Numerical analysis of M/G/1 type queueing systems with Phase-type transition structure*

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Abstract

A number of existing results describe the numerical calculation of the steady state distribution of an M/G/1/ type Markov chain. However, these numerical methods have difficulties when the forward transition structure has a long tail asymptotic. This paper proposes a numerical approximation that can account for the polynomial decay of the steady-state distribution over several orders of magnitude, where the other known methods fail. An important advantage of the proposed approximation is that it uses numerically stable techniques.

Key words: M/G/1 type Markov chain, Phase type distribution, Steady state analysis.

1 Introduction

The M/G/1 type Markov chain is the generalisation of the embedded Markov chain of an M/G/1 queue. The states of the embedded Markov chain of an M/G/1 queue become subsets of states called levels and the transitions between these levels has the same structure as in the embedded Markov chain of an M/G/1 queue.

M/G/1 type Markov chains are studied for a long time (e.g. Neuts [9]) and many finite server queueing problems has been shown to lead to an M/G/1 type Markov chain [9, 6].

Various solution methods were developed for the efficient numerical computation of the steady state distribution of M/G/1 queues (e.g. Baum [1], Meini [7] or Riska and Smirni [10]). A recently published text book provides a survey of the available methods [2]. These methods

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evaluate the solution of an infinite order matrix equation or use infinite matrix summation. The main problem of the available solution methods is that there are cases when the solution is numerically infeasible. The reason for this is that the infinite order matrix equation or summation should be truncated at some point in practice and the computationally feasible truncation points do not provide sufficient accuracy. It is the case when there is a slowly varying decay in the forward transition structure. A computationally feasible truncation of this series can introduce significant error in the final solution.

In this paper we propose to approximate the forward transition structure of M/G/1 type systems using discrete PH (Phase type) distributions and present an numerically stable algorithm for the steady state analysis of the approximate model. We show that the obtained approximate model is in fact a QBD (Quasi Birth-Death) system with finite level size therefore a huge set of efficient numerical methods available for the solution of the model.

A numerical example demonstrate that the proposed method allows to study cases that cannot be analyzed by existing tools because of numerical difficulties.

We note that the proposed method is applicable for all M/G/1 type models with finite level size, since any general transition structure can be approximated arbitrarily well by discrete PH distributions. The complexity of the proposed approximate analysis depends on the order of the PH distribution which allows to tune the accuracy-complexity trade-off. The proposed solution is exact if the forward transitions has a PH distributed structure.

The rest of the paper is organized as follows. Section 2 defines the basic concepts and introduces the notations. Section 3 explains our main idea. Section 4 presents the proposed numerical method and compares its performance to the existing methods by an example. Section 5 concludes the paper.

2 Notations and basic concepts

Discrete phase type distributions

An order m discrete phase type distribution is defined as the time to absorption in a Discrete time Markov chain with m transient and one absorbing state.

The initial probability vector (of the transient states), α, and the transition probability matrix (between the transient states), $\mathbf{T}$ determines the distribution by the mean of the following expression:

$$p(i) = Pr(\text{time to absorption} = i) = \alpha(\mathbf{T}^{i-1} - \mathbf{T}^{i})\mathbb{1} = \alpha\mathbf{T}^{i-1}t, \ i \geq 1, \tag{1}$$

where $\mathbb{1}$ is the column vector of ones, and the column vector t contains the transition probabilities from the transient states to the absorbing one ($t = \mathbb{1} - \mathbf{T}\mathbb{1}$).

Forward transition structure of M/G/1 type Markov chains

The generator of M/G/1 type Markov chains have the following regular block structure [9, 6]:

$$
\mathbf{Q} = \begin{array}{|c|c|c|c|c|}
\hline
\mathbf{L'} & \mathbf{F'_1} & \mathbf{F'_2} & \mathbf{F'_3} & \mathbf{F'_4} \\
\hline
\mathbf{B'} & \mathbf{L} & \mathbf{F_1} & \mathbf{F_2} & \mathbf{F_3} \\
\hline
 & \mathbf{B} & \mathbf{L} & \mathbf{F_1} & \mathbf{F_2} \\
\hline
 & & \mathbf{B} & \mathbf{L} & \mathbf{F_1} \\
\hline
 & & & \ddots & \ddots \\
\hline
\end{array}, \qquad (2)
$$

where the cardinality of the first bock is n' and the other blocks is n. That is, the size of $\mathbf{L'}$ is $n' \times n'$, the size of $\mathbf{F'_i}$ is $n' \times n$, the size of $\mathbf{B'}$ is $n \times n'$ and the size of the other non-zero blocks is $n \times n$. We refer to the set of states associated with the ith block as level i, and a state in this set as phase k.

The forward transition structure of the regular part of an M/G/1 type Markov chain (the second and higher block rows of matrix $\mathbf{Q}$) can be expressed by n^2 transition rates and n^2 discrete probability distributions. We compose matrix $\mathbf{F}$ to describe the forward transition rates and the matrix series $\mathbf{P_i}, i \geq 1$ to describe the forward distributions such that the matrix elements are

$$[\mathbf{P_i}]_{k\ell} = Pr(\text{a forward transition from phase } k \text{ jumps } i \text{ levels and goes to phase } \ell).$$

The mutual relation of $\mathbf{F}, \mathbf{P_i}$ and $\mathbf{F_i}$ is

$$\mathbf{F} = \sum_{i=1}^{\infty} \mathbf{F_i}, \qquad [\mathbf{P_i}]_{k\ell} = \frac{[\mathbf{F_i}]_{k\ell}}{[\mathbf{F}]_{k\ell}} \qquad \Longleftrightarrow \qquad \mathbf{F_i} = \mathbf{F} \circ \mathbf{P_i} \ ,$$

where $\circ$ denotes the element-wise (Hadamard or Schur) matrix multiplication. To describe the forward transitions of the irregular part of the M/G/1 type Markov chain (the first block row of matrix $\mathbf{Q}$) we similarly introduce $\mathbf{F'}$ and $\mathbf{P'_i}$ of size $n' \times n$.

PH approximation of the forward transition structure

The $\mathbf{P_i}$ matrix series describe n^2 discrete distributions. We can approximate each of these distributions by a discrete PH distribution. (Publicly available software tools can be applied for this step [5].)

To exploit the similarities and redundances of the distributions which often occur in practical applications we introduce the following general form to approximate $\mathbf{P_i}$.

$$\tilde{\mathbf{P}}_\mathbf{i} = \sum_{k=1}^{K} \tilde{\mathbf{M}}_\mathbf{k} \, p_k(i), i \geq 1,$$

where K is a finite integer, $\tilde{\mathbf{M}}_\mathbf{k}$ $(1 \leq k \leq K)$ are non-negative matrices of size $n \times n$ and $p_k(i)$ $(1 \leq k \leq K)$ are probability mass functions of discrete phase type distributions. In the worst case $K = n^2$ and the is an $\tilde{\mathbf{M}}_\mathbf{k}$ matrix (such that $[\tilde{\mathbf{M}}_\mathbf{k}]_{ij} = 1$ and all other matrix elements are zero) for each i, j element of the $\mathbf{P_i}$ matrix. $\mathbf{P'_i}$ is approximated similarly.

M/G/1 type Markov chain with PH distributed forward transitions

Based on this discrete PH approximation we compose an approximate M/G/1 type Markov chain:

$$\tilde{\mathbf{Q}} = \begin{array}{|c|c|c|c|c|}\hline \mathbf{L}' & \tilde{\mathbf{F}}'_\mathbf{1} & \tilde{\mathbf{F}}'_\mathbf{2} & \tilde{\mathbf{F}}'_\mathbf{3} & \tilde{\mathbf{F}}'_\mathbf{4} \\ \hline \mathbf{B}' & \mathbf{L} & \tilde{\mathbf{F}}_\mathbf{1} & \tilde{\mathbf{F}}_\mathbf{2} & \tilde{\mathbf{F}}_\mathbf{3} \\ \hline & \mathbf{B} & \mathbf{L} & \tilde{\mathbf{F}}_\mathbf{1} & \tilde{\mathbf{F}}_\mathbf{2} \\ \hline & & \mathbf{B} & \mathbf{L} & \tilde{\mathbf{F}}_\mathbf{1} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}\,, \tag{3}$$

where $\tilde{\mathbf{F}}_\mathbf{i} = \mathbf{F} \circ \tilde{\mathbf{P}}_\mathbf{i}$. To simplify notation we introduce $\mathbf{M_k} = \mathbf{F} \circ \tilde{\mathbf{M}}_\mathbf{k}$, from which

$$\tilde{\mathbf{F}}_\mathbf{i} = \mathbf{F} \circ \sum_{k=1}^{K} \tilde{\mathbf{M}}_\mathbf{k}\ p_k(i) = \sum_{k=1}^{K} \mathbf{M_k}\ p_k(i).$$

The $\tilde{\mathbf{F}}'_\mathbf{i} = \sum_{k=1}^{K'} \mathbf{M}'_\mathbf{k}\ p'_k(i)$ approximation of $\mathbf{F}'_\mathbf{i}$ is obtained in a similar way. In the rest of the paper α_k and $\mathbf{T}_k$ (α'_k and $\mathbf{T'}_k$) denotes the initial probability vector and the transition probability matrix of the $p_k(i)$ ($p'_k(i)$) PH distribution.

3 Analysis of M/G/1 type Markov chain with PH distributed forward transitions

QBD representation with extended block size

We analyse the stationary behaviour of M/G/1 type Markov chains with PH distributed forward transitions using a QBD process whose phase process is composed by 3 blocks. Block 0 represents the phases of the original M/G/1 type process, block 1 and 2 are for describing the PH distributed forward transitions of the irregular and the regular part of the M/G/1 type process, respectively. (See Figure 1 and Figure 2.) This way we extend the set of phases of the original M/G/1 type Markov chain to obtain a Markov chain with QBD structure such that the number of phases of the QBD process remains finite and the steady state distribution of the M/G/1 type Markov chain can be obtained from the steady state distribution of the QBD.

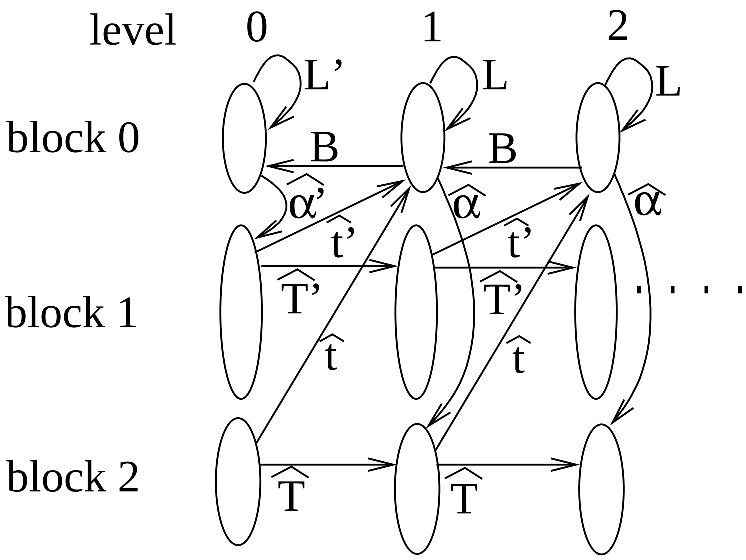

Figure 1: The transition structure of the expanded QBD process

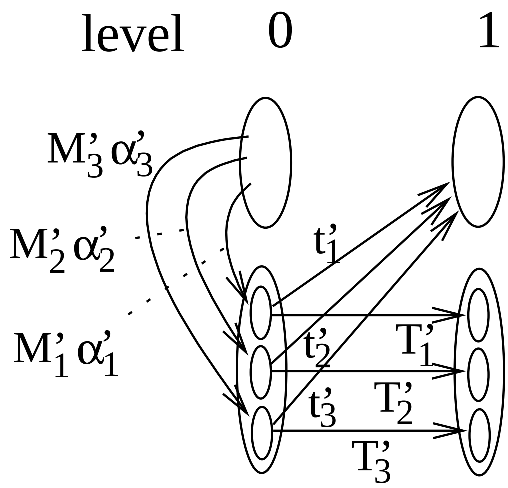

Figure 2: The sub-block structure of $\widehat{\boldsymbol{\alpha}'}$, $\widehat{\mathbf{T}'}$ and $\widehat{\mathbf{t}'}$

The block and sub-block structure of the extended QBD process is

$$\mathbf{Q}' = \begin{array}{|c|c|c|c|c|} \hline \mathbb{B} & \mathbb{A}_0 & & & \\ \hline \mathbb{A}_2 & \mathbb{A}_1 & \mathbb{A}_0 & & \\ \hline & \mathbb{A}_2 & \mathbb{A}_1 & \mathbb{A}_0 & \\ \hline & & \mathbb{A}_2 & \mathbb{A}_1 & \mathbb{A}_0 \\ \hline & & & \ddots & \ddots \\ \hline \end{array} \quad , \tag{4}$$

with

$$\mathbb{B} = \begin{array}{|c|c|c|} \hline \mathbf{L}' & \widehat{\boldsymbol{\alpha}'} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \hline \end{array} \quad , \quad \mathbb{A}_0 = \begin{array}{|c|c|c|} \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widehat{\mathbf{t}'} & \widehat{\mathbf{T}'} & \mathbf{0} \\ \hline \widehat{\mathbf{t}} & \mathbf{0} & \widehat{\mathbf{T}} \\ \hline \end{array} \quad , \tag{5}$$

$$\mathbb{A}_1 = \begin{array}{|c|c|c|}\hline \mathbf{L} & \mathbf{0} & \widehat{\boldsymbol{\alpha}} \\ \hline \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \hline \end{array}, \quad \mathbb{A}_2 = \begin{array}{|c|c|c|}\hline \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \end{array}, \tag{6}$$

where

$$\widehat{\boldsymbol{\alpha}} = \begin{array}{|c|c|c|}\hline \alpha_1 \odot \mathbf{M}_1 & \ldots & \alpha_K \odot \mathbf{M}_K \\ \hline \end{array},$$

$$\widehat{\mathbf{T}} = \begin{array}{|c|c|c|}\hline \mathbf{T_1} \odot \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{T_K} \odot \mathbf{I} \\ \hline \end{array}, \quad \widehat{\mathbf{t}} = \begin{array}{|c|}\hline \mathbf{t_1} \odot \mathbf{I} \\ \hline \vdots \\ \hline \mathbf{t_K} \odot \mathbf{I} \\ \hline \end{array}, \tag{7}$$

and the $\widehat{\boldsymbol{\alpha'}}$, $\widehat{\mathbf{T}'}$, $\widehat{\mathbf{t}'}$ terms are defined similarly. Throughout this paper $\odot$ denotes the Kronecker product.

Theorem 1. *The stationary distribution of the M/G/1 type Markov chain with PH distributed jumps (2) can be calculated from the stationary distribution of the enlarged, but finite phase QBD process (4) as follows:*

$$\boldsymbol{\pi}_i = \frac{\boldsymbol{\pi}'_{i,0}}{\sum_{k=0}^{\infty} \boldsymbol{\pi}'_{k,0}}, \tag{8}$$

where $\boldsymbol{\pi}_i$ is the stationary distribution of the M/G/1 type Markov chain with PH distributed jumps (with generator $\tilde{\mathbf{Q}}$), $\boldsymbol{\pi}'_i$ is the stationary distribution of the enlarged QBD process (with generator $\mathbf{Q}'$) and $\boldsymbol{\pi}'_{i,0}$ denotes the block 0 part of the $\boldsymbol{\pi}'_i$ vector.

Proof: The proof follows the pattern of Section 13.1 of [6]. The main idea of Section 13.1 of [6] is to transform M/G/1 type models into QBD models with an extended phase process. In case of a general M/G/1 process an infinite phase process is introduced in this transformation, but in case of PH distributed batch size a finite extension of the phase process is sufficient as it is presented by the block structure of matrices $\mathbb{A}_0$, $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{B}$ and depicted in Figure 1.

Here we only show that the extended Markov chain (with generator $\mathbf{Q}'$) restricted to block 0 is identical with the M/G/1 type Markov chain with generator $\mathbf{Q}$. Indeed, block 0 of the extended process represents the states of the original process. The rest of the proof is identical with the one in [6] pp. 268-275.

The downward and local transitions of the process restricted to block 0 are readable, e.g. from Figure 1. They are identical with the downward and local transitions of $\mathbf{Q}$. The upward transitions between consecutive visits in block 0 are through block 1 (starting from level 0) or block 2 (starting from level $i \geq 1$).

First we consider the case of starting from level 0. The upward transition rate of the restricted process can be obtained as the product of the exit rate from block 0 and the probability of returning to block 0 in a given state. The exit rate of block 0 is characterized by matrix $\sum_{k=1}^{K'} \mathbf{M'_k}$. Each $\mathbf{M'_k}$ matrix selects one PH structure of block 1 and it remains unchanged during the visit in block 1 (see Figure 2). The probability of starting from block 0 of level 0 moving to block 1 and returning to block 0 at level i ($i \geq 1$) supposed that the kth PH structure is selected equals to $\alpha_k \mathbf{T_k}^{i-1} \mathbf{t_k}$. Hence the upward transitions of the restricted process from level 0 to level i are

$$\sum_{k=1}^{K'} \mathbf{M'_k} \; \alpha_k \mathbf{T_k}^{i-1} \mathbf{t_k} = \sum_{k=1}^{K'} \mathbf{M'_k} \; p'_k(i) = \mathbf{C'_i}.$$

The upward transitions starting from level $i \geq 1$ can be obtained similarly. □

4 Numerical example

We consider a queueing system, where the forward level jumps are described by an order 6 discrete PH distribution that approximates power-tailed discrete distribution for 5 orders of magnitude as it is shown in Figure 3. (Details of the applied PH approximation method can be found in [4].)

The transition matrix and the initial vector of the PH level jumps is

$$\mathbf{I} - \mathbf{T} \approx \begin{pmatrix} 0.170 & 0 & 0 & 0 & 0 & 0 \\ -0.076 & 0.083 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.030 \cdot 10^{-6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.417 \cdot 10^{-5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.607 \cdot 10^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.013 \end{pmatrix},$$

$$\alpha \approx \begin{pmatrix} 0.300 & 0.660 & 2.33 \cdot 10^{-12} & 6.100 \cdot 10^{-9} & 1.583 \cdot 10^{-5} & 0.040 \end{pmatrix}.$$

This representation corresponds to an average level jump of ≈ 16.45 customers. The probability density function of the level jumps (defined by the $\mathbf{F}_i$ matrices) is shown in Figure 3.

The rate matrices corresponding to the backward and local transitions are

$$\mathbf{B} = \begin{pmatrix} 50 & 0 \\ 0 & 5 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} * & 1 \\ 1 & * \end{pmatrix},$$

where the diagonal element of $\mathbf{L}$ are such that the row sum of $\mathbf{Q}$ is zero. Matrix $\mathbf{M}$ can be scaled to set the server utilisation:

$$\mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix}.$$

The numerical examples below are computed with $m = 1$ thus the average utilisation is 0.3.

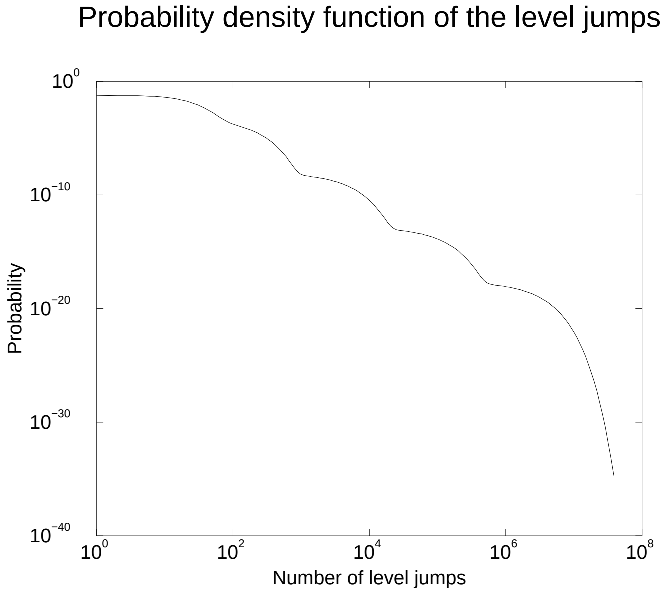

Figure 3: The the pdf of the level jumps in log-log scale

The forward level jump distribution given that there is no customer in the system is the same as the level jump at any other buffer level. That is $\mathbf{T}' = \mathbf{T}$, $\alpha' = \alpha$ and also $\mathbf{M}' = \mathbf{M}$.

The dimension of the QBD matrices based on (5), (6) and (7) is $26x26$.

Computation of the stationary probabilities

The stationary distribution of the expanded QBD process can be computed as $\boldsymbol{\pi}'_n = \boldsymbol{\pi}'_0 \mathbb{R}^n$, where $\mathbb{R}$ is the minimal non-negative solution of the following matrix equation:

$$\mathbf{0} = \mathbb{A}_0 + \mathbb{R}\mathbb{A}_1 + \mathbb{G}^2 \mathbb{A}_2.$$

This equation is usually solved by an iterative substitution method. A collection of solution methods is provided by Latouche and Ramaswami [6].

Here the interesting part of $\boldsymbol{\pi}'_n$ is the first block $\boldsymbol{\pi}'_{n,0}$. The solution has to be normalized such that $\sum_{n=0}^{\infty} \boldsymbol{\pi}'_{n,0} \mathbb{I} = 1$. The required $\boldsymbol{\pi}'_0$ can be obtained by solving the following system of equations:

$$\boldsymbol{\pi}'_0 \left(\mathbb{B} + \mathbb{A}_0 \mathbb{G} \right) = \mathbf{0},$$
$$\boldsymbol{\pi}'_0 \left(\mathbb{I} - \mathbb{R} \right)^{-1} \mathbb{I}^* = 1,$$

where $\mathbb{I}^*$ is a column vector with the same dimension as $\boldsymbol{\pi}'_0$ whose first $|\boldsymbol{\pi}'_{n,0}|$ block contains ones and the rest of the vector elements are zero.

Once $\boldsymbol{\pi}'_0$ is computed, $\boldsymbol{\pi}'_{n+1}$ can be obtained in an iterative way from $\boldsymbol{\pi}'_n$ by $\boldsymbol{\pi}'_{n+1} = \boldsymbol{\pi}'_n \mathbb{R}$. An advantage of this method is that the number of operations needed in one iteration step is independent of n.

Slowly decaying queue length

The queue length of an M/G/1 queuing system with power tailed service time distribution rate has a power tail. The present example demonstrates this property in an M/G/1 type queueing system. Although the level jumps does not have a proper power tail, their function is slowly decaying for several orders of magnitude. Figure 4 shows the queue length distribution in log-log scale calculated with the expanded QBD method.

One can see a region in the distribution of both the decay of the level jumps Figure 3 and the queue length Figure 4 where the decay is almost linear in log-log scale. That is, within this region the distribution functions approximate $c\, n^b$ functions. These regions go though 5 orders of magnitude in both cases. The power of the decay is approximately $b = -3.5$ for the level jumps and $b = -2.5$ for the queue length distribution as it can be expected from the $M/G/1$ analogy.

Numerical solution using the Ramaswami iterative formula

The example presented in this section can be solved numerically using the formulae presented in Section 13.1 in [6]. In the case of this approach, however, the level jumps should be truncated in order to implement the calculations in practice. Therefore, the $(\alpha, \mathbf{T})$ discrete PH distribution is approximated by a distribution with finite support such that

$$p_n = \frac{P(X = n)}{P(X \leq N)}\,, \quad 1 \leq n \leq N, \quad \text{and} \quad p_n = 0, \quad n > N,$$

where X is $(\alpha, \mathbf{T})$ PH distributed random variable.

Equation 13.10 in [6] was solved by iterative substitution initialised with the identity matrix. The $\mathbf{C_0}$ and $\mathbf{C_1}$ matrices of [6] are ($\mathbf{B}$ and ($\mathbf{L}$, $\mathbf{C_n} = p_{n-1}\mathbf{M}$ for $2 \leq n \leq N$ and $\mathbf{C_n} = \mathbf{0}$ for $n > N$. The number of iteration steps until the solution of matrix ($\mathbf{G}$ converged was around 10 for $N = 100$, 1000 and 100000. That is, the number of iterations seems to be independent of N.

Figure 5 shows how the exact queue length distribution can be approximated by the (13.12) method of [6] using different truncation points. As it can be expected, if N is small, the queue length distribution quickly reaches the geometric tail and if N is larger, it follows the exact distribution longer.

Based on the numerical results we can approximate the break point, q, of the numerical solution as a function of the truncation point, N. The following table shows the observed break points as a function of N.

N	100	1000	100000
q	30	300	16000

One of the most important consequences of the introduces results is apparent from the comparison of Figure 4 and 5 and the table with the approximate break points. The (13.12)

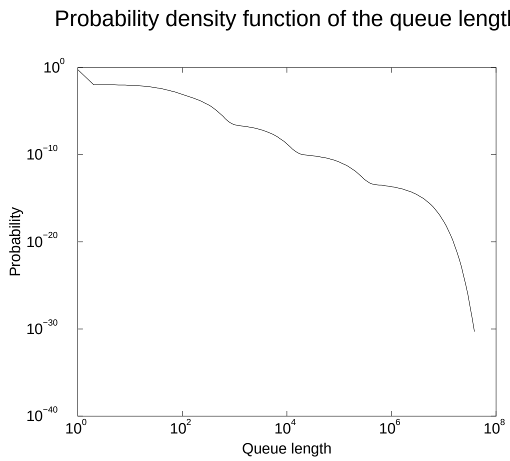

Figure 4: The the pdf of the queue length in log-log scale

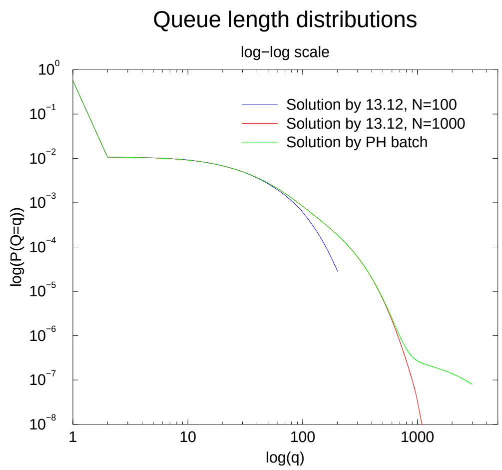

Figure 5: Comparison of the various solutions for the queue length distribution and the exact distribution

method of [6], which is the most commonly applied method for solving M/G/1 type models, can not be used to calculate slowly decaying tail distribution. It seems to us that a numerically feasible way to calculate these kinds of models is the procedure based on the extended QBD model presented in this paper.

5 Conclusion

This paper presents the analysis of M/G/1 type Markov chains with PH distributed transition structure. A numerical example is introduced and evaluated to compare the presented numerical methods with each others and with the most commonly applied general M/G/1 type Markov chain solver.

This numerical example suggests that the best numerical solution of M/G/1 type models with generally distributed heavy tailed jumps is to approximate the jump distribution with a PH one and to use the numerical procedure with the expanded QBD.

It is because the Ramaswami method involve infinite summation when the domain of the jump size distribution is not bounded. However in practice, this summation should be truncated. If the members of the infinite sum have polynomial asymptotics then a finite truncation can introduce significant error as we showed it in this paper.

Our method works better because the extended state space makes it possible to consider Markovian jump size distributions with unbounded domain and it is known that even a heavy tailed distribution can be approximated with a Markovian distribution over several orders of magnitude. That is, our method works on logarithmic scale while the existing methods work on linear scales only.

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