# An algebraic proof of the relation of Markov fluid queues and QBD processes

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**Abstract.** We study the relation of Markov fluid queues and QBD processes in this paper. Ahn and Ramaswami presented results about this relation and provided a stochastic interpretation based reasoning in [1]. In the current work, first we provide an algebraic proof for that relation.
After that, we present a negative result about the potential extension of the QBD based analysis Markov fluid queues to Markov fluid queues with two buffers. We present a 2-dimensional QBD process, which could be a candidate for describing the stationary behaviour of the related Markov fluid queue, but it turns out that the QBD based behaviour is different from the one of the Markov fluid queue.

**Keywords:** Quasi birth death process · Markov fluid queue · queue with 2 buffers

## 1 Introduction

Markov fluid queues (MFQs) have a long tradition in stochastic modeling [5]. MFQs are queueing models where the queue length is continuous and the rate at which the queue length (also referred to as fluid level) changes is modulated by a background continuous time Markov chain (CTMC). Since the seminal applications of MFQs for analyzing high speed networks in [4], they have been applied successfully in many application areas, e.g., [13,18].

Several solution methods have been developed to obtain the stationary distribution of the fluid level in MFQs (e.g., eigenvalue decomposition based [14], Schur decomposition based [3], matrix-analytic [17,12], invariant subspace based [2], etc.).

In [1], Ahn and Ramaswami provide stochastic interpretation based results about the relation of MFQs and quasi birth death (QBD) processes. The relation is based on a stochastic coupling argument and shows a correspondence between the fluid level and the virtual workload in a queue. In this work we present an algebraic proof for the relation between MFQs and QBDs.

In [1] the authors mention that more than one QBD structure can establish the relation with the fluid queue, but they consider only one such QBD structure. In the current work we consider another QBD structure, when we prove the relation of fluid queues and QBD models.

The main advantage of the QBD interpretation of MFQs is the possibility of using efficient methods for QBD [15] analysis also for the analysis of MFQs. Thus, it is natural to ask whether the relation between MFQs and QBDs also can be established for fluid models with more than one buffer like tandem systems [16], fluid-fluid systems [7] or parallel buffers filled by one source but emptied with different rates [10]. Although there are some first results on the stationary analysis of such models [8,6,11], the proposed methods are often computationally expensive and numerically unstable. Thus, a relation to QBDs or level dependent QBDs [9] would be an important step towards an efficient analysis of more general fluid models.

In this paper we introduce two different level dependent QBDs to model fluid models with two parallel buffers. Unfortunately, as we show, the partial differential equations (PDEs) describing the fluid flows in the buffers differ from the PDEs of the original systems. Consequently, the simple relation between QBDs and MFQs that holds for one buffer, no longer holds for two buffers. This is a negative result which, nevertheless, helps to understand the behavior of more complex fluid queues.

The rest of the paper is organized as follows. In the following section we introduce the later used properties of MFQs and QBDs processes. In Section 3 we discuss the relation of MFQs and QBD processes. In Section 4, a queuing model with two fluid buffers is introduced together with its stationary equations, and after that we study a level dependent quasi birth death (LDQBD) process, whose behaviour mimics the one of the MFQ with two fluid buffers. As a result of this analysis we show that the behavior of the two models differ. The paper is finally concluded in Section 5.

## 2 Basic properties of Markov fluid queues and Quasi birth deaths processes

### 2.1 Markov fluid queues

We consider an infinite buffer MFQ $(Y(t), J(t))$, where $J(t) \in \mathcal{S}$ is the state of the background CTMC and $Y(t)$ is the fluid level at time $t$. We assume non-zero fluid rates, such that the rates are positive when the CTMC visits a state in $\mathcal{S}_+$ and negative when it visits a state in $\mathcal{S}_- = \mathcal{S} \setminus \mathcal{S}_+$. That is, if the CTMC stays at state $i \in \mathcal{S}_+$ or the CTMC stays at state $i \in \mathcal{S}_-$ and $Y(t) > 0$, then the fluid level changes at rate $r_i$,

$$\frac{d}{dt} Y(t) = r_i.$$

That is, if $i \in \mathcal{S}_+$, then $r_i > 0$ and the fluid level increases, if $i \in \mathcal{S}_-$ and $Y(t) > 0$, then $r_i < 0$ and the fluid level decreases. If $i \in \mathcal{S}_-$ and $Y(t) = 0$, then the fluid level does not change.

The characterizing matrices of the MFQ are the generator matrix of the CTMC, $\mathbf{Q}$, and the diagonal matrix of the fluid rates, $\mathbf{R}$. The rate dependent decomposition of the characterizing matrices are

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{++} & \mathbf{Q}_{+-} \\ \mathbf{Q}_{-+} & \mathbf{Q}_{--} \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} \mathbf{R}_{+} & \mathbf{0} \\ \mathbf{0} & -\mathbf{R}_{-} \end{bmatrix}. \tag{1}$$

The stationary behaviour of the MFQ is characterized by the following measures: the empty buffer probability $\pi_i = \lim_{t\to\infty} Pr(J(t) = i, Y(t) = 0)$ and the fluid density $f_i(x) = \lim_{t\to\infty} \frac{d}{dx} Pr(J(t) = i, Y(t) < x)$. We note that $Pr(J(t) = i, Y(t) = 0) = 0$ for $i \in \mathcal{S}_+$, because the fluid buffer cannot be empty when the fluid rate is positive.

In this work, we do not look into the solution methods to compute these stationary measures, we only discuss the set of differential, boundary and normalizing equations which have to be satisfied by the stationary solution.

We also introduce the related quantities $\nu_i = \pi_i|r_i|$ and $\phi_i(x) = f_i(x)|r_i|$, which is often referred to as flux. The vectors composed of the state dependent measures are $\pi_- = [\pi_i]_{i\in\mathcal{S}_-}$, $\nu_- = [\nu_i]_{i\in\mathcal{S}_-}$, $f_-(x) = [f_i(x)]_{i\in\mathcal{S}_-}$, $f_+(x) = [f_i(x)]_{i\in\mathcal{S}_+}$, $\phi_-(x) = [\phi_i(x)]_{i\in\mathcal{S}_-}$, and $\phi_+(x) = [\phi_i(x)]_{i\in\mathcal{S}_+}$.

The stationary measures satisfy the following ordinary differential relations [17]

$$\frac{d}{dx} f_+(x)\mathbf{R}_+ = f_+(x)\mathbf{Q}_{++} + f_-(x)\mathbf{Q}_{-+}, \tag{2}$$

$$-\frac{d}{dx} f_-(x)\mathbf{R}_- = f_+(x)\mathbf{Q}_{+-} + f_-(x)\mathbf{Q}_{--}, \tag{3}$$

and, equivalently,

$$\frac{d}{dx} \phi_+(x) = \phi_+(x)\mathbf{T}_{++} + \phi_-(x)\mathbf{T}_{-+}, \tag{4}$$

$$-\frac{d}{dx} \phi_-(x) = \phi_+(x)\mathbf{T}_{+-} + \phi_-(x)\mathbf{T}_{--}, \tag{5}$$

where $\mathbf{T} = |\mathbf{R}^{-1}|\mathbf{Q}$. The initial conditions of these ordinary differential equations [17] are

$$f_+(0)\mathbf{R}_+ = \pi\mathbf{Q}_{-+}, \tag{6}$$

$$-f_-(0)\mathbf{R}_- = \pi\mathbf{Q}_{--}, \tag{7}$$

and, equivalently, we have

$$\phi_+(0) = \nu\mathbf{T}_{-+}, \tag{8}$$

$$-\phi_-(0) = \nu\mathbf{T}_{--}. \tag{9}$$

Finally, the normalizing equations for the density and the flux are

$$1 = \pi\mathbb{1} + \int_0^\infty \left(f_+(x)\mathbb{1} + f_-(x)\mathbb{1}\right) dx \quad \text{and}$$

$$1 = \nu{\mathbf{R}_-}^{-1}\mathbb{1} + \int_0^\infty \left(\phi_+(x){\mathbf{R}_+}^{-1}\mathbb{1} + \phi_-(x){\mathbf{R}_-}^{-1}\mathbb{1}\right) dx. \tag{10}$$

### 2.2 QBD process

Discrete or continuous time QBD processes are discrete or continuous time Markov chains whose states can be efficiently represented by two discrete variables $\{\mathcal{X}, \mathcal{J}\}$, where $\mathcal{X} \in \{0, 1, \dots\}$ is called the level and $\mathcal{J} \in \{1, 2, \dots, N\}$ is called the phase. The direct state transitions of QBD processes are restricted between states of the same level or the neighbouring levels.

We assume (level) homogeneous QBD processes, where matrix $\mathbf{B}$ holds the rates of the level backward transitions, $\mathbf{F}$ the rates of the level forward transitions, and $\mathbf{L}$ the ones of the local transitions, which are not accompanied by the change of the level. At level zero the behavior of the local transitions can differ from the regular ones and the matrix describining these transitions is denoted by $\mathbf{L}'$.

In case of discrete time QBD processes, the one step state transition probability matrix has the following block tri-diagonal structure

$$\mathbf{P} = \begin{pmatrix} \mathbf{L}' & \mathbf{F} & & & \\ \mathbf{B} & \mathbf{L} & \mathbf{F} & & \\ & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} .$$

The stationary distribution of this QBD satisfies the stationary equations

$$p_0 = p_0 \mathbf{L}' + p_1 \mathbf{B}, \tag{11}$$

$$p_n = p_{n-1} \mathbf{F} + p_n \mathbf{L} + p_{n+1} \mathbf{B} \quad \text{for } n \geq 1, \tag{12}$$

where $p_n$ is the stationary probability vector associated with level $n$.

## 3 Relation of MFQs and QBD process with singe buffer

### 3.1 QBD structure proposed in [1]

Ahn and Ramaswami established a relationship between QBD processes of a given structure and MFQs. In their QBD process, the states of the background CTMC of the MFQ are mapped to the phases of the QBD. Following the $\mathcal{S}_+$, $\mathcal{S}_-$ partitioning of the states, the transition matrices of the QBD process proposed in [1] are as follows:

$$\mathbf{B} = \frac{1}{2} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{L} = \frac{1}{2} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_{-+} & \mathbf{P}_{--} \end{bmatrix}, \mathbf{F} = \frac{1}{2} \begin{bmatrix} \mathbf{P}_{++} & \mathbf{P}_{+-} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{13}$$

For level $0$ the local matrix is $\mathbf{L}' = \mathbf{L} + \mathbf{B}$ and matrix $\mathbf{P}$ is defined as $\mathbf{P} = \mathbf{T}/\lambda + \mathbf{I}$, where $\mathbf{T} = |\mathbf{R}^{-1}|\mathbf{Q}$ and $\lambda = \max_{i,j} |T_{i,j}|$.

Based on the stochastic interpretation of the QBD process with these transition matrices and the MFQ characterized by $\mathbf{Q}$ and $\mathbf{R}$, the stationary solution of the MFQ is provided based on the stationary solution of the QBD process. Unfortunately, this solution requires a different scaling of time in $\mathcal{S}_+$ and $\mathcal{S}_-$ [1, Theorem 4]. It is also mentioned in [1] that different QBD structures can be used for establishing such relation between a QBD process and a MFQ. The modified structure we use in this paper allows identical scaling of time in $\mathcal{S}_+$ and $\mathcal{S}_-$.

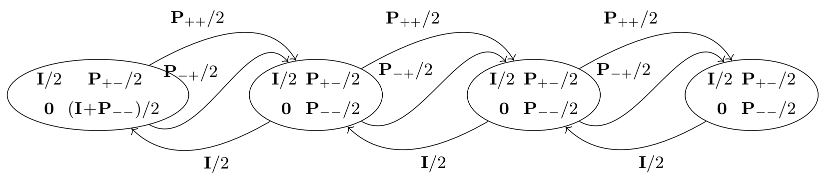


Fig. 1: Block structure of the QBD

### 3.2 QBD process with modified structure

In this work, we study a slightly modified QBD structure, whose analytical treatment is simpler. Let $X(t) = \{N(t), J(t)\}$ be a QBD process where $J(t) \in \mathcal{S}$ is the state of the background CTMC and $N(t)$ is the level of the QBD at time $t$. Based on the $\mathcal{S}_+ - \mathcal{S}_-$ decomposition, the blocks structure of the characterizing matrices of the QBD process are

$$\mathbf{B} = \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{L} = \frac{1}{2}\begin{bmatrix} \mathbf{I} & \mathbf{P}_{+-} \\ \mathbf{0} & \mathbf{P}_{--} \end{bmatrix}, \mathbf{F} = \frac{1}{2}\begin{bmatrix} \mathbf{P}_{++} & \mathbf{0} \\ \mathbf{P}_{-+} & \mathbf{0} \end{bmatrix}. \tag{14}$$

For level 0 the generator matrix is $\mathbf{L}' = \mathbf{L} + \mathbf{B}$. The state transition structure of the QBD is depicted in Figure 1. Arrows from/to the upper half of the ellipses indicate transitions from/to $\mathcal{S}^+$.

Let us define $p_{n,i} = \lim_{t\to\infty} Pr(N(t) = n, J(t) = i)$, which describes the stationary distribution of the QBD. Let $p_n^+$ and $p_n^-$ be the row vectors composed of $p_{n,i}$ for $i \in \mathcal{S}^+$ and $i \in \mathcal{S}^-$, respectively. These vectors satisfy the following stationary equations for $n \geq 1$

$$p_0^+ = 0, \tag{15}$$
$$p_0^- = p_0^-(\mathbf{I}+\mathbf{P}_{--})/2 + p_1^-\mathbf{I}/2, \tag{16}$$
$$p_n^+ = p_{n-1}^+\mathbf{P}_{++}/2 + p_{n-1}^-\mathbf{P}_{-+}/2 + p_n^+\mathbf{I}/2, \tag{17}$$
$$p_n^- = p_n^-\mathbf{P}_{--}/2 + p_n^+\mathbf{P}_{+-}/2 + p_{n+1}^-\mathbf{I}/2, \tag{18}$$

which can be simplified to

$$p_1^- = p_0^-(\mathbf{I} - \mathbf{P}_{--}), \tag{19}$$
$$p_n^+ = p_{n-1}^+\mathbf{P}_{++} + p_{n-1}^-\mathbf{P}_{-+}, \tag{20}$$
$$p_n^-(2\mathbf{I} - \mathbf{P}_{--}) = p_n^+\mathbf{P}_{+-} + p_{n+1}^-\mathbf{I}. \tag{21}$$

### 3.3 Algebraic proof of the relation of MFQs and QBD processes

The following theorem relates the stationary behaviour of the QBD processes defined in (14) with $\mathbf{L}' = \mathbf{L} + \mathbf{B}$ and the MFQ with characterizing matrices defined in (1).

**Theorem 1.** *When* $\mathbf{T} = |\mathbf{R}^{-1}|\mathbf{Q}$*,* $\lambda = \max_{i,j} |T_{i,j}|$*,* $\mathbf{P} = \mathbf{T}/\lambda + \mathbf{I}$*, and* $p_n^+$*,* $p_n^-$ *is a non-zero solution of* (19) - (21)*, then*

$$\hat{\phi}_\pm(x) = \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} p_n^\pm \;\; \textit{and} \;\; \hat{\nu} = p_0^- \tag{22}$$

*satisfy the differential equations* (4) *and* (5) *with boundary conditions* (8) *and* (9)*.*

The theorem states that the solution of the MFQ is a mixture of Erlang distributions of order $n$ and rate $\lambda$ weighted according to the stationary distribution of level $n$ of the QBD process for $n \geq 1$ and the empty buffer probability of the MFQ is related to the stationary distribution of level $0$ of the QBD process.

*Proof.* When $\mathbf{P} = \mathbf{T}/\lambda + \mathbf{I}$, we have $\mathbf{P}_{++} = \mathbf{T}_{++}/\lambda + \mathbf{I}$, $\mathbf{P}_{+-} = \mathbf{T}_{+-}/\lambda$, $\mathbf{P}_{--} = \mathbf{T}_{--}/\lambda + \mathbf{I}$, $\mathbf{P}_{-+} = \mathbf{T}_{-+}/\lambda$.

Substituting this into (19)-(21), for $n \geq 1$, we get

$$\lambda p_1^- = -p_0^- \mathbf{T}_{--}, \tag{23}$$

$$\lambda p_i^+ = p_{i-1}^+(\mathbf{T}_{++} + \lambda \mathbf{I}) + p_{i-1}^- \mathbf{T}_{-+}, \tag{24}$$

$$p_i^-(\lambda \mathbf{I} - \mathbf{T}_{--}) = p_i^+ \mathbf{T}_{+-} + p_{i+1}^- \lambda \mathbf{I}. \tag{25}$$

For $i \geq 1$, we have

$$\frac{d}{dx} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} = \mathcal{I}_{\{i>1\}} \frac{\lambda^i x^{i-2} e^{-\lambda x}}{(i-2)!} - \frac{\lambda^{i+1} x^{i-1} e^{-\lambda x}}{(i-1)!}, \tag{26}$$

from which $\hat{\phi}_\pm(x)$ satisfies

$$\begin{aligned} \frac{d}{dx}\hat{\phi}_\pm(x) &= \sum_{i=2}^{\infty} \frac{\lambda^i x^{i-2} e^{-\lambda x}}{(i-2)!} p_i^\pm - \lambda \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p_i^\pm \\ &= \lambda \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p_{i+1}^\pm - \lambda \hat{\phi}_\pm(x). \end{aligned} \tag{27}$$

Multiplying (24) by $\frac{\lambda^{i-1} x^{i-2} e^{-\lambda x}}{(i-2)!}$ and summing up from $i = 2$ to $\infty$ gives

$$\begin{aligned} &\lambda \sum_{i=2}^{\infty} \frac{\lambda^{i-1} x^{i-2} e^{-\lambda x}}{(i-2)!} p_i^+ \\ &= \sum_{i=2}^{\infty} \frac{\lambda^{i-1} x^{i-2} e^{-\lambda x}}{(i-2)!} \left(p_{i-1}^+(\mathbf{T}_{++} + \lambda \mathbf{I}) + p_{i-1}^- \mathbf{T}_{-+}\right), \text{ and} \\ &\frac{d}{dx}\hat{\phi}_+(x) + \lambda \hat{\phi}_+(x) = \hat{\phi}_+(x)(\mathbf{T}_{++} + \lambda \mathbf{I}) + \hat{\phi}_-(x)\mathbf{T}_{-+}, \text{ and} \\ &\frac{d}{dx}\hat{\phi}_+(x) = \hat{\phi}_+(x)\mathbf{T}_{++} + \hat{\phi}_-(x)\mathbf{T}_{-+}. \end{aligned} \tag{28}$$

Multiplying (25) by $\frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!}$ and summing up from $i = 1$ to $\infty$ gives

$$\sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p_i^- (\lambda \mathbf{I} - \mathbf{T}_{--}) = \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \left( p_i^+ \mathbf{T}_{+-} + p_{i+1}^- \lambda \mathbf{I} \right), \text{ and}$$

$$\hat{\phi}_-(x)(\lambda \mathbf{I} - \mathbf{T}_{--}) = \hat{\phi}_+(x)\mathbf{T}_{+-} + \frac{d}{dx}\hat{\phi}_-(x) + \lambda \hat{\phi}_-(x), \text{ and}$$

$$-\frac{d}{dx}\hat{\phi}_-(x) = \hat{\phi}_+(x)\mathbf{T}_{+-} + \hat{\phi}_-(x)\mathbf{T}_{--}, \tag{29}$$

where, from (27), we used that

$$\sum_{i=2}^{\infty} \frac{\lambda^i x^{i-2} e^{-\lambda x}}{(i-2)!} p_i^{\pm} = \lambda \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p_{i+1}^{\pm} = \frac{d}{dx}\hat{\phi}_{\pm}(x) + \lambda \hat{\phi}_{\pm}(x). \tag{30}$$

For the initial conditions, we start from the definition of $\hat{\phi}_{\pm}(x)$ given in (22), from which

$$\hat{\phi}_{\pm}(0) = \lambda p_1^{\pm}. \tag{31}$$

Substituting $p_1^-$ from (23) and $p_1^+$ from (24) (and using that $\hat{\nu} = p_0^-$ and $p_0^+ = 0$) gives

$$\hat{\phi}_+(0) = \hat{\nu}\mathbf{T}_{-+}, \tag{32}$$

$$-\hat{\phi}_-(0) = \hat{\nu}\mathbf{T}_{--}. \tag{33}$$

□

Theorem 1 does not imply that $\phi_{\pm}(x) = \hat{\phi}_{\pm}(x)$ and $\nu = \hat{\nu}$, because the normalizing condition of the MFQ and the QBD process differ. $\phi(x)$ and $\nu$ satisfy the normalizing equation (10), while $\hat{\phi}(x)$ and $\hat{\nu}$ are normalized as follows

$$\sum_{i=0}^{\infty} (p_i^+ \mathbb{1} + p_i^- \mathbb{1}) = \hat{\nu} + \int_{x=0}^{\infty} \hat{\phi}_+(x)\mathbb{1} + \hat{\phi}_-(x)\mathbb{1} dx = 1. \tag{34}$$

## 4 Two fluid buffers

In this section we investigate if the relation of QBD processes and MFQs can be extended for simple MFQs with two buffers using the same approach as in the previous section.

### 4.1 Markov fluid queue

We consider a MFQ with two infinite buffers $(J(t), Y_1(t), Y_2(t))$, where $J(t)$ is the state of the background CTMC and $Y_i(t)$ is the fluid level of buffer $i$ ($i \in \{1, 2\}$) at time $t$. The state space of the CTMC is composed of two disjoint subsets $\mathcal{S}_+$ and $\mathcal{S}_- = \mathcal{S} \setminus \mathcal{S}_+$ such that the fluid level of both buffers increases at rate 1 in $\mathcal{S}_+$ and in $\mathcal{S}_-$, the fluid level of buffer 1 decreases with rate 1 and the fluid level of buffer 2 decreases with rate $r_2 < 1$, if the buffers are non-empty. That is, the characterizing matrices of the MFQ are $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{++} & \mathbf{Q}_{+-} \\ \mathbf{Q}_{-+} & \mathbf{Q}_{--} \end{bmatrix}$, $\mathbf{R}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}$ and $\mathbf{R}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{r_2 I} \end{bmatrix}$. A trajectory of the fluid levels is depicted in Figure 2. An important consequence of this model behaviour is that $Y_1(t) \le Y_2(t)$ for $t > 0$. As a consequence, the MFQ is stable if buffer 2 is stable. Figure 2 indicates 4 possible cases:

- $Y_2(t) > Y_1(t) > 0$,
- $Y_2(t) > 0, Y_1(t) = 0$,
- $Y_2(t) = Y_1(t) = 0$,
- $Y_2(t) = Y_1(t) > 0$.

The stationary measures associated with these 4 cases are as follows

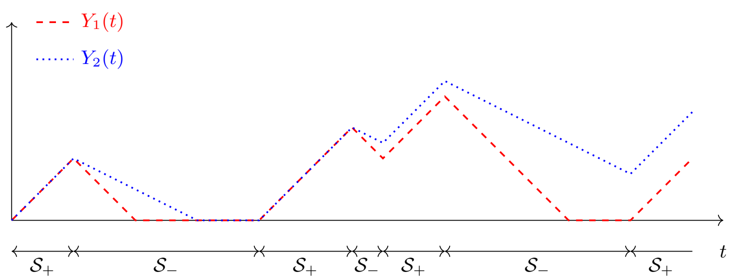


Fig. 2: Evolution of the buffer contents with $r_2 = 0.5$

$$W_i(x, y) = \lim_{t\to\infty} \frac{d}{dx}\frac{d}{dy} Pr(J(t) = i, Y_1(t) < x, Y_2(t) < y),$$

$$U_i(y) = \lim_{t\to\infty} \frac{d}{dy} Pr(J(t) = i, Y_1(t) = 0, Y_2(t) < y),$$

$$\pi_i = \lim_{t\to\infty} Pr(J(t) = i, Y_1(t) = Y_2(t) = 0),$$

$$V_i(x) = \lim_{t\to\infty} \frac{d}{dx} Pr(J(t) = i, Y_1(t) = Y_2(t) < x).$$

The vectors composed of the state dependent measures are $\pi = [\pi_i]_{i\in\mathcal{S}_-}$, $U(y) = [U_i(y)]_{i\in\mathcal{S}_-}$, $V(x) = [V_i(x)]_{i\in\mathcal{S}_+}$, $W_+(x, y) = [W_i(x, y)]_{i\in\mathcal{S}_+}$, and $W_-(x, y) = [W_i(x, y)]_{i\in\mathcal{S}_-}$.

The stationary solution satisfies the following equations

$$\pi^+ = \mathbf{0}\,; \qquad \mathbf{0} = \pi^- \mathbf{Q}_{--} + r_2 U(0); \tag{35}$$

$$\frac{\partial}{\partial x} V(x) = V(x)\mathbf{Q}_{++}, \tag{36}$$

with initial condition

$$V(0) = \pi^{-}\mathbf{Q}_{-+}; \tag{37}$$

$$-r_2 \frac{\partial}{\partial y} U(y) = U(y)\mathbf{Q}_{--} - W^{-}(0,y); \tag{38}$$

$$\frac{\partial}{\partial x} W^{+}(x,y) + \frac{\partial}{\partial y} W^{+}(x,y) = W^{+}(x,y)\mathbf{Q}_{++} + W^{-}(x,y)\mathbf{Q}_{-+}, \tag{39}$$

with initial conditions

$$W^{+}(x,x) = \mathbf{0} \quad \text{and} \quad W^{+}(0,y) = U(y)\mathbf{Q}_{-+}; \tag{40}$$

$$-\frac{\partial}{\partial x} W^{-}(x,y) - r_2 \frac{\partial}{\partial y} W^{-}(x,y) = W^{+}(x,y)\mathbf{Q}_{+-} + W^{-}(x,y)\mathbf{Q}_{--}, \tag{41}$$

with initial condition

$$r_2 W^{-}(x,x) = V^{+}(x)\mathbf{Q}_{+-}. \tag{42}$$

### 4.2 Level dependent quasi birth death process

In this section we introduce a level dependent quasi birth death (LDQBD) process whose structure is meant to represent the behaviour of the MFQ with two buffers and we check if the stationary behaviour of the MFQ with two buffers and the LDQBD process are related.

The block structure of the process is depicted in Figure 3 applying the same graphical representations of the transitions associated with $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ as in Figure 1, that is, the arrows from/to the upper half of the ellipses indicate transitions from/to $\mathcal{S}^{+}$ and the lower half of the ellipses are related to $\mathcal{S}^{-}$.

Assuming, that the blocks of states of this process (indicated by ellipses in the figure) are such that the associated *levels* increase along the horizontal axis, and all of the blocks along a vertical line compose the *phases* of the given level, the obtained stochastic process is a LDQBD process whose state transitions can be described with the transition probability matrix

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}^{(0)} & \mathbf{F}^{(0)} & & & \\ \mathbf{B}^{(1)} & \mathbf{L}^{(1)} & \mathbf{F}^{(1)} & & \\ & \mathbf{B}^{(2)} & \mathbf{L}^{(2)} & \mathbf{F}^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{43}$$

where the size of the matrices of the different levels increases level-by-level. For the detailed internal structure of the non-zero blocks of (43) we refer to Figure 3.

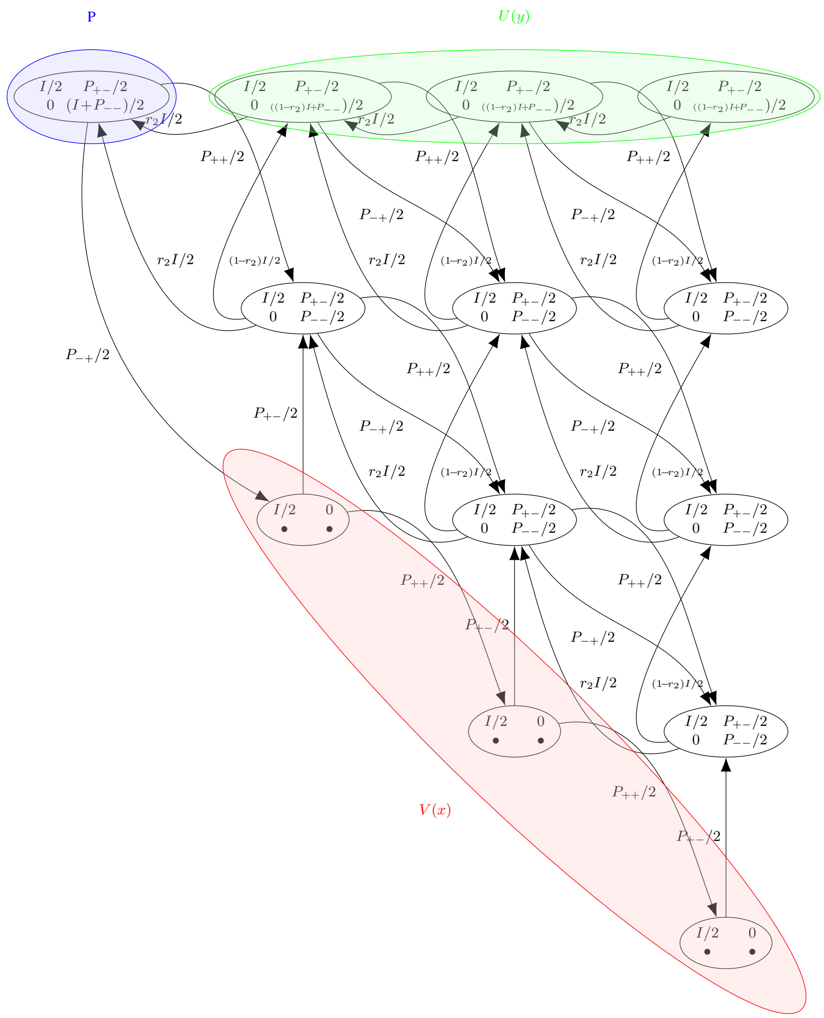


Fig. 3: Block structure of the LDQBD

We decompose the stationary probability vector of the LDQBD into the following blocks: $p^+_{00}, p^-_{00},\ p^+_{10}, p^-_{10}, p^+_{11}, p^-_{11}, p^+_{12}, p^-_{12}, \ldots,\ p^+_{i0}, p^-_{i0}, \ldots, p^+_{i,i+1}, p^-_{i,i+1}, p^+_{i+1,0}, p^-_{i+1,0}, \ldots$. The stationary probability of the transient states of the LDQBD is zero, from which

$$p^+_{i-1,0} = 0 \text{ and } \ p^-_{i,i+1} = 0 \quad \text{for } i \geq 1. \tag{44}$$

According to Figure 3, for $i \geq 1$ and $i \geq j \geq 1$, the decomposed vectors satisfy the following stationary equations

$$p^-_{00} = p^-_{00}(\mathbf{I}+\mathbf{P}_{--})/2 + (p^-_{10} + p^-_{11})r_2\mathbf{I}/2, \tag{45}$$

$$p^-_{i0} = p^-_{i0}((1-r_2)\mathbf{I}+\mathbf{P}_{--})/2 + p^-_{i1}(1-r_2)\mathbf{I}/2 + (p^-_{i+1,0} + p^-_{i+1,1})r_2\mathbf{I}/2, \tag{46}$$

$$p^+_{ij} = p^+_{ij}\mathbf{I}/2 + p^+_{i-1,j-1}\mathbf{P}_{++}/2 + p^-_{i-1,j-1}\mathbf{P}_{-+}/2, \tag{47}$$

$$\begin{aligned} p^-_{ij} &= p^-_{ij}\mathbf{P}_{--}/2 + p^+_{ij}\mathbf{P}_{+-}/2 + p^-_{i+1,j+1}r_2\mathbf{I}/2 \\ &\quad + \mathcal{I}_{\{j<i\}}p^-_{i,j+1}(1-r_2)\mathbf{I}/2 + \mathcal{I}_{\{j=i\}}p^-_{i,j+1}\mathbf{P}_{+-}/2, \end{aligned} \tag{48}$$

$$p^+_{1,2} = p^+_{1,2}\mathbf{I}/2 + p^-_{0,0}\mathbf{P}_{-+}/2, \tag{49}$$

$$p^+_{i+1,i+2} = p^+_{i+1,i+2}\mathbf{I}/2 + p^+_{i,i+1}\mathbf{P}_{++}/2. \tag{50}$$

### 4.3 Relation of the MFQ and the LDQBD

In this section we follow the same approach as in Theorem 1 and check if the stationary behaviour of the LDQBD is related to the one of the MFQ.

**Theorem 2.** *Assuming* $\mathbf{P} = \mathbf{Q}/\lambda + \mathbf{I}$, $\tilde{\pi} = p^-_{0,0}$, $\tilde{V}(x) = \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p^+_{i,i+1}$, $\tilde{U}(y) = \sum_{i=1}^{\infty} \frac{\lambda^i y^{i-1} e^{-\lambda y}}{(i-1)!} p^-_{i,0}$, $\tilde{W}(x,y)^{\pm} = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \frac{\lambda^i y^{i-1} e^{-\lambda y}}{(i-1)!} \frac{\lambda^j x^{j-1} e^{-\lambda x}}{(j-1)!} p^{\pm}_{i,j}$, $\tilde{V}(x)$, $\tilde{U}(x)$ *and* $\tilde{W}(x,y)$ *satisfy*

$$\frac{d}{dx}\tilde{V}(x) = \tilde{V}(x)\mathbf{Q}_{++}. \tag{51}$$

$$-r_2\frac{d}{dy}\tilde{U}(y) = \tilde{U}(y)\mathbf{Q}_{--} + \underbrace{(1-r_2)\tilde{W}(0,y)^- + r_2\left(\frac{d}{dx}\tilde{W}(0,y)^- + \lambda\tilde{W}(0,y)^-\right)}_{\textit{different from MFQ}}, \tag{52}$$

$$\begin{aligned} \frac{\partial}{\partial y}\tilde{W}(x,y)^+ + \frac{\partial}{\partial x}\tilde{W}(x,y)^+ &= \tilde{W}(x,y)^+\mathbf{Q}_{++} + \tilde{W}(x,y)^-\mathbf{Q}_{-+} \\ &\quad - \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\tilde{W}(x,y)^+}_{\textit{different from MFQ}}, \end{aligned} \tag{53}$$

$$-\frac{\partial}{\partial x}\tilde{W}(x,y)^- - r_2\frac{\partial}{\partial y}\tilde{W}(x,y)^- = \tilde{W}(x,y)^-\mathbf{Q}_{--} + \tilde{W}(x,y)^-\mathbf{Q}_{--}$$
$$\underbrace{+\varepsilon(x,y)\mathbf{Q}_{+-} + \frac{r_2}{\lambda}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\tilde{W}(x,y)^-}_{\text{different from MFQ}}, \quad (54)$$

*where* $\varepsilon(x,y) = \sum_{i=2}^{\infty} \frac{\lambda^{i-1}x^{i-2}e^{-\lambda x}}{(i-2)!}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}p^-_{i-1,i}$.

The proof is provided in the appendix.

While $\tilde{V}(x)$ satisfies the same differential equation as $V(x)$ in (36), unfortunately, $\tilde{U}(x)$, $\tilde{W}(x,y)^+$ and $\tilde{W}(x,y)^-$ are characterized by different differential equations than (38), (39) and (41), respectively. It means that the stationary distribution of the MFQ with two buffers cannot be established based on this LDQBD.

## 5 Conclusion

We revisited the relation of MFQs and QBD processes in order to extend it for MFQs with two buffers. To this end, we replaced the stochastic intuition based discussion of [1] with an algebraic one and provided an algebraic proof of the relation of MFQs and QBD processes. For the analysis of a simple MFQ with two buffers we introduced a LDQBD, whose structure mimics the behaviour of the queue in the same way as in the single buffer case. Unfortunately, the accurate algebraic approach indicated that the stationary behaviour of the MFQs with two buffers and the LDQBD process differ.

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## A Proof of Theorem 2

Using $\mathbf{P}_{++} = \mathbf{Q}_{++}/\lambda + \mathbf{I}$, $\mathbf{P}_{+-} = \mathbf{Q}_{+-}/\lambda$, $\mathbf{P}_{--} = \mathbf{Q}_{--}/\lambda + \mathbf{I}$, $\mathbf{P}_{-+} = \mathbf{Q}_{-+}/\lambda$, for $i \geq j \geq 1$, equations (45)-(50) can be simplified to

$$-p^-_{00}\mathbf{Q}_{--} = \lambda r_2(p^-_{10} + p^-_{11}), \tag{55}$$

$$p^-_{i0}(\lambda r_2\mathbf{I} - \mathbf{Q}_{--}) = \lambda(1-r_2)p^-_{i1} + \lambda r_2(p^-_{i+1,0} + p^-_{i+1,1}), \tag{56}$$

$$\lambda p^+_{ij} = p^+_{i-1,j-1}(\lambda\mathbf{I} + \mathbf{Q}_{++}) + p^-_{i-1,j-1}\mathbf{Q}_{-+}, \tag{57}$$

$$\begin{aligned} p^-_{ij}(\lambda\mathbf{I} - \mathbf{Q}_{--}) = {} & p^+_{ij}\mathbf{Q}_{+-} + \lambda r_2 p^-_{i+1,j+1} \\ & + \mathcal{I}_{\{j<i\}}\lambda(1-r_2)p^-_{i,j+1} + \mathcal{I}_{\{j=i\}}p^-_{i,j+1}\mathbf{Q}_{+-}, \end{aligned} \tag{58}$$

$$\lambda p^+_{1,2} = p^-_{0,0}\mathbf{Q}_{-+}, \tag{59}$$

$$\lambda p^+_{i+1,i+2} = p^+_{i,i+1}(\lambda\mathbf{I} + \mathbf{Q}_{++}). \tag{60}$$

Multiplying (60) by $\frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!}$ and summing up from $i = 1$ to $\infty$ gives

$$\sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \lambda p^+_{i+1,i+2} = \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p^+_{i,i+1}(\lambda\mathbf{I} + \mathbf{Q}_{++}), \tag{61}$$

$$\frac{d}{dx}\tilde{V}(x) + \lambda\tilde{V}(x) = \tilde{V}(x)(\lambda\mathbf{I} + \mathbf{Q}_{++}), \tag{62}$$

which results in (51).

By definition $\tilde{W}(0,x)^{\pm} = \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \lambda p_{i,1}^{\pm}$. Multiplying (56) by $\frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!}$ and summing up from $i=1$ to $\infty$ gives

$$\sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} p_{i0}^{-} (\lambda r_2 \mathbf{I} - \mathbf{Q}_{--}) = \sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \left( \lambda (1-r_2) p_{i1}^{-} + \lambda r_2 (p_{i+1,0}^{-} + p_{i+1,1}^{-}) \right),$$

$$\tilde{U}(x)(\lambda r_2 \mathbf{I} - \mathbf{Q}_{--}) = \underbrace{\sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \lambda (1-r_2) p_{i1}^{-}}_{(1-r_2)\tilde{W}(0,x)^{-}} + \underbrace{\sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \lambda r_2 p_{i+1,1}^{-}}_{r_2\left(\frac{d}{dx}\tilde{W}(0,x)^{-} + \lambda \tilde{W}(0,x)^{-}\right)}$$
$$+ \underbrace{\sum_{i=1}^{\infty} \frac{\lambda^i x^{i-1} e^{-\lambda x}}{(i-1)!} \lambda r_2 p_{i+1,0}^{-}}_{r_2\left(\frac{d}{dx}\tilde{U}(x) + \lambda \tilde{U}(x)\right)},$$

which results in (51).

For the computation of $\tilde{W}(x,y)$ we need the following lemma.

**Lemma 1.** *The derivatives of* $\tilde{W}(x,y)^{\pm} = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{\lambda^i y^{i-1} e^{-\lambda y}}{(i-1)!} \frac{\lambda^j x^{j-1} e^{-\lambda x}}{(j-1)!} p_{i,j}^{\pm}$ *satisfy*

$$\frac{\partial}{\partial y} \tilde{W}(x,y)^{\pm} + \lambda \tilde{W}(x,y)^{\pm} = \sum_{i=2}^{\infty} \sum_{j=1}^{i} \frac{\lambda^i y^{i-2} e^{-\lambda y}}{(i-2)!} \frac{\lambda^j x^{j-1} e^{-\lambda x}}{(j-1)!} p_{i,j}^{\pm}, \tag{63}$$

$$\frac{\partial}{\partial x} \tilde{W}(x,y)^{\pm} + \lambda \tilde{W}(x,y)^{\pm} = \sum_{i=1}^{\infty} \sum_{j=2}^{i} \frac{\lambda^i y^{i-1} e^{-\lambda y}}{(i-1)!} \frac{\lambda^j x^{j-2} e^{-\lambda x}}{(j-2)!} p_{i,j}^{\pm} \tag{64}$$

$$= \sum_{i=2}^{\infty} \sum_{j=2}^{i-1} \frac{\lambda^{i-1} y^{i-2} e^{-\lambda y}}{(i-2)!} \frac{\lambda^j x^{j-2} e^{-\lambda x}}{(j-2)!} p_{i-1,j}^{\pm}, \tag{65}$$

*and*

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \tilde{W}(x,y)^{\pm} + \lambda \frac{\partial}{\partial y} \tilde{W}(x,y)^{\pm} + \lambda \frac{\partial}{\partial x} \tilde{W}(x,y)^{\pm} + \lambda^2 \tilde{W}(x,y)^{\pm}$$
$$= \sum_{i=2}^{\infty} \sum_{j=2}^{i} \frac{\lambda^i y^{i-2} e^{-\lambda y}}{(i-2)!} \frac{\lambda^j x^{j-2} e^{-\lambda x}}{(j-2)!} p_{i,j}^{\pm}. \tag{66}$$

*Proof.* The statements of the lemma can be obtained by substituting the definition of $\tilde{W}(x,y)^{\pm}$. We omit the details of the proof here.

Multiplying (57) by $\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}$ and summing up from $i=2$ to $\infty$ and $j=2$ to $i$ and utilizing (66) gives

$$\underbrace{\sum_{i=2}^{\infty}\sum_{j=2}^{i}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}\lambda p_{ij}^{+}}_{\frac{1}{\lambda}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\tilde{W}(x,y)^{+}+\frac{\partial}{\partial y}\tilde{W}(x,y)^{+}+\frac{\partial}{\partial x}\tilde{W}(x,y)^{+}+\lambda\tilde{W}(x,y)^{+}}$$
$$=\underbrace{\sum_{i=2}^{\infty}\sum_{j=2}^{i}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}\left(p_{i-1,j-1}^{+}(\lambda\mathbf{I}+\mathbf{Q}_{++})+p_{i-1,j-1}^{-}\mathbf{Q}_{-+}\right)}_{\tilde{W}(x,y)^{+}(\lambda\mathbf{I}+\mathbf{Q}_{++})+\tilde{W}(x,y)^{-}\mathbf{Q}_{-+}}$$

which results in (53).

For $2\le j\le i$ we rewrite (58) as

$$\begin{aligned}&p_{i-1,j-1}^{-}(\lambda\mathbf{I}-\mathbf{Q}_{--})-p_{i-1,j-1}^{+}\mathbf{Q}_{+-}\\&\quad=\lambda r_2 p_{i,j}^{-}+\mathcal{I}_{\{j<i\}}\lambda(1-r_2)p_{i-1,j}^{-}+\mathcal{I}_{\{j=i\}}p_{i-1,j}^{-}\mathbf{Q}_{+-},\end{aligned}\tag{67}$$

Multiplying (67) by $\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}$ and summing up from $i=2$ to $\infty$ and $j=2$ to $i$ and utilizing Lemma 1 gives

$$\underbrace{\sum_{i=2}^{\infty}\sum_{j=2}^{i}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}p_{i-1,j-1}^{-}}_{\tilde{W}(x,y)^{-}}(\lambda\mathbf{I}-\mathbf{Q}_{--})$$
$$-\underbrace{\sum_{i=2}^{\infty}\sum_{j=2}^{i}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}p_{i-1,j-1}^{+}}_{\tilde{W}(x,y)^{+}}\mathbf{Q}_{+-}$$
$$=\underbrace{\sum_{i=2}^{\infty}\sum_{j=2}^{i}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}\lambda r_2 p_{i,j}^{-}}_{r_2\left(\frac{1}{\lambda}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\tilde{W}(x,y)^{-}+\frac{\partial}{\partial y}\tilde{W}(x,y)^{-}+\frac{\partial}{\partial x}\tilde{W}(x,y)^{-}+\lambda\tilde{W}(x,y)^{-}\right)}$$
$$+\underbrace{\sum_{i=2}^{\infty}\sum_{j=2}^{i-1}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{j-1}x^{j-2}e^{-\lambda x}}{(j-2)!}\lambda(1-r_2)p_{i-1,j}^{-}}_{(1-r_2)\left(\frac{\partial}{\partial x}\tilde{W}(x,y)^{-}+\lambda\tilde{W}(x,y)^{-}\right)}$$
$$+\underbrace{\sum_{i=2}^{\infty}\frac{\lambda^{i-1}y^{i-2}e^{-\lambda y}}{(i-2)!}\frac{\lambda^{i-1}x^{i-2}e^{-\lambda x}}{(i-2)!}p_{i-1,i}^{-}}_{\varepsilon(x,y)}\mathbf{Q}_{+-},$$

that is

$$\tilde{W}(x,y)^{-}(\lambda\mathbf{I}-\mathbf{Q}_{--}) - \tilde{W}(x,y)^{+}\mathbf{Q}_{+-}$$
$$= r_2 \left( \frac{1}{\lambda}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\tilde{W}(x,y)^{-} + \frac{\partial}{\partial y}\tilde{W}(x,y)^{-} + \frac{\partial}{\partial x}\tilde{W}(x,y)^{-} + \lambda\tilde{W}(x,y)^{-} \right)$$
$$+ (1-r_2)\left( \frac{\partial}{\partial x}\tilde{W}(x,y)^{-} + \lambda\tilde{W}(x,y)^{-} \right) + \varepsilon(x,y)\mathbf{Q}_{+-},$$

which results in (54).