International Journal of Performability Engineering, Vol.7, No. 5, September 2011, pp. 441-454.

Printed in India

Stochastic Petri Nets with Low Variation Matrix Exponentially Distributed Firing Time

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(Received on November 21, 2010 and revised on May 18, 2011)

Abstract: Matrix exponential (ME) distributions with low squared coefficient of variation (scv) are such that the density function becomes zero at some points in $(0,\infty)$. For such distributions there is no equivalent finite dimensional PH representation, which inhibits the application of existing methodologies for the numerical analysis of stochastic Petri nets (SPNs) with this kind of ME distributed firing time. To overcome the limitations of existing methodologies we apply the flow interpretation of ME distributions and study the transient and the stationary behaviour of stochastic Petri nets with ME distributed firing times via ordinary differential and linear equations, respectively. The main result of this study is a theory stating that all kinds of ME distributions can be used like phase type (PH) distributions in stochastic Petri nets and the numerical computation of transient or stationary measures is possible with methods similar to those used for Markov models.

***Keywords**: Stochastic Petri net, phase type distribution, matrix exponential distribution*

1 Introduction

The method of extended Markov chain (EMC) [9] is a widely used analysis technique for stochastic Petri nets with PH distributed firing times. It is based on the generation of a Markov chain that describes the behaviour of the marking process and additionally the phase processes of the involved PH distributions. The resulting Markov chain can then be analyzed with established numerical techniques for transient or stationary analysis.

Following the general results in [8] it was likely that in a stochastic model ME distributions can be used in place of PH distributions and several results will carry over. There are some results to this direction, but it is not easy to prove results in the general setting because probabilistic arguments associated with PH distributions do no longer hold. In [4] it has been shown that matrix geometric methods can be applied for quasi birth death processes (QBDs) with rational arrival processes (RAPs) [2], which can be viewed as an extension of ME distributions to arrival processes. To prove that the matrix geometric relations hold, the authors of [4] use an interpretation of RAPs that has been proposed in [2]. However, the resulting proofs are limited to QBDs.

The closest related result considers a *subclass* of SPNs with ME distributed firing times [6]. That paper proves the applicability of an extended system of differential equations for the transient analysis and an extended system of linear equations for stationary analysis in case of ME distributions with strictly positive density in $(0,\infty)$. Due to the similarity to the EMC based solution we refer to this solution method as EMC-like

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solution. [6] proofs that the EMC-like solution is applicable for SPNs with ME distributed firing times whose density is *strictly positive* in $(0,\infty)$.

In this paper we extend the result of [6] for the case of ME distributed firing times whose density might be zero in $(0,\infty)$. The importance of this extension comes from the fact that the density of important and practically convenient ME distributions is zero in some points in $(0,\infty)$. Some of the most important examples of these ME distributions are the ME distributions with low scv as it is detailed below.

The methodology applied in this paper is different from the one used in [6]. The proof of [6] is based on the fact that any ME distribution with strictly positive density on $(0,\infty)$ can be represented as a PH distribution with a potentially larger vector-matrix pair. This approach is not applicable for ME distributed firing time whose density might be zero on $(0,\infty)$, since these ME distributions cannot be represented as a PH distribution with finite dimension [3]. Instead, we provide a proof of the applicability of EMC-like solution based on the flow interpretation of ME distributions provided by Bladt and Neuts in [5].

The paper is organised as follows. Sect. 2 presents the definition of ME distributions and some important results about their representations. Examples of ME distributions with low scv are reported in Sect. 3. In Sect. 4 we introduce SPNs with ME distributed firing time and give the necessary elements for their analysis. Sect. 5 provides examples to show applications of the approach. Finally, Sect. 6 concludes the paper.

2 Matrix Exponential Distributions

We quickly recall the basic definition of ME [5] and PH [10] distributions for completeness.

Definition 1 *Let* X *be a random variable with cumulative distribution function (cdf)* $F_X(x) = Pr(X < x) = 1 - \alpha e^{\mathbf{A}x}\bar{1}$ *where* α *is a row vector of size* n, $\mathbf{A}$ *is a matrix of size* $n \times n$ *and* $\bar{1}$ *is the column vector of ones of size* n. *Then we say that* X *is matrix exponentially distributed with representation* $\alpha, \mathbf{A}$, *or shortly, ME(*$\alpha, \mathbf{A}$*) distributed.*

Definition 2 *If* X *is an ME(*$\alpha, \mathbf{A}$*) distributed random variable and* α *and* $\mathbf{A}$ *have the properties that* $\alpha_i \geq 0$, $\alpha\bar{1} = 1$ *(there is no probability mass at* $t = 0$*),* $A_{ii} < 0$, $A_{ij} \geq 0$ *for* $i \neq j$, $\mathbf{A}\bar{1} \leq 0$ *and* $\mathbf{A}$ *is non-singular, then we say that* X *is phase type distributed with representation* $\alpha, \mathbf{A}$, *or shortly, PH(*$\alpha, \mathbf{A}$*) distributed.*

Using the notation $a = -\mathbf{A}\bar{1}$, the probability density function (pdf) and the moments of X are, respectively, $f_X(x) = \alpha e^{\mathbf{A}x} a$ and $\mu_n = E(X^n) = n!\alpha(-\mathbf{A})^{-n}\bar{1}$.

2.1 A Convenient Subclass of ME Distributions

One of the main problems of working with ME distributions is that the monotone increasing property of $F_X(x)$ (or the non-negativity of $f_X(x)$) is hard to check. Although there are special subclasses of ME distributions whose construction ensures that the associated PDF is non-negative.

Definition 3 *The set of ME distributions with pdf* $f(t) = a(t) / \int_0^\infty a(t)dt$ *where*

$a(t) = \sum_i (r_i^2(t) + ts_i^2(t))e^{-\lambda_i t} + \sum_i (q_i^2(t) + tw_i^2(t))e^{-\mu_i t}\cos^2(\omega_i t + \phi_i)$,

and $r_i(t), s_i(t), q_i(t), w_i(t)$ *are arbitrary finite polynomials of* t *and* $\lambda_i, \mu_i, \omega_i, \phi_i$ *are*

positive real numbers, is referred to as ME distributions with quadratic polynomials.
ME distributions with quadratic polynomials are guaranteed to have non-negativity of the pdf. Additionally, as it is discussed in [7], some important extreme ME distributions belong this class: numerical investigations suggests that the order n ME distributions with real eigenvalues and minimal scv and the order $2k+1$ ME distributions with real and complex eigenvalues and minimal scv belong to the class of ME distributions with quadratic polynomials. Based on [7], Table 1 lists the minimal scv of ME distributions with quadratic polynomials of different order. To the best of our knowledge the values presented in Table 1 represent the minimal scv of the whole ME class (with or without complex eigenvalues) of the given order, but there is no proof or counter example (an ME with lower scv) is available up to now which verify or destroy this conjecture. To avoid invalid statements below we are going to talk about ME distribution with low scv, but we think that it can be read as ME distribution with minimal scv.

PH distributions of order n are known to have scv greater or equal to $1/n$. Consequently, the $1/\text{min\_scv}$ parameters in Table 1 indicate the minimal size of a PH distribution to approximate such a low coefficient of variation. This property of the class of ME distributions with quadratic polynomials makes their use very efficient for approximating distributions with low scv.

Table 1: Minimal Squared Coefficient of Variation of ME Distributions with Quadratic Polynomials

Order	min_scv	1/min_scv	min_scv	1/min_scv
	Real poles		Real and Complex poles	
3	0.276583	3.61556	0.200902	4.97756
4	0.19333	5.17251	0.149808	6.6752
5	0.138453	7.22266	0.0812643	12.3055
6	0.108623	9.20619		
7	0.0861277	11.6107	0.04288	23.3209
8	0.0717026	13.9465		
9	0.0600486	16.6532	0.0261569	38.2309
10	0.0518365	19.2914		
11	0.0449173	22.2632	0.017494	57.1625
12	0.0397335	25.1677		
13	0.0352403	28.3766	0.0124696	80.1951
14	0.031726	31.5199		
15	0.0286172	34.944	0.00931281	107.379

2.2 Interpretation of Matrix Exponential Distributions via Flows

In [5] the authors provide a stochastic interpretation of ME distributions via flows. This interpretation is the following for ME($\alpha, \mathbf{A}$) of size n. Consider n doubly infinite containers of liquid whose initial contents are $\alpha_1,...,\alpha_n$ and an additional container whose content is zero initially. Assume that liquids flow from container i to container j, with $1 \le i, j \le n, i \ne j$, at constant rate given by $\mathbf{A}(i, j)$. That means that if the ith container has c amount of liquid at time u then $c\mathbf{A}(i, j)du$ amount of liquid flows from the ith to jth container in the interval $[u, u+du]$. Further, from container $i, 1 \le i \le n,$ liquid flows toward the $n+1$th container at constant rate given by the ith entry of a.

Let us denote by $v_i(u), 1 \le i \le n+1$ the level of liquid in container i at time u. As shown in [5], the vector $v(u) = | v_1(u),...,v_n(u) |$ refering to the first n containers follows the set of ordinary differential equations (ODE): $dv(u)/du = v(u)\mathbf{A}$ with initial condition $v(0) = \alpha$. The solution of these ODEs is $v(u) = \alpha \exp(\mathbf{A}u)$. Then it is easy to see that the following relations hold between the levels of the liquids in the containers and a random variable X distributed according to ME($\alpha, \mathbf{A}$):

$$1 - F_X(u) = Pr(X > u) = \sum_{i=1}^{n} v_i(u) = 1 - v_{n+1}(u),$$

i.e., the total amount of liquid present in containers $v_1, ..., v_n$ at time u corresponds to the probability that X is greater then u, and $f_X(u) = v(u)a$, i.e., the pdf of X can be connected to the level of the liquids through the vector a and we will refer to the quantity $v(u)a$ as the firing potential. It follows that the aging of a ME distributed random variable can be captured by the real valued vector $v(u)$.

3 Examples of ME Distributions with Low Coefficient of Variation

3.1 Order 3 ME Distributions with Complex Eigenvalues

First we consider the order 3 ME structure with low scv reported in [7]. It is $f(t) =$

$$ue^{-at} 2\cos^2\left(\frac{\omega t + \phi}{2}\right) = ue^{-at}(1 + \cos(\omega t + \phi)) = e^{-at}(u + u\cos(\phi)\cos(\omega t) - u\sin(\phi)\sin(\omega t))$$

Where, from $\int f(t)dt = 1$, we have $u = a(a^2 + \omega^2)/(a^2 + \omega^2 + a^2\cos(\phi) - a\omega\sin(\phi)).$ On the other hand we have the following real matrix representation

$$\begin{aligned}
f'(t) &= \alpha e^{\mathbf{A}x}(-\mathbf{A})\bar{1} = (g, c+d, c-d)\ \exp\left[\begin{pmatrix} -a & 0 & 0 \\ 0 & -a & -\omega \\ 0 & \omega & -a \end{pmatrix} t\right] \begin{pmatrix} a \\ a+\omega \\ a-\omega \end{pmatrix} = \\
&= (g, c-id, c+id)\ \exp\left[\begin{pmatrix} -a & 0 & 0 \\ 0 & -a-i\omega & 0 \\ 0 & 0 & -a+i\omega \end{pmatrix} t\right] \begin{pmatrix} a \\ a+i\omega \\ a-i\omega \end{pmatrix} = \\
&= age^{-at} + (c-id)(a+i\omega)e^{-at}e^{-i\omega t} + (c+id)(a-i\omega)e^{-at}e^{i\omega t} = \\
&= e^{-at}(ag + 2(ac + \omega d)\cos(\omega t) - 2(ad - \omega c)\sin(\omega t))
\end{aligned}$$

where from the first matrix representation to the second one a similarity transformation is applied. Having a, ω and ϕ fixed, from $f(t) \equiv f'(t)$ we have $ag = u,\ 2(ac + \omega d) = u\cos(\phi),\ 2(ad - \omega c) = u\sin(\phi),$ from which

$$g = \frac{u}{a},\ c = \frac{a^2\cos(\phi) - a\omega\sin(\phi)}{2(a^2 + \omega^2 + a^2\cos(\phi) - a\omega\sin(\phi))},\ d = \frac{a\omega\cos(\phi) + a^2\sin(\phi)}{2(a^2 + \omega^2 + a^2\cos(\phi) - a\omega\sin(\phi))}.$$

With $a = 1$, $\phi = -3.47863$, and $\omega = 1.03593$ the minimal scv of this structure is obtained and it is $0.200902 \sim 1/5$ [7]. The pdf of this distribution is depicted in Fig. 1 and, to indicate the flexibility of this class of distributions, Fig. 2 depicts the pdf obtained at $a = 1$, $\phi = 2$, and $\omega = 20$.

3.2 Order 3 ME Distributions with Real Eigenvalues

The order 3 ME structure with real eigenvalues and with minimal scv is [7]

$$f(t) = e^{-at}((w_1 t + w_0)^2 + v_0^2 t) = e^{-at}(w_1^2 t^2 + (w_1 w_0 + v_0^2)t + w_0^2)$$

where $w_1 = (-aw_0 - \sqrt{2a^3 - 2av_0^2 - a^2w_0^2})/2$ ensures $\int f(t)dt = 1$. Starting from an Erlang type matrix representation we have

$$f'(t) = \alpha e^{\mathbf{A}x}(-\mathbf{A})\bar{1} = (x_1, x_2, x_3)\ \exp\left[\begin{pmatrix} -a & a & 0 \\ 0 & -a & a \\ 0 & 0 & -a \end{pmatrix} t\right]\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} = e^{-at}\left(\frac{a^3t^2x_1}{2!} + \frac{a^2tx_2}{1!} + \frac{a^1x_3}{0!}\right).$$

From the identity of the coefficients of t^i we have $x_1 = \frac{2w_1^2}{a^3}$, $x_2 = \frac{w_1w_0 + v_0^2}{a^2}$, $x_3 = \frac{w_0^2}{a}$.

The minimal scv with real poles is obtained at $w_0 = 0.358998$ and it is 0.276583. The pdf of this distribution is depicted in Fig. 3.

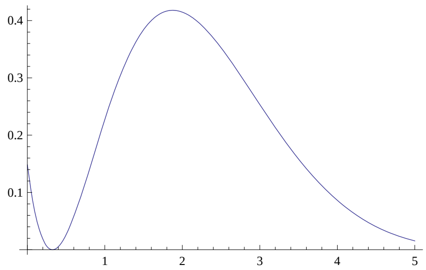

Figure 1: Probability density function of ME3 with $a = 1$, $\phi = -3.47863$,

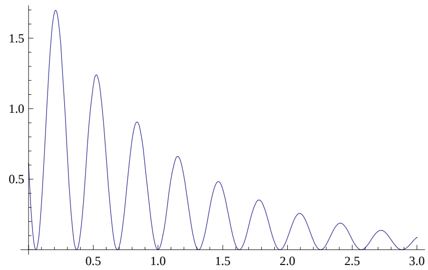

Figure 2: Probability density function of ME3 with $a = 1$, $\phi = 2$, $\omega = 20$

3.3 Higher order ME Distributions with Complex Eigenvalues

Let the density of a ME distribution be $f(t) = ue^{-at}\prod_{i=1}^{k} 2\cos^2\left(\frac{\omega t + \phi_i}{2}\right) = ue^{-at}\prod_{i=1}^{k}(1 + \cos(\omega t + \phi_i)).$ If $n = 2k + 1 = 7$, $a = 1$, $\omega = 0.884919$ $\phi_1 = 3.29263, \phi_2 = 3.90442, \phi_3 = 4.86219$ and u is set such that $\int f(t)dt = 1$ then the scv of this distribution is $0.04288 < 1/23$.

The analytical treatment of this case is rather cumbersome, but it can be avoided by a numerical approach to obtain the associated matrix representation. The moments of the distribution can be computed from $m_i = \int t^i f(t)dt$ for $i = 1, \ldots, n$, Based on these moments a matrix representation of f(t) can be obtained in a two steps numerical method. In the first step we generate matrix $\mathbf{A}$ such that it exhibits the same exponential coefficients, i.e., eigenvalues, as $f(t)$. We have

$$\mathbf{A} = \begin{pmatrix} -a & & & & & \\ & -a & -\omega & & & \\ & \omega & -a & & & \\ & & & \ddots & & \\ & & & & -a & -k\omega \\ & & & & k\omega & -a \end{pmatrix},$$

i.e., the eigenvalues of $\mathbf{A}$ are $-a$ and $\{-a + i\omega, -a - i\omega\}$ for $i = 1, 2, \ldots, k$. In the second step we obtain vector α by solving the linear system $i!\alpha(-\mathbf{A})^{-i}\bar{1} = m_i$ for $i = 0, \ldots, n-1$.

We applied this numerical procedure for $n = 7$ ($k = 3$) and obtained the ME distribution with matrix representation $(\alpha, \mathbf{A})$ where α ={9.77827, -4.71506, -6.56599, 0.172583, 2.40449, 0.213925, -0.288218} and $\mathbf{A}$ is defined by its structure.

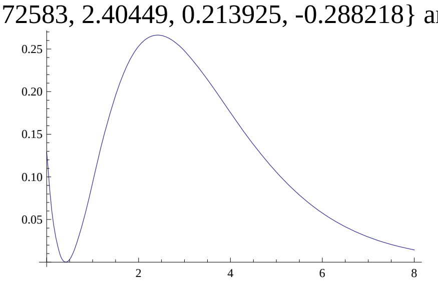

Figure 3: Probability Density Function of ME3 with real poles and $w_0 = 0.358998$

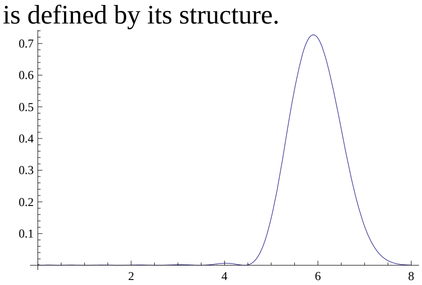

Figure 4: Probability Density Function of the order 15 ME with scv= 0.00931281

The approach works for $n=11$ ($k=5$). With $a=1$, $\phi_1=3.61267$, $\phi_2=3.23369$, $\phi_3=4.21678$, $\phi_4=4.96537$, $\phi_5=5.81516$, $\omega=0.800546$, and u set such that $\int f(t)dt=1$, the scv of this distribution is $0.017494<1/57$. The initial vector of its matrix representation is α ={25.312, -4.86661, -23.9613, -7.24198, 8.29355, 3.77278, 0.862056, -0.214874, -0.938152, -0.0976547, 0.0802795}.

The same approach works also for $n=15$ ($k=7$). With $a=1$, $\phi_1=3.47395$, $\phi_2=4.45106$, $\phi_3=3.20593$, $\phi_4=3.90857$, $\phi_5=5.06454$, $\phi_6=5.73043$, $\phi_7=6.4419$, $\omega=0.745921$, and u set such that $\int f(t)dt=1$, the scv of this distribution is $0.00931281<1/107$ and its pdf is depicted in Fig. 4. The initial vector of its matrix representation is α ={53.1771, 5.41155, -55.1427, -28.5444, 8.56841, 7.01384, 12.2088, 4.40013, -3.67725, -1.432, -1.22016, -0.215864, 0.384661, 0.0534409, 0.0144613}.

4 Stochastic Petri Nets with ME Distributed Firing Times

In this section we introduce stochastic Petri nets in which the firing times of the transitions are ME distributed. We consider first Petri nets and their reachability graph. Afterwards, the reachability graph is expanded by considering detailed state information to describe the age of the enabled transitions. We start by briefly presenting some basic definitions and results for Petri nets following [6].

Definition 4 *A Petri net is a five tuple* $PN=(P,T,I,O,M_0)$ *where* P *is a set of places,* T *is a set of transitions such that* $P\cap T=\emptyset$, $I: P\times T\rightarrow N$ *is the input function,* $O: T\times P\rightarrow N$ *is the output function, and* $M_0: P\rightarrow N$ *is the initial marking.*

We assume an ordering the set of transitions such that for $t,t'\in T$ with $t\neq t'$ either $t<t'$ or $t>t'$ holds. Denote by $\bullet t=\{p \mid p\in P\wedge I(p,t)>0\}$ and $t\bullet=\{p \mid p\in P\wedge O(t,p)>0\}$ the input and output bag of transition t, respectively. A marking M is a vector of length $|P|$ whose elements represent the token population of each place. $M(p)$ denotes the p-th element of this vector. Marking M_0 defines the initial token population. Transition t is enabled in marking M if and only if $M(p)\geq I(p,t)$ for all $p\in\bullet t$. If t is enabled in marking M and fires, then a new marking M' with $M'(p)=M(p)-I(p,t)+O(t,p)$ is generated. For this event we use the notation $M\xrightarrow{t}M'$. We assume that $M\xrightarrow{t}M'$ implies $M\neq M'$. The extension to $M\xrightarrow{t}M'$ is straightforward but requires a more complicate notation. The set of markings available from M_0 with repeated application of relation $\xrightarrow{t}$ defines the reachability set RS of the Petri net. The reachability graph RG is a directed and labeled graph with vertex set RS and an arc labeled with t between $M, M' \in \mathsf{RS}$ if and only if $M \xrightarrow{t} M'$. Further assumptions about RS and RG, like finiteness or strong connectivity will be made later when necessary.

Let $Ena(M) = \{t \mid t \in T \text{ and for all } p \in P : M(p) \geq I(p,t)\}$ be the set of enabled transitions in marking M. The concept underlying our definition of newly enabled transitions is denoted as enabling memory in [1]. The general approach is applicable for age memory policy as well. Only the structure of the state descriptor and the definition of the resetting or maintaining the memory in (2) has to be modified in that case. Furthermore, we assume single server semantics for all transitions.

4.1 Flow Interpretation of SPN with ME Distributed Firing Times

Hereinafter we show that the behaviour of a PN with ME timings can be described through the behaviour of the levels of the liquids associated with the ME distributed transitions of the net. This is done by associating each marking M with a vector $v(u,M)$ providing at time u the joint state (i.e., the joint liquid levels) of the ME distributions of the transitions that are enabled in marking M. We denote by n_t, α_t, $\mathbf{A}_t$ and $a_t = (-\mathbf{A}_t)\bar{1}$ the size, the initial vector, the generator and the closing vector of the ME distribution associated with transition t. Using these notations we can present the main theorem of the paper.

Theorem 1 *$v(u,M)$ satisfies the vector differential equation*

$$\frac{dv(u,M)}{du} = v(u,M) \bigoplus_{t \in Ena(M)} A_t + \sum_{t: M' \xrightarrow{t} M} v(u,M') \bigotimes_{t' \in T} R_{t',t}(M',M) \tag{1}$$

where

$$R_{t',t}(M',M) = \begin{cases} I_{n_{t'}} & \text{if } t' \neq t \text{ and } t' \in Ena(M') \cap Ena(M), \\ \alpha_{t'} & \text{if } t' \neq t, t' \notin Ena(M') \text{ and } t' \in Ena(M), \\ \bar{1}_{n_{t'}} & \text{if } t' \neq t, t' \in Ena(M') \text{ and } t' \notin Ena(M), \\ a_t & \text{if } t' = t, \text{ and } t \notin Ena(M), \\ a_t \alpha_t & \text{if } t' = t \text{ and } t \in Ena(M), \\ 1 & \text{otherwise}, \end{cases} \tag{2}$$

where I_n is the $n \times n$ identity matrix and $\bar{1}_n$ is the column vector of ones of length n. The initial condition is $v(0,M_0) = \bigotimes_{t \in Ena(M_0)} \alpha_t$ and $v(0,M) = 0$ for $\forall M \neq M_0$.

Proof. To prove the theorem we present the scalar equations governing the system behaviour. Unfortunately, it requires the introduction of complicated notations referring to the elements of complicated multidimensional vectors and matrices. We denote by K_M the number of active ME distributions in marking M, and by $n_{M,i}, \alpha_{M,i}$, $\mathbf{A}_{M,i}$ and $a_{M,i} = (-\mathbf{A}_{M,i})\bar{1}$ the size and the descriptors of the ith active ME distribution in marking M. The entries of $\alpha_{M,i}$, $\mathbf{A}_{M,i}$ and $a_{M,i}$, in order to avoid heavy subscripting, will be indicated in parenthesis, i.e., for example, the jth entry of $\alpha_{M,i}$ as $\alpha_{M,i}(j)$ and the entry

(j,k) of $\mathbf{A}_{M,i}$ as $\mathbf{A}_{M,i}(j,k)$. The index of transition t in marking M will be denoted by $p_{M,t}$, *i.e.*, if the ith active ME distribution in marking M is t then $p_{M,t} = i$.

The vector $v(u,M)$ is of length $\prod_{t\in Ena(M)} n_t$ and its entries are organised according to lexicographical order (also referred to as the mixed-base scheme). This order is naturally generated by the Kronecker product operation of the vectors representing the level of the containers associated with the active transitions in marking M at time u. For the elements of the vector the lexicographical order means that, having a vector of indices $l = |l_1, l_2, ..., l_{K_M}|$ with $1 \le l_i \le n_{M,i}, 1 \le i \le K_M$ identifying a container for each enabled transition of marking M, the entry of $v(u,M)$ that describes the joint state of these containers is in position $(...((l_1-1)n_{M,2} + l_2 - 1)n_{M,3}...)n_{M,K_M} + l_{K_M} - 1 = \sum_{k=1}^{K_M}(l_k-1)\prod_{i=k+1}^{K_M} n_{M,k}$ (where, for simplicity of notation, the empty product equals to 1). A given entry of the vectors $v(u,M)$ will be a container itself and the vectors $v(u,M)$ provide the expanded state space of the containers of the individual transitions. The entry of $v(u,M)$ corresponding to the vector of containers $|l_1, l_2, ..., l_{K_M}|$ will be denoted by $v(u,M,|l_1,...,l_{K_M}|)$.

The entries of $v(u,M)$ will be such that the level of the jth container of the ith enabled transition of marking M can be recovered by the sum

$$v(u,M,i,j) = \sum_{l_1=1}^{n_{M,1}}\sum_{l_2=1}^{n_{M,2}}\cdots\sum_{l_{i-1}=1}^{n_{M,i-1}}\sum_{l_{i+1}=1}^{n_{M,i+1}}\cdots\sum_{l_{K_M}=1}^{n_{M,K_M}} v(u,M,|l_1,l_2,..,l_{i-1},j,l_{i+1},...,l_{K_M}|). \tag{3}$$

Further, the probability of marking M at time u will be given by the total amount of liquid present in $v(u,M)$ as

$$\pi(u,M) = \sum_{l_1=1}^{n_{M,1}}\cdots\sum_{l_{K_M}=1}^{n_{M,K_M}} v(u,M,|l_1,...,l_{K_M}|).$$

The initial condition for the vectors $v(u,M)$ is given as

$$v(0,M_0) = \bigotimes_{t\in Ena(M_0)} \alpha_t \quad \textit{and} \quad \forall M \ne M_0 : v(0,M) = \underline{0}. \tag{4}$$

with which it is easy to see that $v(0,M_0,i,j) = \alpha_{M_0,i}(j), \forall i,j : 1 \le i \le K_{M_0}, 1 \le j \le n_{M_0,i}$ and $v(0,M,i,j) = 0, \forall M,i,j : M \ne M_0, 1 \le i \le K_M, 1 \le j \le n_{M,i}$, i.e., (4) provides correct initial setting of the levels of the liquids.

In order to describe correctly the evolution of the PN, the evolution of $v(u,M)$, $\forall M$ and $u \ge 0$, has to be such that the level of the jth container of the ith enabled transition of M given by $v(u,M,i,j)$, $1 \le i \le K_M, 1 \le j \le n_{M,i}$, and computed according to (3) satisfies the following conditions.

1. The level $v(u,M,i,j)$ is decreased at rate $\mathbf{A}_{M,i}(j,j)$.

2. There is an exchange of liquids from containers $v(u,M,i,k), k \ne j$, to container $v(u,M,i,j)$ with rate $\mathbf{A}_{M,i}(k,j)$.

3. For $\forall t,t' : M' \xrightarrow{t} M, t' \ne t$

- the firing potential of t at time u has to be equal to $\sum_{j=1}^{n_t} v(u, M', p_{M',t}, j) a_t(j)$ and, accordingly, in the interval $[u, u+du]$ the amount of liquid flowing from the containers of $v(u, M')$ to the containers of $v(u, M)$ is $du \sum_{j=1}^{n_t} v(u, M', p_{M',t}, j) a_t(j)$;

- if $t \in Ena(M)$ then the flow from the containers of $v(u, M')$ to the containers of $v(u, M)$ has to be distributed among the levels $v(u, M, p_{M,t}, i), 1 \le i \le n_t$ according to α_t;

- if $t' \in Ena(M')$ and $t' \in Ena(M)$ then the flow from $v(u, M')$ to $v(u, M)$ has to be distributed among the levels $v(u, M, p_{M,t'}, i), 1 \le i \le n_{t'}$ as it was distributed among the levels $v(u, M', p_{M',t'}, i), 1 \le i \le n_{t'}$, i.e., the state (age) of t' has to be maintained;

- if $t' \notin Ena(M')$ and $t' \in Ena(M)$ then the liquid flowing from $v(u, M')$ to $v(u, M)$ has to be distributed among the levels $v(u, M, p_{M,t'}, i), 1 \le i \le n_{t'}$ according to $\alpha_{t'}$, i.e., the state of t' has to be initialised;

- if $t' \notin Ena(M)$ then t' has no impact on the flow from $v(u, M')$ to $v(u, M)$.

In the following we provide a set of ODEs which describes the evolution of each container of $v(u, M)$ for every marking M of the PN. The fact that these ODEs satisfy the conditions listed above can be seen by straightforward but cumbersome algebraic steps based on the summation provided in (3).

For a vector $l = |l_1, l_2, ..., l_{K_M}|$ of marking M and an index $i, 1 \le i \le K_M$, we will denote by $f(M, l, i)$ the set of vectors which differs from l at most in position i, i.e., $f(M, l, i) = \{|l_1, ..., l_{i-1}, k, l_{i+1}, ..., l_{K_M}| : 1 \le k \le n_{M,i}, k \ne l_i\}$; note that $l \notin f(M, l, i)$.

Within a given marking M, liquids flow to the container $|l_1, ..., l_{K_M}|$ from another container $|k_1, ..., k_{K_M}|$ of marking M if $\exists i : k \in f(M, l, i)$ and at rate $\mathbf{A}_{M,i}(k_i, l_i)$. Liquid flows away instead from container $|l_1, ..., l_{K_M}|$ of marking M at rate $-\sum_{i=1}^{K_M} \mathbf{A}_{M,i}(l_i, l_i)$.

The containers of other markings, $M' \ne M$, from which liquids flow toward container $l = |l_1, ..., l_{K_M}|$ of marking M can be identified as follows. From a container $k = |k_1, ..., k_{K_{M'}}|$ of marking M' fluid flows to $|l_1, ..., l_{K_M}|$ of marking M if the following conditions hold

1. $\exists t : M' \xrightarrow{t} M$;
2. $\forall t' \ne t$: if $t' \in Ena(M') \cap Ena(M)$ then we must have $l_{p_{M,t'}} = k_{p_{M',t'}}$;
3. $\forall t' \ne t$: if $t' \notin Ena(M')$ and $t' \in Ena(M)$ then we must have $\alpha_{t'}(l_{p_{M,t'}}) \ne 0$;
4. if $t \notin Ena(M)$ then we must have $a_t(k_{M',t}) \ne 0$;
5. if $t \in Ena(M)$ then we have must $a_t(k_{M',t}) \ne 0$ and $\alpha_t(l_{M,t}) \ne 0$.

Condition (1) simply states that there must be a transition that takes the system from marking M' to marking M. As described by condition (2), if t' is not the transition that fires and it is enabled both in M and M' then the liquid describing the state of the ME distribution of transition t' flows from a container of M' to the corresponding one in M in such a way that the age (state) of the transition is maintained. The remaining three conditions have an effect also on the rate at which liquid flows from $|l_1,...,l_{K_M}|$ of marking M to $|k_1,...,k_{K_{M'}}|$ of marking M'. In particular, as condition (3) states it, if t' is not enabled in M' but it is enabled in M then liquid flows only toward those containers of M that corresponds to local containers of t' that have to be initialised to a non-zero level. The effect of t' on the rate of the flow is given by $\alpha_{t'}(l_{p_{M,t'}})$. If transition t is not enabled in marking M then it contributes to the flow, according to condition (4), only if its local container has a flow toward its fictitious container representing the termination of the activity associated with t. The associated rate is $a_t(k_{M',t})$. Condition (5) states that, if transition t is enabled in M then it contributes to the flow if its local container $k_{M',t}$ has a flow toward its fictitious container and its local container $l_{M,t}$ is with non-zero initial liquid level. The associated rate is $a_t(k_{M',t})\alpha_t(l_{M,t}) \neq 0$.

We denote by $g(M,l)$ the set of couples $(M',k), M' \neq M$ for which there is a flow from container $k = |k_1,...,k_{K_{M'}}|$ of marking M' to container $l = |l_1,...,l_{K_M}|$ of marking M.

Based on the above description we can write the change of the level of the liquid present in container $l = |l_1,...,l_{K_M}|$ of marking M as

$$\frac{v(u,M,|l_1,...,l_{K_M}|)}{du} = v(u,M,|l_1,...,l_{K_M}|)\sum_{i=1}^{K_M}\mathbf{A}_{M,i}(l_i,l_i) +$$
$$\sum_{i=1}^{K_M}\sum_{k\in f(M,l,i)} v(u,M,|k_1,...,k_{K_M}|)\mathbf{A}_{M,i}(k_i,l_i) + \tag{5}$$
$$\sum_{(M',k)\in g(M,l),M'\xrightarrow{t}M} (v(u,M',|k_1,..,k_{K_{M'}}|)\cdot a_t(k_{M',t})\cdot(\alpha_t(l_{M,t}))_{t\in Ena(M)}\cdot \prod_{\substack{t'\neq t,t'\notin Ena(M'),\\ t'\in Ena(M)}}\alpha_{t'}(l_{p_{M,t'}}))$$

where $(\alpha_t(l_{M,t}))_{t\in Ena(M)}$ gives $\alpha_t(l_{M,t})$ if $t \in Ena(M)$ and 1 otherwise. In matrix notation (5) can be written as (1), which completes the proof.

A positive consequence of Theorem 1 is that the transient behaviour of a PN with ME distributed firing times can be analysed based on similar differential equations as in case of Markovian PN models, but due to the more general structure of ME distributions the constant coefficients of the differential equation do not obey sign restrictions.

4.2 Stationary Behaviour

Let us denote the stationary solution by $w(M) = \lim_{u\to\infty} v(u,M)$.

Theorem 2 $w(M)$ *satisfies following balance equation*

$$0 = w(M)\bigoplus_{t\in Ena(M)} A_t + \sum_{t:M'\xrightarrow{t}M} w(M')\bigotimes_{t'\in T} R_{t',t}(M',M) \tag{6}$$

where $R_{t',t}(M',M)$ is defined in (2).

The proof of the theorem is obtained simply from the transient differential equation by taking the $u \to \infty$ limit. Equation (10) is a system of linear equations, where the constant matrix obtained from A_t and $R_{t',t}(M',M)$ can contain positive and negative elements at any position. The normalized solution of (10) is unique if 0 is a unique eigenvalue of the constant matrix, in this case the stationary marking distribution can be obtained as the properly normalized solution of this set of linear equations.

5 Two SPN Examples

The first net, a SPN with synchronised activities, is intended to compare Erlang and ME distributions in modeling low coefficients of variation. The basic net is shown in Fig. 5. We assume that the initial marking is given by putting 10 tokens at the places $p1$ and $p2$.

First, we consider a configuration where transition $t1$ has an exponentially distributed firing time with mean 1.0 and the transitions $t2$ and $t3$ have identically distributed ME or Erlang distributed firing times. We apply the ME distributions with 3, 7, 11 and 15 phases which have been defined above and compare them with Erlang distributions with the same number of phases. In all cases we assume that the mean firing time of the distributions is 1 and the coefficient of variation is as low as possible. Table 2 contains the number of states and the number of non-zero elements in the overall generator matrix. The number of states depends only on the number of phases and not on the non-zero structure of the matrices for the distributions. The number of non-zero elements in the resulting generating matrix depends on the number of non-zero elements in the matrix and the initial vector of the distributions. Since the ME distributions have more non-zero elements in its vector and matrix, the generator matrix becomes more dense when using ME instead of Erlang distributions.

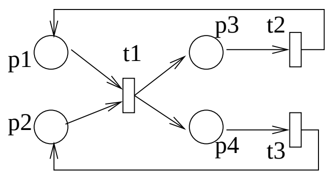

Figure 5: SPN with synchronised activities

Table 2: Number of states and non-zero elements for different number of phases for distributions of $t2$ and $t3$.

		Erlang	ME
phases	states	non zeros	non zeros
3	961	3605	8309
7	5041	19077	82077
11	12321	46741	290821
15	22801	86597	703661

As the measure of interest we consider first the token distribution in place $p3$. For deterministically distributed firing times of the transitions $t2$ and $t3$ with the same mean, the model is equivalent to a M/D/1/10 queueing model with mean inter-arrival and mean service time equal to 1. Fig. 6 and 7 contain the results for different configurations and include for comparison the results for an M/D/1/10 queue and for the same net with exponentially distributed firing times. It can be clearly seen that with the same number of phases, ME distributions approximate deterministic distributions much better than Erlang distributions do. In particular, the ME distributions with 11 and 15 phases that have a very small coefficient of variation approximate the token distribution of the model with deterministic distributions quite well, only for population 9 and 10 the probabilities are underestimated.

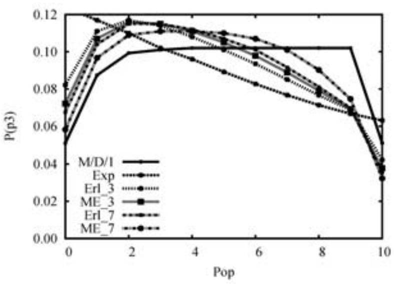

Figure 6: Token Distribution at place $p3$ for the Erlang and ME distributions with 3 and 7 phases

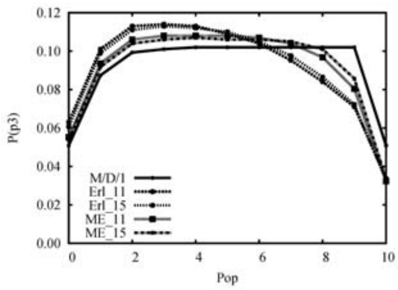

Figure 7: Token Distribution at place $p3$ for the Erlang and ME distributions with 11 and 15 phases

Additionally, we analyse the transient behaviour of the net when all transitions have ME or Erlang distributed firing times with 15 phases and mean 1. The results is shown in Fig. 8. The ME distribution results in a good approximation of a step function which would occur in a deterministic system.

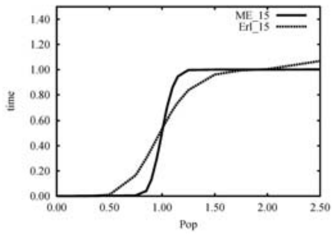

Figure 8: Transient Token Population in place $p3$

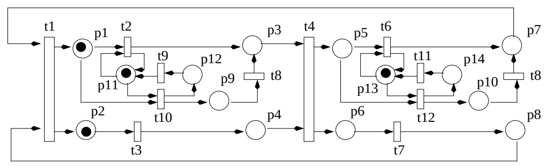

Figure 9: SPN Model of Production Cells

The second example we consider, a SPN model of production cells, is from [11] and can be seen in Fig. 9. The net describes two consecutive production cells with two types of material to be processed. The machines are subject to failures (transitions $t10$, $t12$) and repairs ($t9, t11$). For further explanations of the model we refer to [11]. We assume that the transitions $t9$, $t10$, $t11$ and $t12$ have exponentially distributed firing times with rate 1.0 for transitions $t9$ and $t11$ and rate 0.2 for transitions $t10$ and $t12$. The remaining transitions have ME firing times with mean 1.0 for $t1,\ldots,t6$ and mean 0.5 for $t7$ and $t8$.

In Fig. 10 the transient population of place $p2$ (that is the mean number of tokens at p_2) in the interval $[0,12]$ is shown for ME distributions with 3 and 15 phases. Furthermore, the net has been analysed with exponentially distributed firing times at all transitions. It can be seen that with exponentially distributed firing times for all transitions the population converges quickly towards the steady state, whereas the ME distributions show a cyclic behaviour which is smoothed by the firing of the exponential distributions $t10$ and $t12$, i.e., by failures during the processing step. For the ME distribution with 15 states the transient population contains peaks that describe cycles without failures and those with a single failure. (The probability of two failures during a cycle is so small that there is not a corresponding peak in the transient probabilities.) This behaviour is not visible if exponential distributions or with ME distributions with 3 phases are used.

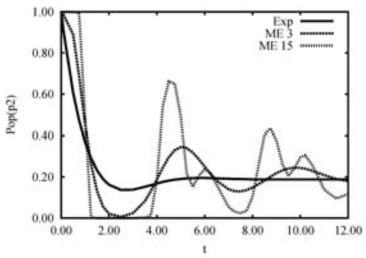

Figure 10: Transient Token Population in Place $p2$

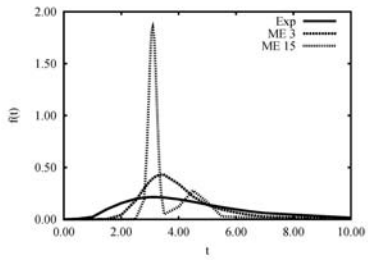

Figure 11: Sojourn Time Density of a Single Processing Step

Finally, the density of the sojourn time for a single run of the net is computed. The run starts in the initial marking and ends when both tokens reached the places $p7$ and $p8$. Fig. 11 shows the density function which is smooth for exponentially distributed firing times. For the ME distribution with 15 phases we can again observe the two modes with and without failure.

6 Conclusions and Future Work

In this paper we discussed a methodology to evaluate the transient and the stationary parameters of stochastic Petri nets with firing times whose scv is low. In particular, we proposed the use of ME distributions. We presented ME distributions with low scv and proved the widespread conjecture that SPNs with ME firing times can be solved applying the extension approach developed for SPNs with PH firing times. We demonstrated through numerical examples the benefits of using ME distributions with low scv instead of using Erlang distributions, which are the PH distributions with lowest possible scv.

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