ is given by

$$Tput(\tau, t) = \sum_{M:t\in Ena(M)} \nu(\tau, M) \left(\mathbb{1}_{m_{<t}} \bigotimes -\mathbf{G}_t \, \mathbb{1}_{m_t} \bigotimes \mathbb{1}_{m_{>t}} \right) \ , \tag{21}$$

where $m_{<t} = \prod_{t'\in Ena(M), t'<t} m_t$ and $m_{>t} = \prod_{t'\in Ena(M), t'>t} m_t$. The results for $SPN{-}ME$ are computed similarly after substituting $\nu(\tau, M)$ by $\pi(\tau, M)$, m_t by n_t and $(\gamma_t, \mathbf{G}_t)$ by $(\alpha_t, \mathbf{A}_t)$.

The following theorem shows the relation between $\nu(\tau, M)$ and $\pi(\tau, M)$ when Theorem 2 describes the relation between all distributions in $SPN{-}ME$ and $SPN{-}PH$.

Theorem 5 *If $\nu(0), \mathbb{G}$ and $\pi(0), \mathbb{A}$ are generated from the same SPN using* $\mathrm{ME}(\alpha_t, \mathbf{A}_t)$ *and* $\mathrm{PH}(\gamma_t, \mathbf{G}_t)$ *such that $\gamma_t \mathbf{V}_t = \alpha_t$, $\mathbf{G}_t \mathbf{V}_t = \mathbf{V}_t \mathbf{A}_t$ and $\mathbf{V}_t \, \mathbb{1}_{n_t} = \mathbb{1}_{m_t}$, then*

$$\forall M \in \mathcal{RS}, \forall \tau \geq 0 : \ \pi(\tau, M) = \nu(\tau, M) \left(\bigotimes_{t\in Ena(M)} \mathbf{V}_t \right) \ .$$

Furthermore, $Prob(\tau, M)$ and $Tput(\tau, t)$ remain the same in both nets.

Proof. Starting from (14) we have

$$\pi(0, M_0) = \bigotimes_{t\in Ena(M_0)} \alpha_t = \bigotimes_{t\in Ena(M_0)} \gamma_t \mathbf{V}_t =$$
$$\left(\bigotimes_{t\in Ena(M_0)} \gamma_t \right) \left(\bigotimes_{t\in Ena(M_0)} \mathbf{V}_t \right) = \nu(0, M_0) \left(\bigotimes_{t\in Ena(M_0)} \mathbf{V}_t \right) \ . \tag{22}$$

Together with $\pi(0, M) = \underline{0}$ and $\nu(0, M) = \underline{0}$ for $M \neq M_0$ this implies that the theorem holds for $\tau = 0$. Due to this and (19) it is sufficient to show for all $M \in \mathcal{RS}$ that

$$\mathbf{a}^{(k)}(M) = \mathbf{g}^{(k)}(M) \left(\bigotimes_{t\in Ena(M)} \mathbf{V}_t \right) \ \Rightarrow \ \mathbf{a}^{(k+1)}(M) = \mathbf{g}^{(k+1)}(M) \left(\bigotimes_{t\in Ena(M)} \mathbf{V}_t \right) \tag{23}$$

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where

$$\mathbf{g}^{(k+1)}(M) = \mathbf{g}^{(k)}(M)\mathbb{G}(M, M) + \sum_{M':M' \xrightarrow{t} M} \sum_{t \in Ena(M')} \mathbf{g}^{(k)}(M')\mathbb{G}_t(M', M)$$

and $\mathbf{a}^{(k+1)}(M)$ is computed in the same way using the matrices $\mathbb{A}(M, M)$ and $\mathbb{A}(M', M)$. We show that the required relation holds after all separate multiplications with the submatrices which implies that it also holds after summation. We begin with the multiplication with the diagonal blocks. Observe that

$$\mathbb{G}(M, M) = \bigoplus_{t \in Ena(M)} \mathbf{G}_t = \sum_{t \in Ena(M)} \mathbf{I}_{m_{<t}, m_{<t}} \bigotimes \mathbf{G}_t \bigotimes \mathbf{I}_{m_{>t}, m_{>t}}$$

such that

$$\mathbf{g}^{(k)}(M)\mathbb{G}(M, M)\left(\bigotimes_{t \in Ena(M)} \mathbf{V}_t\right) =$$

$$\mathbf{g}^{(k)}(M) \sum_{t \in Ena(M)} \left(\left(\bigotimes_{t' \in Ena(M), t' < t} \mathbf{I}_{m_t, m_t}\right) \otimes \mathbf{G}_t \otimes \left(\bigotimes_{t' \in Ena(M), t' > t} \mathbf{I}_{m_t, m_t}\right)\right)\left(\bigotimes_{t \in Ena(M)} \mathbf{V}_t\right) =$$

$$\mathbf{g}^{(k)}(M) \sum_{t \in Ena(M)} \left(\left(\bigotimes_{t' \in Ena(M), t' < t} \mathbf{V}_t\right) \otimes \mathbf{G}_t \mathbf{V}_t \otimes \left(\bigotimes_{t' \in Ena(M), t' > t} \mathbf{V}_t\right)\right) =$$

$$\mathbf{g}^{(k)}(M) \sum_{t \in Ena(M)} \left(\left(\bigotimes_{t' \in Ena(M), t' < t} \mathbf{V}_t\right) \otimes \mathbf{V}_t \mathbf{A}_t \otimes \left(\bigotimes_{t' \in Ena(M), t' > t} \mathbf{V}_t\right)\right) =$$

$$\mathbf{g}^{(k)}(M) \left(\bigotimes_{t \in Ena(M)} \mathbf{V}_t\right) \sum_{t \in Ena(M)} \left(\left(\bigotimes_{t' \in Ena(M), t' < t} \mathbf{I}_{n_t, n_t}\right) \otimes \mathbf{A}_t \otimes \left(\bigotimes_{t' \in Ena(M), t' > t} \mathbf{I}_{n_t, n_t}\right)\right) =$$

$$\mathbf{a}^{(k)}(M)\mathbb{A}(M, M) .$$

For the non-diagonal blocks we have to show for $M' \xrightarrow{t} M$ that

$$\left(\bigotimes_{t' \in Ena(M')} \mathbf{V}_{t'}\right)\left(\bigotimes_{t' \in T} \mathbf{R}_{t',t}^A(M', M)\right) = \left(\bigotimes_{t' \in T} \mathbf{R}_{t',t}^G(M', M)\right)\left(\bigotimes_{t' \in Ena(M)} \mathbf{V}_{t'}\right) \tag{24}$$

Matrix $\mathbf{R}_{t',t}$ is defined in (16). We consider the different possibilities:

- if $t' \neq t$ and $t' \in Ena(M') \cap Ena(M)$ then

$$\mathbf{V}_{t'} \mathbf{R}_{t',t}^A(M', M) = \mathbf{V}_{t'}\mathbf{I}_{n_{t'}, n_{t'}} = \mathbf{I}_{m_{t'}, m_{t'}}\mathbf{V}_{t'} = \mathbf{R}_{t',t}^G(M', M)\mathbf{V}_{t'},$$

  because $\mathbf{I}_{m_{t'}, m_{t'}}\mathbf{V}_{t'} = \mathbf{V}_{t'}$ and $\mathbf{V}_{t'}\mathbf{I}_{n_{t'}, n_{t'}} = \mathbf{V}_{t'}$,
- if $t' \neq t$, $t' \notin Ena(M')$ and $t' \in Ena(M)$ then

$$\mathbf{R}_{t',t}^A(M', M) = \mathbf{R}_{t',t}^G(M', M)\mathbf{V}_{t'}$$

  because $\gamma_{t'}\mathbf{V}_{t'} = \alpha_{t'}$,

- if $t' \neq t$, $t' \in Ena(M')$ and $t' \notin Ena(M)$ then

$$\mathbf{V}_{t'} \ \mathbf{R}^A_{t',t}(M', M) = \mathbf{R}^G_{t',t}(M', M)$$

because $\mathbf{V}_{t'} \ \mathbb{1}_{n_t} = \mathbb{1}_{m_t}$,
- if $t' = t$, $t \in Ena(M')$ and $t \notin Ena(M)$ then

$$\mathbf{V}_t \ \mathbf{R}^A_{t,t}(M', M) = -\mathbf{V}_t \mathbf{A}_t \ \mathbb{1}_{n_t} = -\mathbf{G}_t \mathbf{V}_t \ \mathbb{1}_{n_t} = -\mathbf{G}_t \ \mathbb{1}_{m_t} = \mathbf{R}^G_{t,t}(M', M),$$

- if $t' = t$ and $t' \in Ena(M') \cap Ena(M)$ then

$$\mathbf{V}_t \ \mathbf{R}^A_{t,t}(M', M) = -\mathbf{V}_t \mathbf{A}_t \ \mathbb{1}_n \alpha_t \ = -\mathbf{G}_t \mathbf{V}_t \ \mathbb{1}_n \alpha_t$$
$$= -\mathbf{G}_t \ \mathbb{1}_{m_t} \alpha_t = -\mathbf{G}_t \ \mathbb{1}_{m_t} \gamma_t \mathbf{V}_t = \mathbf{R}^G_{t,t}(M', M) \mathbf{V}_t,$$

This completes the proof of the inductive relation between the solution vectors in (23). The relation between the vectors implies that the probability for every marking is the same since

$$Prob_{ME}(\tau, M) = \nu(\tau, M) \left(\bigotimes_{t \in Ena(M)} \mathbb{1}_{m_t} \right)$$
$$= \nu(\tau, M) \left(\bigotimes_{t \in Ena(M)} \mathbf{V}_t \ \mathbb{1}_{n_t} \right) = \nu(\tau, M) \left(\bigotimes_{t \in Ena(M)} \mathbf{V}_t \right) \left(\bigotimes_{t \in Ena(M)} \mathbb{1}_{n_t} \right)$$
$$= \pi(\tau, M) \left(\bigotimes_{t \in Ena(M)} \mathbb{1}_{n_t} \right) = Prob_{PH}(\tau, M)$$

For the throughput of transition t we obtain

$$Tput_{PH}(\tau, t) =$$
$$\sum_{M: t \in Ena(M)} \nu(\tau, M) \left(\left(\bigotimes_{t' \in Ena(M): t' < t} \mathbb{1}_{m_{t'}} \right) \otimes \mathbf{G}_t \ \mathbb{1}_{m_t} \otimes \left(\bigotimes_{t' \in Ena(M): t' > t} \mathbb{1}_{m_{t'}} \right) \right) =$$
$$\sum_{M: t \in Ena(M)} \nu(\tau, M) \left(\left(\bigotimes_{t' \in Ena(M): t' < t} \mathbf{V}_{t'} \ \mathbb{1}_{n_{t'}} \right) \otimes \mathbf{G}_t \mathbf{V}_t \ \mathbb{1}_{n_t} \otimes \left(\bigotimes_{t' \in Ena(M): t' > t} \mathbf{V}_{t'} \ \mathbb{1}_{n_{t'}} \right) \right) =$$
$$\sum_{M: t \in Ena(M)} \nu(\tau, M) \left(\bigotimes_{t' \in Ena(M)} \mathbf{V}_{t'} \right) \left(\bigotimes_{t' \in Ena(M): t' < t} \mathbb{1}_{n_{t'}} \right) \otimes \mathbf{A}_t \ \mathbb{1}_{n_t} \otimes \left(\bigotimes_{t' \in Ena(M): t' > t} \mathbb{1}_{n_{t'}} \right) =$$
$$\sum_{M: t \in Ena(M)} \pi(\tau, M) \left(\bigotimes_{t' \in Ena(M): t' < t} \mathbb{1}_{n_{t'}} \right) \otimes \mathbf{A}_t \ \mathbb{1}_{n_t} \otimes \left(\bigotimes_{t' \in Ena(M): t' > t} \mathbb{1}_{n_{t'}} \right) =$$
$$Tput_{ME}(\tau, t) \ .$$

$$\tag{25}$$

$\blacksquare$

Theorem 6 *If $\nu(0), \mathbb{G}$ and $\pi(0), \mathbb{A}$ are generated from the same SPN using* $\mathrm{ME}(\alpha_t, \mathbf{A}_t)$ *and* $\mathrm{PH}(\gamma_t, \mathbf{G}_t)$ *such that* $\gamma_t = \alpha_t \mathbf{W}_t$, $\mathbf{W}_t \mathbf{G}_t = \mathbf{A}_t \mathbf{W}_t$ *and* $\mathbb{1}_{n_t} =$

$\mathbf{W}_t \, \mathbb{1}_{m_t}$ *then*

$$\forall M \in \mathcal{RS}, \forall \tau \geq 0 : \ \nu(\tau, M) = \pi(\tau, M) \left(\bigotimes_{t \in Ena(M)} \mathbf{W}_t \right) .$$

Furthermore $Prob(\tau, M)$ and $Tput(\tau, t)$ remain the same in both nets.

Proof. The proof follows the same pattern as the one of Theorem 5. ∎

The previous two theorems show that ME distributions and matrix $\mathbb{A}$ of the resulting net can be used in (19) to compute the transient vector. The resulting vector may not be a probability vector, it may contain negative elements but its elements sum up to 1 and marking probabilities as well as transition throughput are computed exactly from the vector. Since the general model has no probabilistic interpretation, uniformization with its nice probabilistic interpretation [10] can only be applied for PH distributions and not in the general case, but other methods to solve linear differential equations like the adaptive solvers from the Gnu Scientific Library [9] or the extended randomization method [17] may be applied as well for the solution of (19).

4.2 Stationary Distribution

In principle we can apply the transient results also for stationary analysis by letting $\tau \to \infty$. However, here we consider explicitly the stationary distribution π and ν, and assume that $\pi \mathbb{A} = \mathbf{0}$ or $\nu \mathbb{G} = \mathbf{0}$ with $\pi \, \mathbb{1} = \nu \, \mathbb{1} = 1$ exists. We assume in the following theorems that one of the vectors π and ν exists uniquely which implies in the Markov case that the process is ergodic.

Theorem 7 *If $\mathbb{G}$ and $\mathbb{A}$ are generated from the same SPN using $\mathrm{ME}(\alpha_t, \mathbf{A}_t)$ and $\mathrm{PH}(\gamma_t, \mathbf{G}_t)$ such that $\gamma_t \mathbf{V}_t = \alpha_t$, $\mathbf{G}_t \mathbf{V}_t = \mathbf{V}_t \mathbf{A}_t$ and $\mathbf{V}_t \, \mathbb{1}_{n_t} = \mathbb{1}_{m_t}$, and π and ν, the solution of $\pi \mathbb{A} = \mathbf{0}$, $\nu \mathbb{G} = \mathbf{0}$ with $\pi \, \mathbb{1} = \nu \, \mathbb{1} = 1$, exist uniquely, then*

$$\forall M \in \mathcal{RS} : \ \nu(M) \left(\bigotimes_{t \in Ena(M)} \mathbf{V}_t \right) = \pi(M) .$$

Furthermore, the stationary marking probability, $Prob(M)$, throughput, $Tput_{ME}(t)$, remain the same in both nets.

Proof. Let $\nu(M)$ and $\pi(M)$ be the block of the stationary vectors associated with marking M. From (24) for all $M \in \mathcal{RS}$ we have

$$\left(\bigotimes_{t' \in Ena(M')} \mathbf{V}_{t'} \right) \mathbb{A}(M', M) = \mathbb{G}(M', M) \left(\bigotimes_{t' \in Ena(M)} \mathbf{V}_{t'} \right)$$

from which

$$\sum_{M':M'\xrightarrow{t}M} \nu(M')\mathbb{G}(M',M)\left(\bigotimes_{t'\in Ena(M)}\mathbf{V}_{t'}\right) =$$

$$\sum_{M':M'\xrightarrow{t}M} \nu(M')\left(\bigotimes_{t'\in Ena(M')}\mathbf{V}_{t'}\right)\mathbb{A}(M',M) = \sum_{M':M'\xrightarrow{t}M}\pi(M')\mathbb{A}(M',M)$$

and thus $\nu\mathbb{G} = \mathbf{0} \Rightarrow \pi\mathbb{A} = \mathbf{0}$. Since π is unique, also $\pi\mathbb{A} = \mathbf{0} \Rightarrow \nu\mathbb{G} = \mathbf{0}$ holds.

Furthermore,

$$Prob_{ME}(M) = \nu(M)\left(\bigotimes_{t\in Ena(M)}\mathbb{1}_{m_t}\right) = \nu(M)\left(\bigotimes_{t\in Ena(M)}\mathbf{V}_t\,\mathbb{1}_{n_t}\right) =$$

$$\pi(M)\left(\bigotimes_{t\in Ena(M)}\mathbb{1}_{n_t}\right) = Prob_{PH}(M)$$

for all $M \in \mathcal{RS}$ such that $\nu\,\mathbb{1} = 1 \Leftrightarrow \pi\,\mathbb{1} = 1$.

The identity of $Tput_{ME}(t)$ and $Tput_{PH}(t)$ is obtained by the same steps as in (25). ∎

Theorem 8 *If $\mathbb{G}$ and $\mathbb{A}$ are generated from the same SPN using $\mathrm{ME}(\alpha_t, \mathbf{A}_t)$ and $\mathrm{PH}(\gamma_t, \mathbf{G}_t)$ such that $\gamma_t = \alpha_t\mathbf{W}_t$, $\mathbf{W}_t\mathbf{G}_t = \mathbf{A}_t\mathbf{W}_t$ and $\mathbb{1}_{n_t} = \mathbf{W}_t\mathbb{1}_{m_t}$ and π and ν exist uniquely, then*

$$\forall M \in \mathcal{RS}: \ \nu(M) = \pi(M)\left(\bigotimes_{t\in Ena(M)}\mathbf{W}_t\right).$$

Furthermore, the stationary marking probability, $Prob(M)$, throughput, $Tput(t)$, remain the same in both nets.

Proof. The proof of Theorem 8 follows the same pattern as the one of Theorem 7. ∎

In both cases described in the previous theorems a system of linear equations has to be solved. In the Markovian case the resulting matrix is weakly diagonal dominant which does not hold in general for matrix $\mathbb{A}$. This has implications on the convergence behavior of numerical solvers but does not modify the general problem.
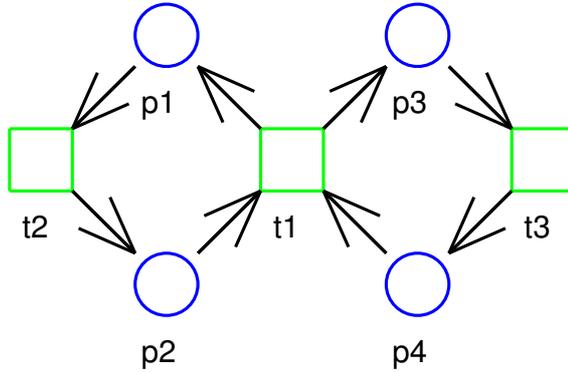
Fig. 1. Example net.

	SPN−PH		SPN−ME	
N	states	non-zeros	states	non-zeros
1	77	230	18	64
2	615	3536	81	550
3	1825	11546	198	1540
4	3707	24260	369	3034
5	6261	41678	594	5032

Table 1
 Number of states and non-zero entries for *SPN−PH* and *SPN−ME* and different token populations.

5 An Example

In the following we consider a small example model which is not intended to describe a real application. However, it allows us to show that the substitution of PH distributions by equivalent ME distributions may result in a non-Markovian process with a smaller state space such that numerical solution methods benefit from the substitution. The example net is shown in Figure 1. It contains three transitions with ME/PH distributed firing times. Distributions of the net are chosen either as ME distributions with 3 states each or as equivalent PH distributions with 6, 8, and 7 states, respectively. The concrete distributions are shown in the appendix.

The number of states and the number of non-zero entries in the matrices $\mathbb{A}$ and

	Uniformization		RKF_45		RK_8PD	
N	iterations	time	iterations	time	iterations	time
1	5679	0.1	11605	0.2	18189	0.3
2	6566	0.8	13507	1.5	21152	2.1
3	6566	2.1	13490	4.1	21230	6.5
4	6566	3.8	13501	8.4	21217	13.2
5	6566	6.8	13514	14.1	21230	22.4

Table 2

Effort for analyzing $SPN{-}PH$ up to time $t = 100$ with different solvers.

	RKF_45		RK_8PD	
N	iterations	time	iterations	time
1	7573	0.1	11414	0.1
2	9104	0.3	13482	0.5
3	9560	0.6	14273	0.8
4	9650	1.0	14844	1.3
5	9814	1.4	15118	1.9

Table 3

Effort for analyzing $SPN{-}ME$ up to time $t = 100$ with different solvers.

$\mathbb{G}$ resulting from this SPN can be found in Table 1 for initial markings with $N = 1, \ldots, 5$ tokens in each of the places $p1$ and $p3$. It can be noticed that the state space of $SPN{-}PH$ is about an order of magnitude larger than the state space of $SPN{-}ME$ and the number of non-zero elements, which determines the effort for vector matrix multiplications, is about 8 times larger.

For both nets a transient analysis up to time $t = 100$ is performed with different solution techniques. For $SPN{-}PH$ we use uniformization [10] and two differential equation solvers from the Gnu Scientific Library [9], namely the embedded Runge-Kutta-Fehlberg (4, 5) method ($RKF{\_}45$) and the embedded Runge-Kutta Prince-Dormand (8,9) method ($RK{\_}8PD$). Table 2 contains the analysis effort for $SPN{-}PH$ with different initial markings. The table includes the number of iterations and the CPU time in seconds needed on a standard laptop. Uniformization runs with an error $\epsilon = 10^{-8}$, the ϵ-values for the two differential equation solver are chosen such that the Euclidean norm of the difference between the marking probability vector computed by uniformization and by the differential equation solvers is less than 10^{-6}. Obviously uniformization is the most efficient solver requiring the smallest number of iterations and the shortest solution time.

Table 3 includes the iteration numbers and the effort to analyze $SPN{-}ME$ up to time $t = 100$ with the same accuracy as $SPN{-}PH$. Since $\mathbb{A}$ is not a stochastic matrix, uniformization cannot be applied but the two differential equation solvers still work for this example. It can be seen that the solvers require fewer iterations on $\mathbb{A}$ than on $\mathbb{G}$ which is a result of the smaller difference in the transition rates in the matrices of the ME distributions. However, the number of iterations is still larger than the number of iterations required by uniformization for $\mathbb{G}$. Nevertheless, since $\mathbb{A}$ is much smaller than $\mathbb{G}$ and contains fewer non-zero elements, the solution of $SPN{-}ME$ with RKF_45 is about 5 times faster than the solution of $SPN{-}PH$ with uniformization.

It is, of course, not possible to draw general conclusions from a few experiments but it is relatively obvious that if ME distributions are available to model some real process, then it is preferable to use them directly in a model instead of expanding the distributions to PH distributions with a larger state space. This is especially attractive for distributions with a low coefficient of variation that can be conveniently modelled by ME distributions.

We present here only results for transient analysis but results for steady state analysis are similar. Since for steady state analysis no specific efficient method like uniformization that requires Markovian models exists, the use of ME distributions is even more attractive in this case.

6   Conclusions

In this paper we focused on the solution of SPNs with ME distributed firing time and showed that the same expansion and solution techniques are applicable as for the analysis of SPNs with PH distributed firing time if the procedures do not utilize the non-negativity of the computed quantities.

The proposed approach is beneficial if the firing time distributions of SPNs can be described with a low order ME distributions whose equivalent PH representation has a higher order.

A simple example demonstrates the computational benefit of using small ME representations instead of using larger PH representations. The computational benefit holds even for a small gain in the size of the ME representation and it increases when the size of the model increases.

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Appendix: Firing Time Distributions for the Example SPN

The firing time of transition $t1$ in $SPN{-}ME$ is $\mathrm{ME}(\alpha_1, \mathbf{A}_1)$ distributed with

$$\alpha_1 = (0.8, -0.4, 0.6) \ , \quad \mathbf{A}_1 = \begin{pmatrix} -10.0 & 10.0 & 0.0 \\ 0.0 & -20.0 & 20.0 \\ 0.0 & 25.0 & -30.0 \end{pmatrix} .$$

For this ME distribution a PH distribution $(\gamma_1, \mathbf{G}_1)$ with 6 states can be generated with the approach from [13].

$$\gamma_1 = (0.79086, 0.10382, 0.03514, 0.00313, 0.01010, 0.05696) \ ,$$

$$\mathbf{G}_1 = \begin{pmatrix} -0.958 & 0.958 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -21.95 & 21.95 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -21.95 & 21.95 & 0.0 & 0.0 \\ 0.0 & 7.204 & 0.0 & -21.95 & 14.75 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -52.67 & 52.67 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -52.67 \end{pmatrix} .$$

The second and third distribution are originated from $\mathrm{ME}(\alpha, \mathbf{A})$ with

$$\alpha = (0.2, 0.3, 0.5) \ , \quad \mathbf{A} = \begin{pmatrix} -1.0 & 0.0 & 0.0 \\ 0.0 & -3.0 & h \\ 0.0 & -h & -3.0 \end{pmatrix} .$$

It is known (see [8,16]) that for $h \le 2.3$ an ME distribution results from the above matrix and for $h \le 0.55$ the distribution can be represented as a PH distribution with 3 states. The ME distribution for transition $t2$ is associated with the above ME distribution with $h = 1.15$ which can be expanded to a

PH$(\gamma_2, \mathbf{G}_2)$ distribution with 8 states and

$$\gamma_2 = (0.00031, 0.02684, 0.01689, 0.00515, 0.01179, 0.70035, 0.24954, 0.61914) \ ,$$

$$\mathbf{G}_2 = \begin{pmatrix} -1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2.336 & 2.336 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2.336 & 2.336 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.429 & 0.0 & -2.336 & 1.907 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -4.571 & 4.571 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -4.571 & 4.571 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -4.571 & 4.571 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -4.571 \end{pmatrix} .$$

For $h = 1$ we obtain with the approach from [13] the following equivalent PH$(\gamma_3, \mathbf{G}_3)$ with 7 states that is associated with transition $t3$.

$$\gamma_3 = (0.00887, 0.02508, 0.01518, 0.00351, 0.03284, 0.21118, 0.70333) \ ,$$

$$\mathbf{G}_3 = \begin{pmatrix} -1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -2.4227 & 2.4227 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -2.4227 & 2.4227 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.2623 & 0.0 & -2.4227 & 2.1603 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3.9811 & 3.9811 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.9811 & 3.9811 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.9811 \end{pmatrix} .$$