Analysis of a two-state Markov fluid model with 2 buffers*

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Abstract. Single buffer Markov fluid models are well understood in the literature, but the extension of those results for multiple buffers is still an open research problem. In this paper we consider one of the simplest Markov fluid models (MFM) with 2 buffers of infinite capacity, where the fluid rates ensure that the fluid level of buffer 1 is never larger than fluid level of buffer 2. In spite of these restrictions, the stationary analysis is non straightforward with the available analysis tools. We provide an analysis approach based on the embedded time points at the busy-idle cycles of buffer 1.

Keywords: Markov fluid model with 2 buffers · embedded process · Laplace transform

1 Introduction

In the 1980's, the evolution of the telecommunication systems turned the attention of traffic engineers towards queueing models with "continuous" buffers, which are commonly referred to as fluid queues [2]. The associated fertile research effort resulted in many solution techniques for such queues from spectral decomposition based ones [7] to matrix analytic method based ones [8] by the 2010's, the computational methods for various performance measures of fluid models have also been enhanced [9,3].

During this evolution of single buffer fluid models and related solution techniques, many practical problems popped up where multiple fluid buffers are present, but the solution of such models is not available in general. In some special cases, e.g., when one of the buffer is allowed to be negative as well, promising analytical approaches are proposed [4,5], but results are still not available for one of the simplest fluid models, a system which has two infinite buffers whose levels restricted to be non-negative and ita fluid rates are modulated by a background Markov chain.

In order to pave the road for the analysis of fluid models with 2 buffers, in this paper, we focus on a rather simple fluid model with two infinite buffers, whose levels

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are restricted to be non-negative, the fluid rates are such that the fluid level of buffer 1 is never larger than the level of buffer 2 and the background Markov chain has two states.

The rest of the paper is organized as follows. Section 2 introduces the considered fluid model and Section 3 presents its analysis at embedded time points. Based on the embedded measures, Section 4 provides the time stationary measures. Section 5 provides a numerical example whose results are compared with simulation results. Section 6 concludes the paper.

2 Model description

We consider a system with 2 fluid buffers of infinite capacity whose fluid level is governed by a two-state background Markov chain (BMC). Let $X_1(t)$, $X_2(t)$ and $\phi(t)$ be the fluid level in buffer 1, buffer 2 and the state of the BMC, respectively. The state space of the BMC is composed of a state with positive fluid rates, $\mathcal{S}_+ = \{1\}$ and a state with negative fluid rates, $\mathcal{S}_- = \{2\}$. The generator matrix of the BMC is

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}. \tag{1}$$

The fluid accumulation is such that

$$\frac{d}{dt}X_1(t) = \frac{d}{dt}X_2(t) = 1 \text{ when } \phi(t) \in \mathcal{S}_+$$

and for $i = \{1, 2\}$

$$\frac{d}{dt}X_i(t) = -r_i < 0 \text{ when } \phi(t) \in \mathcal{S}_- \ \&\ X_i(t) > 0 \quad \text{and}$$
$$\frac{d}{dt}X_i(t) = 0 \text{ when } \phi(t) \in \mathcal{S}_- \ \&\ X_i(t) = 0$$

where $r_1 > r_2 > 0$. As a consequence $X_1(t) \leq X_2(t)$ holds for $\forall t > 0$ if $X_1(0) \leq X_2(0)$, which we assume in the paper. A trajectory of the system evolution is depicted in Figure 1.

Our final goal is to compute the following time stationary measures

$$\begin{aligned}
\tilde{W}_+(x,y) &= \lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_+, X_1(t) < x, X_2(t) < y) \\
\tilde{W}_-(x,y) &= \lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_-, X_1(t) < x, X_2(t) < y), \\
\tilde{V}(x) &= \lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_+, X_1(t) = X_2(t) < x), \\
\tilde{U}(y) &= \lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_-, X_1(t) = 0, X_2(t) < y), \\
P &= \lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_-, X_1(t) = X_2(t) = 0),
\end{aligned}$$

and $W_+(x,y) = \frac{d}{dx}\frac{d}{dy}\tilde{W}_+(x,y)$, $W_-(x,y) = \frac{d}{dx}\frac{d}{dy}\tilde{W}_-(x,y)$, $V(x,y) = \frac{d}{dx}\tilde{V}(x)$, $U(x,y) = \frac{d}{dy}\tilde{U}(y)$, for $x, y > 0$. Due to the sign of the fluid rates the other measures are zero, e.g., $\lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_+, X_1(t) = 0, X_2(t) = 0) = 0$.

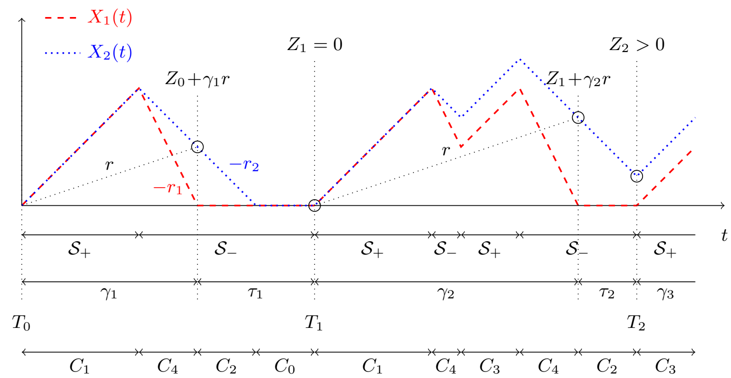


Fig. 1: Evolution of the buffer contents during the busy-idle cycles of buffer 1

3 Performance analysis of the fluid model

3.1 Utilization of the buffers and stability condition

The stationary probabilities of the BMC are $\lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_+) = \frac{\mu}{\lambda+\mu}$ and $\lim_{t\to\infty} Pr(\phi(t) \in \mathcal{S}_-) = \frac{\lambda}{\lambda+\mu}$.

Assuming $X_1(0) = 0$, $\phi(0) \in \mathcal{S}_+$, we define the length of the busy period of buffer 1 (while the buffer is non-empty) as $\gamma = \min(t > 0 : X_1(t) = 0)$.

Theorem 1. *During the γ long busy period of buffer 1 the time spent in $\mathcal{S}_+$ and $\mathcal{S}_-$ are $\frac{r_1\gamma}{1+r_1}$ and $\frac{\gamma}{1+r_1}$.*

Proof. Since $X_1(0) = X_1(\gamma) = 0$, the fluid increase during the sojourn in $\mathcal{S}_+$ equals to the fluid decrease during the sojourn in $\mathcal{S}_-$ and $\frac{r_1\gamma}{1+r_1}(1) = \frac{\gamma}{1+r_1}(r_1)$.

The utilization of the buffer 1 is the stationary probability that the buffer is non-empty, that is $\rho_1 = \lim_{t\to\infty} Pr(X_1(t) > 0)$.

Theorem 2. *The utilization of the buffer 1 is $\rho_1 = \frac{\mu(1+\frac{1}{r_1})}{\lambda+\mu}$.*

Proof. Let $X_1(0) = 0$, $T > 0$ and $\check{T} = \max(t < T : X_1(t) = 0)$. For a stable queue, $\lim_{T\to\infty} \frac{\check{T}}{T} = 1$ and this way

$$\lim_{T\to\infty} \frac{E\left(\int_{t=0}^{T} \mathcal{I}_{\{X_1(t)>0\}} dt\right)}{T} = \lim_{T\to\infty} \frac{E\left(\int_{t=0}^{\check{T}} \mathcal{I}_{\{X_1(t)>0\}} dt\right)}{\check{T}}$$

$$= \lim_{T\to\infty} \frac{E\left(\int_{t=0}^{\check{T}} \mathcal{I}_{\{X_1(t)>0,\phi(t)\in\mathcal{S}_+\}} dt\right)}{\check{T}} + \lim_{T\to\infty} \frac{E\left(\int_{t=0}^{\check{T}} \mathcal{I}_{\{X_1(t)>0,\phi(t)\in\mathcal{S}_-\}} dt\right)}{\check{T}}$$

$$= \lim_{T\to\infty} \frac{E\left(\int_{t=0}^{\check{T}} \mathcal{I}_{\{\phi(t)\in\mathcal{S}_+\}} dt\right)}{\check{T}} + \frac{1}{r_1} \lim_{T\to\infty} \frac{E\left(\int_{t=0}^{\check{T}} \mathcal{I}_{\{\phi(t)\in\mathcal{S}_+\}} dt\right)}{\check{T}} = \frac{\mu}{\lambda+\mu}\left(1+\frac{1}{r_1}\right)$$

where $\mathcal{I}_{\{A\}}$ is the indicator of event A (i.e., $\mathcal{I}_{\{A\}} = 1$ only when A is true) and $\int_{t=0}^{T} \mathcal{I}_{\{X_1(t)>0\}} dt$ denotes the time while $X_1(t) > 0$ in the $(0, T)$ interval. In the first term of the second step, we utilized that the fluid buffer is always non-empty when $\phi(t) \in \mathcal{S}_+$. In the second term of the second step, similar to Theorem 1, we utilized the fact that the fluid increase equals to the fluid decrease in $(0, \check{T})$. In the last step, we utilized that the stationary probability of state 1 is $\frac{\mu}{\lambda+\mu}$.

Similarly, the utilization of the buffer 2 is $\rho_2 = \lim_{t\to\infty} Pr(X_2(t) > 0) = \frac{\mu(1+\frac{1}{r_2})}{\lambda+\mu}$. Since $r_1 > r_2$, $\rho_1 < \rho_2$. The stability of both buffers are ensured when $\rho_2 < 1$, i.e., $\lambda r_2 > \mu$. We assume stable buffers along this paper.

Due to the $X_1(t) \leq X_2(t)$ inequality, for the joint stationary probabilities we have

$$P = \lim_{t\to\infty} Pr(X_1(t) = 0, X_2(t) = 0) = \lim_{t\to\infty} Pr(X_2(t) = 0) = 1 - \rho_2, \tag{2}$$

$$\lim_{t\to\infty} Pr(X_1(t) > 0, X_2(t) = 0) = 0, \tag{3}$$

$$\int_0^\infty U(y)dy = \lim_{t\to\infty} Pr(X_1(t) = 0, X_2(t) > 0) = \rho_2 - \rho_1, \tag{4}$$

$$\begin{aligned}\int_0^\infty \int_0^\infty W_+(x, y) + W_-(x, y)dydx &= \lim_{t\to\infty} Pr(X_1(t) > 0, X_2(t) > 0) \\ &= \lim_{t\to\infty} Pr(X_1(t) > 0) = \rho_1.\end{aligned} \tag{5}$$

That is, one of the performance measures of interest, P, is given by $P = 1-\rho_2$. Further more, from Theorem 1 we have

$$\begin{aligned}\int_0^\infty \int_0^\infty W_+(x, y)dydx &= \lim_{t\to\infty} Pr(X_1(t) > 0, X_2(t) > 0, \phi(t) \in \mathcal{S}_+) \\ &= \rho_1 \frac{r_1}{1 + r_1} = \frac{\mu}{\lambda + \mu},\end{aligned} \tag{6}$$

$$\begin{aligned}\int_0^\infty \int_0^\infty W_-(x, y)dydx &= \lim_{t\to\infty} Pr(X_1(t) > 0, X_2(t) > 0, \phi(t) \in \mathcal{S}_-) \\ &= \rho_1 \frac{1}{1 + r_1} = \frac{\mu}{r_1(\lambda + \mu)}.\end{aligned} \tag{7}$$

For the analysis of the remaining measures of interest we apply an embedded process based approach.

3.2 Analysis of embedded time points

Let $X_1(0) = 0$, $X_2(0) = x$, $\phi(0) \in \mathcal{S}_+$, $\gamma = \min(t > 0 : X_1(t) = 0)$.

In $(0, \gamma)$, the time spent in $\mathcal{S}_+$ and $\mathcal{S}_-$ are $\frac{r_1\gamma}{1+r_1}$ and $\frac{\gamma}{1+r_1}$ (because the fluid inflow equals the fluid out flow during $(0, \gamma)$ and consequently, the ratio of time spent in $\mathcal{S}_+$ and $\mathcal{S}_-$ is $r_1 : 1$). This way

$$X_2(\gamma) = x + (1)\frac{r_1\gamma}{1 + r_1} + (-r_2)\frac{\gamma}{1 + r_1} = x + \frac{\gamma(r_1 - r_2)}{1 + r_1} = x + \gamma r, \tag{8}$$

where $r = \frac{r_1 - r_2}{1+r_1}$. Intuitively, r is the increase rate of buffer 2 during the busy period of buffer 1, as it is depicted in Figure 1.

After time γ, buffer 1 remains idle for a random amount of time denoted by τ.

During the idle period of buffer 1, $X_2(t)$ decreases with rate r_2. At the end of the idle period of buffer 1, $X_2(\gamma + \tau) = \max\left(0, x + \frac{\gamma(r_1 - r_2)}{1+r_1} - r_2\tau\right)$.

Let $T_0 = 0, T_1, \ldots$ be the beginning of the busy-idle cycles of buffer 1 and $Z_n = X_2(T_n)$ the fluid level of buffer 2 in those instances. In Figure 1, the first busy-idle cycle of buffer 1 is such that $Z_1 = 0$ and the second one is such that $Z_2 > 0$.

For Z_n we have

$$\begin{aligned} Z_{n+1} &= \max\left(0, Z_n + \frac{\gamma_n(r_1 - r_2)}{1 + r_1} - r_2\tau_n\right) \\ &= \max\left(0, Z_n + r\gamma_n - r_2\tau_n\right) \end{aligned} \tag{9}$$

We are interested in the stationary behaviour of $Z = \lim_{n\to\infty} Z_n$. Let $p = Pr(Z = 0)$, $\tilde{Z}(x) = Pr(Z < x)$ and $Z(x) = \frac{d}{dx}\tilde{Z}(x)$ for $x > 0$, and we also introduce the Laplace transform of $Z(x)$, $Z^*(s) = \int_{x=0}^{\infty} e^{-sx} Z(x) dx$. For stable fluid models these quantities satisfy the normalizing equation

$$p + \int_{u=0}^{\infty} Z(u) du = p + Z^*(0) = 1. \tag{10}$$

At this point, we would like to emphasize the difference between the time stationary and embedded stationary measures, that is

$$\begin{aligned} &\lim_{t\to\infty} Pr(X_1(t) = 0, X_2(t) = 0) = 1 - \rho_2 \\ &\neq \lim_{n\to\infty} Pr(X_1(T_n) = 0, X_2(T_n) = 0) = \lim_{n\to\infty} Pr(Z_n = 0) = p. \end{aligned}$$

According to the behaviour of the BMC in (1), τ is exponentially distributed with rate μ. The return measure of buffer 1,

$$\Psi(t) = \frac{d}{dt} Pr(\gamma < t | X_1(0) = 0),$$

is available in LT domain. With our BMC (commonly the related expressions are given in matrix from in the literature, e.g. in [1], which we specify here according to (1)) and fluid rates, $\Psi^*(s) \triangleq E(e^{-s\gamma}) = \int_0^\infty e^{-st}\Psi(t)dt$ is

$$\Psi^*(s) = \frac{s(r_1 + 1) + r_1\lambda + \mu - \sqrt{(s(r_1 + 1) + r_1\lambda + \mu)^2 - 4r_1\lambda\mu}}{2\mu}, \tag{11}$$

and $E(\gamma) = -\Psi^{*'}(0) = \frac{1+r_1}{\lambda r_1 - \mu}$.

Theorem 3. *At the beginning of stationary busy period of buffer 1 the buffer content in buffer 2 is characterized by*

$$Z^*(s) = \frac{p\mu\left(1 - \Psi^*(rs)\right)}{r_2 s - \mu + \mu\Psi^*(rs)}. \tag{12}$$

Proof. Considering that the density of γ is $\Psi(x)$ and the density and the CDF of τ are $f(x)$ and $F(x)$, from (9) we have

$$Z(x) = p\int_{y=x/r}^{\infty} \Psi(y)\frac{1}{r_2}f\left(\frac{ry-x}{r_2}\right)dy \\ + \int_{u=0}^{\infty} Z(u)\int_{y=(x-u)^+/r}^{\infty} \Psi(y)\frac{1}{r_2}f\left(\frac{u+ry-x}{r_2}\right)dydu \tag{13}$$

$$p = p\int_{y=0}^{\infty} \Psi(y)\left(1-F\left(\frac{ry}{r_2}\right)\right)dy \\ + \int_{u=0}^{\infty} Z(u)\int_{y=0}^{\infty} \Psi(y)\left(1-F\left(\frac{u+ry}{r_2}\right)\right)dydu. \tag{14}$$

Considering $f(x) = \mu e^{-\mu x}$ and $F(x) = e^{-\mu x}$, from (16) we have

$$p = p\int_{y=0}^{\infty} \Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)dy + \int_{u=0}^{\infty} Z(u)\int_{y=0}^{\infty} \Psi(y)\exp\left(-\mu\frac{u+ry}{r_2}\right)dydu$$
$$= p\int_{y=0}^{\infty} \Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)dy \\ + \int_{u=0}^{\infty} Z(u)\exp\left(-\mu\frac{u}{r_2}\right)\int_{y=0}^{\infty} \Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)dydu$$
$$= p\Psi^*\left(\frac{r\mu}{r_2}\right) + Z^*\left(\frac{\mu}{r_2}\right)\Psi^*\left(\frac{r\mu}{r_2}\right) = \left(p + Z^*\left(\frac{\mu}{r_2}\right)\right)\Psi^*\left(\frac{r\mu}{r_2}\right)$$

That is

$$p = \left(p + Z^*\left(\frac{\mu}{r_2}\right)\right)\Psi^*\left(\frac{r\mu}{r_2}\right)$$

and

$$Z^*\left(\frac{\mu}{r_2}\right) = \frac{p\left(1-\Psi^*\left(\frac{r\mu}{r_2}\right)\right)}{\Psi^*\left(\frac{r\mu}{r_2}\right)} \tag{15}$$

Similarly, from (13) we have

$$Z(x) = p\frac{\mu}{r_2}\int_{y=x/r}^{\infty} \Psi(y)\exp\left(-\mu\frac{ry-x}{r_2}\right)dy \\ + \frac{\mu}{r_2}\int_{u=0}^{\infty} Z(u)\int_{y=(x-u)^+/r}^{\infty} \Psi(y)\exp\left(-\mu\frac{u+ry-x}{r_2}\right)dydu \\ = p\frac{\mu}{r_2}\int_{y=x/r}^{\infty} \Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)\exp\left(\mu\frac{x}{r_2}\right)dy \\ + \frac{\mu}{r_2}\int_{u=0}^{\infty} Z(u)\int_{y=(x-u)^+/r}^{\infty} \Psi(y)\exp\left(-\mu\frac{u}{r_2}\right)\exp\left(-\mu\frac{ry}{r_2}\right)\exp\left(\mu\frac{x}{r_2}\right)dydu$$

A multiplying both sides with e^{-sx} and integrating from 0 to ∞ gives

$$\begin{aligned}
Z^*(s) =\ & p\frac{\mu}{r_2}\int_{x=0}^{\infty} e^{-sx}\int_{y=x/r}^{\infty}\Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)\exp\left(\mu\frac{x}{r_2}\right)dydx\\
&+\frac{\mu}{r_2}\int_{x=0}^{\infty} e^{-sx}\int_{u=0}^{\infty}Z(u)\int_{y=(x-u)^+/r}^{\infty}\Psi(y)\exp\left(-\mu\frac{u+ry-x}{r_2}\right)dydudx\\
&= V_1^*(s)+V_2^*(s).
\end{aligned}$$

Using $\int_{x=0}^{\infty}\int_{y=x/r}^{\infty}\bullet dydx = \int_{y=0}^{\infty}\int_{x=0}^{ry}\bullet dxdy$, for the first term we have

$$\begin{aligned}
V_1^*(s) &= p\frac{\mu}{r_2}\int_{y=0}^{\infty}\Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)\underbrace{\int_{x=0}^{ry}\exp\left(\mu\frac{x}{r_2}-sx\right)dx}\,dy\\
&= p\frac{\mu}{r_2}\int_{y=0}^{\infty}\Psi(y)\exp\left(-\mu\frac{ry}{r_2}\right)\underbrace{\frac{r_2}{r_2s-\mu}\left(1-\exp\left(-y\frac{r(r_2s-\mu)}{r_2}\right)\right)}dy\\
&=\frac{p\mu}{r_2s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)-\Psi^*\left(\frac{r\mu}{r_2}+\frac{r(r_2s-\mu)}{r_2}\right)\right)\\
&=\frac{p\mu}{r_2s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)-\Psi^*(rs)\right).
\end{aligned}$$

For the second term we first refine the integrals

$$\begin{aligned}
&\int_{x=0}^{\infty}\int_{u=0}^{\infty}\int_{y=(x-u)^+/r}^{\infty}\bullet\, dydudx = \int_{u=0}^{\infty}\int_{x=0}^{\infty}\int_{y=(x-u)^+/r}^{\infty}\bullet\, dydxdu\\
&=\int_{u=0}^{\infty}\int_{x=0}^{u}\int_{y=0}^{\infty}\bullet\, dydxdu+\int_{u=0}^{\infty}\int_{x=u}^{\infty}\int_{y=(x-u)/r}^{\infty}\bullet\, dydxdu\\
&=\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{x=0}^{u}\bullet\, dxdudy+\int_{u=0}^{\infty}\int_{y=0}^{\infty}\int_{x=u}^{u+yr}\bullet\, dxdydu\\
&=\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{x=0}^{u}\bullet\, dxdudy+\int_{y=0}^{\infty}\int_{u=0}^{\infty}\int_{x=u}^{u+yr}\bullet\, dxdudy
\end{aligned}$$

Based on this, the second term is

$$
\begin{aligned}
V_2^*(s) = {} & \frac{\mu}{r_2} \int_{y=0}^{\infty} \Psi(y) \exp\left(-y\frac{r\mu}{r_2}\right) \int_{u=0}^{\infty} Z(u) \exp\left(-u\frac{\mu}{r_2}\right) \underbrace{\int_{x=0}^{u} \exp\left(-x\left(s-\frac{\mu}{r_2}\right)\right) dx}\, dudy \\
& + \frac{\mu}{r_2} \int_{y=0}^{\infty} \Psi(y) \exp\left(-y\frac{r\mu}{r_2}\right) \int_{u=0}^{\infty} Z(u) \exp\left(-u\frac{\mu}{r_2}\right) \underbrace{\int_{x=u}^{u+yr} \exp\left(-x\left(s-\frac{\mu}{r_2}\right)\right) dx}\, dudy \\
& = \frac{\mu}{r_2} \int_{y=0}^{\infty} \Psi(y) \exp\left(-y\frac{r\mu}{r_2}\right) \int_{u=0}^{\infty} Z(u) \exp\left(-u\frac{\mu}{r_2}\right) \underbrace{\frac{r_2}{r_2 s-\mu}\left(1-\exp\left(-u\left(s-\frac{\mu}{r_2}\right)\right)\right)} dudy \\
& + \frac{\mu}{r_2} \int_{y=0}^{\infty} \Psi(y) \exp\left(-y\frac{r\mu}{r_2}\right) \int_{u=0}^{\infty} Z(u) \exp\left(-u\frac{\mu}{r_2}\right) \\
& \quad \cdot \underbrace{\frac{r_2}{r_2 s-\mu}\left(\exp\left(-u\left(s-\frac{\mu}{r_2}\right)\right)-\exp\left(-u\left(s-\frac{\mu}{r_2}\right)-yr\left(s-\frac{\mu}{r_2}\right)\right)\right)}_{\frac{r_2}{r_2 s-\mu}\exp(-us)\exp(u\frac{\mu}{r_2})\left(1-\exp(-yrs)\exp\left(yr\frac{\mu}{r_2}\right)\right)} dudy \\
& = \frac{\mu}{r_2 s-\mu}\Psi^*\left(\frac{r\mu}{r_2}\right)\left(Z^*\left(\frac{\mu}{r_2}\right)-Z^*(s)\right) \\
& + \frac{\mu}{r_2 s-\mu} \int_{y=0}^{\infty} \Psi(y) \exp\left(-y\frac{r\mu}{r_2}\right) \int_{u=0}^{\infty} Z(u) \exp(-us)\, dudy \\
& - \frac{\mu}{r_2 s-\mu} \int_{y=0}^{\infty} \Psi(y) \exp(-yrs) \int_{u=0}^{\infty} Z(u) \exp(-us)\, dudy \\
= {} & \frac{\mu}{r_2 s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)\left(Z^*\left(\frac{\mu}{r_2}\right)-Z^*(s)\right)+\Psi^*\left(\frac{r\mu}{r_2}\right)Z^*(s)-\Psi^*(sr)\,Z^*(s)\right) \\
& = \frac{\mu}{r_2 s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)Z^*\left(\frac{\mu}{r_2}\right)-\Psi^*(sr)\,Z^*(s)\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
Z^*(s) = {} & \frac{p\mu}{r_2 s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)-\Psi^*(rs)\right) \\
& + \frac{\mu}{r_2 s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)Z^*\left(\frac{\mu}{r_2}\right)-\Psi^*(rs)\,Z^*(s)\right),
\end{aligned}
$$

Using (15)

$$
\begin{aligned}
Z^*(s) = {} & \frac{p\mu}{r_2 s-\mu}\left(\Psi^*\left(\frac{r\mu}{r_2}\right)-\Psi^*(rs)\right)+\frac{\mu}{r_2 s-\mu}\left(p\left(1-\Psi^*\left(\frac{r\mu}{r_2}\right)\right)-\Psi^*(rs)\,Z^*(s)\right) \\
= {} & \frac{p\mu}{r_2 s-\mu}\left(-\Psi^*(rs)\right)+\frac{\mu}{r_2 s-\mu}\left(p-\Psi^*(rs)\,Z^*(s)\right) \\
= {} & \frac{p\mu}{r_2 s-\mu}\left(1-\Psi^*(rs)\right)-\frac{\mu}{r_2 s-\mu}Z^*(s)\,\Psi^*(rs) \\
= {} & \frac{\mu}{r_2 s-\mu}\left(p-\bar{Z}^*(s)\Psi^*(rs)\right)
\end{aligned}
$$

where $\bar{Z}^*(s) = p + Z^*(s)$.

$$
\begin{aligned}
\bar{Z}^*(s) &= p + \frac{\mu}{\mu - r_2 s}\left(-p+\bar{Z}^*(s)\Psi^*(rs)\right) \\
&= \left(1-\frac{\mu}{\mu - r_2 s}\right)p + \frac{\mu}{\mu - r_2 s}\bar{Z}^*(s)\Psi^*(rs)
\end{aligned}
$$

which results in (12).

Corollary 1. *The only unknown in* (12) *can be obtained as*

$$p = 1 - \frac{\mu(r_1 - r_2)}{r_2(\lambda r_1 - \mu)} \tag{16}$$

Proof. Since both, the numerator and the denominator of (12) are zero at $s = 0$, we use the L'Hopital's rule to obtain $Z^*(0)$

$$Z^*(0) \triangleq \lim_{s\to 0} Z^*(s) = \frac{\frac{d}{ds} p\mu\left(1 - \Psi^*(rs)\right)\Big|_{s\to 0}}{\frac{d}{ds} r_2 s - \mu + \mu\Psi^*(rs)\Big|_{s\to 0}} = \frac{-p\mu r \Psi^{*'}(0)}{r_2 + \mu r\Psi^{*'}(0)}. \tag{17}$$

Using $1-p = Z^*(0)$ from (10), $r = \frac{r_1 - r_2}{1+r_1}$ and $E(\gamma) = -\Psi^{*'}(0) = \frac{1+r_1}{\lambda r_1 - \mu}$, we further have

$$1 - p = \frac{-p\mu r\Psi^{*'}(0)}{r_2 + \mu r\Psi^{*'}(0)} = \frac{p\mu(r_1 - r_2)}{r_2(\lambda r_1 - \mu) - \mu(r_1 - r_2)},$$

from which the corollary comes.

4 Time stationary behaviour

We aim to obtain the remaining time stationary measures based on the embedded stationary measure $Z(x)$. For the joint distribution of the buffers we have the following cases:

C0) $X_1(t) = X_2(t) = 0, \phi(t) \in \mathcal{S}_-$: this case can be obtained from the stationary analysis of buffer 2 in isolation. The associated stationary measure is P.
C1) $X_1(t) = X_2(t) > 0, \phi(t) \in \mathcal{S}_+$: this case can arise after an idle period of buffer 2. The associated stationary measure is $V(x)$.
C2) $X_1(t) = 0, X_2(t) > 0, \phi(t) \in \mathcal{S}_-$: this case can arise during the idle period of buffer 1. The associated stationary measure is $U(y)$.
C3) $X_2(t) \geq X_1(t) > 0, \phi(t) \in \mathcal{S}_+$: this case can arise during the busy period of buffer 1. The associated stationary measure is $W_+(x, y)$.
C4) $X_2(t) \geq X_1(t) > 0, \phi(t) \in \mathcal{S}_-$: this case can arise during the busy period of buffer 1. The associated stationary measure is $W_-(x, y)$.

4.1 Analysis of C1)

As it is demonstrated in Figure1, case C1) can occur only when $\phi(t) \in \mathcal{S}_+$, in those busy-idle cycles of buffer 1 where buffer 2 is idle at the beginning of the cycle. According to the ergodicity of the model

$$\tilde{V}(x) = \frac{E\left(\int_{t=0}^{\gamma} \mathcal{I}_{\{X_2(t)=X_1(t)<x\}} dt\right)}{E\left(\gamma + \tau\right)}, \tag{18}$$

where $E(\gamma+\tau) = E(\gamma) + E(\tau) = \frac{1+r_1}{\lambda r_1 - \mu} + \frac{1}{\mu}$.

We compute the numerator of the right hand side from the following conditional relation. Let Φ be the sojourn time during the first visit in $\mathcal{S}_+$ in a busy-idle cycle of buffer 1. If $\Phi = y$ and $X_2(0) = 0$, we have

$$\int_{t=0}^{\gamma} \mathcal{I}_{\{X_2(t)=X_1(t)<x\}} dt = \begin{cases} y & \text{if } y < x \\ x & \text{if } y > x \end{cases}$$

Let $F_\Phi(y) = \Pr(\Phi < y) = 1 - e^{-\lambda y}$ be the CDF of Φ and $\Pr(X_2(0) = 0) = p$. Then

$$E\left(\int_{t=0}^{\gamma} \mathcal{I}_{\{X_2(t)=X_1(t)<x\}} dt\right) = p\int_{y=0}^{x} y dF_\Phi(y) + p\int_{y=x}^{\infty} x dF_\Phi(y) \tag{19}$$

$$= p\int_{y=0}^{x} y\lambda e^{-\lambda y} dy + px\int_{y=x}^{\infty} \lambda e^{-\lambda y} dy = \frac{p(1-e^{-\lambda x})}{\lambda}, \tag{20}$$

from which

$$\tilde{V}(x) = \frac{E\left(\int_{t=0}^{\gamma} \mathcal{I}_{\{X_2(t)=X_1(t)<x\}} dt\right)}{E(\gamma+\tau)} = \frac{p(1-e^{-\lambda x})}{\lambda E(\gamma+\tau)}, \tag{21}$$

and $V(x) = \frac{d}{dx}\tilde{V}(x) = \frac{pe^{-\lambda x}}{E(\gamma+\tau)}$.

4.2 Analysis of C2)

According to Figure1, case C2) can occur only when $\phi(t) \in \mathcal{S}_-$. We recall from (9) that $Z = \max(0, Z + r\gamma - r_2\tau)$, where $r = \frac{r_1 - r_2}{1+r_1}$, γ is the busy time of buffer 1 and τ is the idle time of buffer 1 in $\mathcal{S}_-$.

Similar to case C1), we compute $\tilde{U}(x)$ based on the analysis of the stationary busy-idle cycle of buffer 1 using the ergodicity property.

$$1 - \tilde{U}(x) = \frac{E\left(\int_{t=\gamma}^{\gamma+\tau} \mathcal{I}_{\{X_2(t)>x, X_1(t)=0\}} dt\right)}{E(\gamma+\tau)}. \tag{22}$$

The numerator of the right hand side is obtained from the following conditional relation. If $\tau = h, Z + r\gamma = y$ then

$$\int_{t=\gamma}^{\gamma+\tau} \mathcal{I}_{\{X_2(t)>x, X_1(t)=0\}} dt = \int_{t=0}^{\tau} \mathcal{I}_{\{X_2(t)>x, X_1(t)=0 | X_2(0)=y, X_1(0)=0\}} dt$$

$$= \begin{cases} 0 & \text{if } y < x, \\ \frac{y-x}{r_2} & \text{if } x < y < x + hr_2, \\ h & \text{if } y > x + hr_2. \end{cases} \tag{23}$$

Let $G(y) = \Pr(Z + r\gamma < y)$ and $T(h) = \Pr(\tau < h) = 1 - e^{-\mu h}$ be the CDF of $Z + r\gamma$ and τ, respectively. Than

$$E\left(\int_{t=\gamma}^{\gamma+\tau} \mathcal{I}_{\{X_2(t)>x, X_1(t)=0\}} dt\right) \tag{24}$$

$$= \int_{h=0}^{\infty} \int_{y=0}^{\infty} E\left(\int_{t=\gamma}^{\gamma+\tau} \mathcal{I}_{\{X_2(t)>x, X_1(t)=0\}} dt | \tau = h, Z + r\gamma = y\right) dG(y) dT(h) \tag{25}$$

$$= \int_{h=0}^{\infty} \int_{y=x}^{x+r_2 h} \frac{y-x}{r_2} dG(y) dT(h) + \int_{h=0}^{\infty} h \underbrace{\int_{y=x+hr_2}^{\infty} dG(y)}_{1-G(x+hr_2)} dT(h) \tag{26}$$

$$= \int_{h=0}^{\infty} \int_{y=0}^{r_2 h} \frac{y}{r_2} dG(y+x) \mu e^{-\mu h} dh + \int_{h=0}^{\infty} (1 - G(x + hr_2)) h \mu e^{-\mu h} dh \tag{27}$$

$$= \int_{y=0}^{\infty} \frac{y}{r_2} \underbrace{\int_{h=y/r_2}^{\infty} \mu e^{-\mu h} dh}_{e^{-\mu y/r_2}} dG(y+x) + \int_{h=0}^{\infty} (1 - G(x + hr_2)) h \mu e^{-\mu h} dh \tag{28}$$

$$= \int_{y=0}^{\infty} \frac{y}{r_2} e^{-\mu y/r_2} G'(y+x) dy + \frac{1}{\mu} - \int_{h=0}^{\infty} G(x + hr_2) h \mu e^{-\mu h} dh \tag{29}$$

Finally,

$$\begin{aligned} U(x) &= -\frac{d}{dx} \frac{\int_{y=0}^{\infty} \frac{y}{r_2} e^{-\mu y/r_2} G'(y+x) dy + \frac{1}{\mu} - \int_{h=0}^{\infty} G(x + hr_2) h \mu e^{-\mu h} dh}{E(\gamma + \tau)} \\ &= \frac{\int_{h=0}^{\infty} G'(x + hr_2) h \mu e^{-\mu h} dh - \int_{y=0}^{\infty} \frac{y}{r_2} e^{-\mu y/r_2} G''(y+x) dy}{E(\gamma + \tau)}. \end{aligned} \tag{30}$$

For $G(x)$, we have

$$\begin{aligned} G^*(s) &\triangleq \int_{x=0}^{\infty} e^{-sx} dG(x) = \int_{x=0}^{\infty} e^{-sx} G'(x) dx \\ &= E(e^{-s(Z+r\gamma)}) = E(e^{-sZ}) E(e^{-sr\gamma}) = (p + Z^*(s)) \Psi^*(sr). \end{aligned} \tag{31}$$

and $\int_{x=0}^{\infty} e^{-sx} G''(x) dx = sG^*(s) - G'(0)$, where $G'(0) = \lim_{s\to\infty} sG^*(s) = \frac{\lambda p r_1}{r_1 - r_2}$. Since $G'(x)$ and $G''(x)$ are given only in Laplace transform domain, a numerical inverse Laplace transformation (NILT) is required (e.g., using [6]) to compute $U(x)$.

4.3 Analysis of case C3) and C4)

Theorem 4. *If $X_1(0) = 0$, $\phi(0) = \mathcal{S}_+$ and $t < \gamma$ then*

$$X_2(t) = X_2(0) + tr + X_1(t)(1 - r),$$

where $r = \frac{r_1 - r_2}{1 + r_1}$.

Proof. If $X_1(0) = 0, \phi(0) = \mathcal{S}_+$ and $X_1(t) = x$ for $t < \gamma$ then the process spent $\frac{r_1 t + x}{r_1+1}$ time in $\mathcal{S}_+$ and $\frac{t-x}{r_1+1}$ time in $\mathcal{S}_-$ in the $(0, t)$ time interval, since $\frac{r_1 t+x}{r_1+1} + \frac{t-x}{r_1+1} = t$ and

$$X_1(t) = \underbrace{0}_{\text{initial fluid level}} + \underbrace{\frac{r_1 t + x}{r_1 + 1}(1)}_{\text{fluid increase}} + \underbrace{\frac{t - x}{r_1 + 1}(-r_1)}_{\text{fluid decrease}} = x.$$

In the mean time, buffer 2 increases with $\frac{r_1 t+x}{r_1+1} \times 1$ and decreases with $\frac{t-x}{r_1+1} \times r_2$, that is

$$X_2(t) = X_2(0) + \frac{r_1 t + x}{r_1 + 1} - \frac{t - x}{r_1 + 1} r_2 = X_2(0) + tr + x(1 - r)$$

For $\circ \in \{+, -\}$, let the transient probabilities of buffer 1 during the first busy period be defined as

$$\Theta_\circ(t, x) = Pr(\phi(t) = \mathcal{S}_\circ, X_1(t) < x, t < \gamma | X_1(0) = 0, \phi(0) = \mathcal{S}_+),$$

and $\theta_\circ(t, x) = \frac{d}{dx}\Theta_\circ(t, x)$. Their Laplace transforms, $\theta^*_\circ(s, x) = \int_0^\infty e^{-st}\theta_\circ(t, x)dt$ are available in LT domain as

$$\theta^*_+(s, x) = e^{xK(s)} \quad \text{and} \quad \theta^*_-(s, x) = e^{xK(s)}\Psi(s), \tag{32}$$

where $K(s) = -\lambda - s + \mu\Psi(s)$. Similar to (11), $\theta^*_+(s, x)$, $\theta^*_-(s, x)$, and $K(s)$ are obtained from the matrix expressions of those fluid measures, e.g. in [1], which are specified here according to (1).

Applying the ergodic property again we have

$$\tilde{W}_+(x, y) = \frac{E\left(\int_{t=0}^{\gamma} \mathcal{I}_{\{X_1(t)<x, X_2(t)<y, \phi(t)\in\mathcal{S}_+\}} dt\right)}{E\left(\gamma + \tau\right)}. \tag{33}$$

The numerator of the right hand side is obtained from the following conditional relation. If $X_1(0) = 0$, $\phi(0) = \mathcal{S}_+$ and $t < \gamma$ then, according to Theorem 4, $X_2(t) = z + tr + X_1(t)(1 - r)$. This way,

$$\mathcal{I}_{\{X_1(t)<x, X_2(t)<y\}} = \begin{cases} 1 & \text{if } X_1(t) < \min(x, \frac{y-z-tr}{1-r}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $t^*(z) = \max\left(0, \frac{y-z-(1-r)x}{r}\right)$ then, for $t > 0$

$$\min\left(x, \frac{y - z - tr}{1 - r}\right) = \begin{cases} x & \text{if } t \le t^*(z), \\ \frac{y-z-tr}{1-r} & \text{if } t^*(z) < t. \end{cases}$$

Using this and $\tilde{Z}(x) = \Pr(Z < x)$, we write

$$
\begin{aligned}
&E\left(\int_{t=0}^{\gamma} \mathcal{I}_{\{X_1(t)<x, X_2(t)<y, \phi(t)\in\mathcal{S}_+\}} dt\right) \qquad (34)\\
&= \int_{t=0}^{\infty} \Pr(X_1(t) < x, X_2(t) < y, \phi(t) \in \mathcal{S}_+, t < \gamma) dt\\
&= \int_{z=0}^{\infty} \int_{t=0}^{t^*(z)} \Pr(X_1(t) < x, \phi(t) \in \mathcal{S}_+, t < \gamma) dt d\tilde{Z}(z)\\
&\qquad + \int_{z=0}^{\infty} \int_{t=t^*(z)}^{\infty} \Pr\left(X_1(t) < \frac{y - z - tr}{1 - r}, \phi(t) \in \mathcal{S}_+, t < \gamma\right) dt d\tilde{Z}(z)\\
&= \int_{z=0}^{\infty} \int_{t=0}^{t^*(z)} \Theta_+(t, x) dt d\tilde{Z}(z) + \int_{z=0}^{\infty} \int_{t=t^*(z)}^{\infty} \Theta_+\left(t, \frac{y - z - tr}{1 - r}\right) dt d\tilde{Z}(z)
\end{aligned}
$$

which can be obtained from the transient measures of buffer 1.

4.4 Steps of the numerical procedure

Based on the above detailed analysis approach the stationary distribution of the model is obtained from the model parameters (λ, μ, r_1, r_2, which satisfy $r_1 > r_2$ and the stability condition, $\lambda r_2 > \mu$) in the following steps:

1. compute P from (2),
2. compute p and $Z^*(s)$ from (16) and (12),
3. compute $V(x)$ from (21) using $E(\gamma + \tau) = \frac{1+r_1}{\lambda r_1 - \mu} + \frac{1}{\mu}$,
4. compute $G'(x)$ and $G''(x)$ from (31) via NILT,
5. compute $U(x)$ from (30),
6. compute $\Theta_+(t, x)$ and $\Theta_-(t, x)$ from (32) via NILT,
7. compute $W(x, y)$ using (34).

5 Numerical example

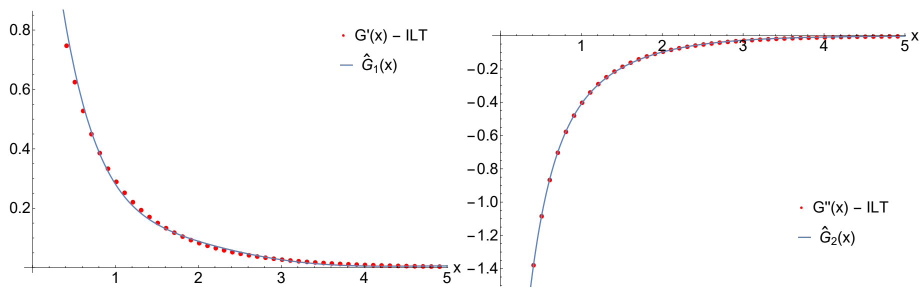
To demonstrate the application of the procedure in Section 4.4, we present a numerical example which we also compare with simulation results. We consider a system with parameters $\lambda = 2$, $\mu = 1$, $r_1 = 3$, and $r_2 = 1$. The parameters in Steps 1, 2, and 3 of Section 4.4 can be calculated analytically. To compute $U(x)$ in Step 5, we need to calculate the integrals in (30) numerically, since $G'(x)$ and $G''(x)$ are only available as a result of NILT in a finite number of points. The required integrals could be calculated using, e.g., the Simpson integral formula. Instead, to accelerate the computation, we calculate $G'(x)$ and $G''(x)$ in 20 points for $G'(x)$ and 50 points for $G''(x)$ and fit functions $\hat{G}_1(x)$ and $\hat{G}_2(x)$ to them, respectively. Our numerical investigations show that a polynomial of form

$$\hat{G}_1(x) = \sum_{i=0}^{11} a_i x^i$$

provides a close fit for $G'(x)$. On the other hand, while $\hat{G}_2(x) = \frac{d\hat{G}_1(x)}{dx}$ gives acceptable results, a much better fit can be achieved using the form

$$\hat{G}_2(x) = b_0 + b_1 e^{-c_1 x} + b_2 e^{-c_2 x}.$$

Figure 2 shows the result of the fitting.



(a) Approximation of $G'(x)$ (b) Approximation of $G''(x)$

Fig. 2: Approximation of $G'(x)$ and $G''(x)$

Using the obtained $\hat{G}_1(x)$ and $\hat{G}_2(x)$ functions we can approximate $U(x)$ according to (30). To verify the results, we implemented a model specific simulation tool which computes $V(x)$ and $U(y)$ by discrete event simulation. We repeated the simulation 100000 times from simulation time 0 to 500 (where the time unit is defined by $\lambda = 2$ and $\mu = 1$).

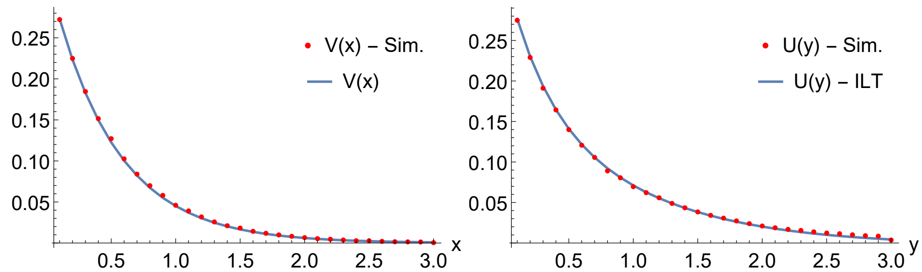
Figure 3 compares the results of the procedure in Section 4.4 with the ones obtained from discrete event simulation. We note that in the figure the solid line corresponding to $V(x)$ is the result of fully analytical calculation, while $U(y)$ is approximated by simulation and the combination of NILT and numerical function fitting. Thus, Figure 3a shows the precision of the simulation, while Figure 3b shows that the simulation and the proposed numerical procedure gives quite similar results, thus verifying the obtained formulas and the validity of the proposed approach.

6 Conclusion

The paper presents an embedded time points based analysis of a fluid model with two buffers, whose BMC is composed by two states and the fluid rates ensure that the content of one buffer is never less than the content of the other. This restriction significantly simplifies the analysis of fluid models with two buffers.

The evaluated numerical example verifies that the results of the proposed computational method closely fit with simulation results.

In the future, we intend to extend the analysis of this model with general BMC. The generalization of the embedded time points based approach does not seem to be straight



(a) $V(x)$ (b) $U(y)$

Fig. 3: Results for $U(y)$ and $V(x)$

forward because we miss an equation for obtaining the state probabilities of the BMC at embedded time points. Instead, we look for alternative analysis approaches which might be applicable for this fluid model also with general BMC.

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