Matrix exponential distributions with minimal coefficient of variation

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Outline

- Motivation
- Matrix exponential distributions
- Properties of ME distributions
- Positive ME subclasses
- Numerical results
- Conclusions

Motivation

Markov models are effectively used in performance analysis.

Phase type (PH) approximation can be applied if non-exponential event times are present.

Event times with low coefficient of variation requires high order PH distributions, since

$$cv^2_{PH} = \frac{\mu_0 \mu_2}{\mu_1^2} - 1 \geq \frac{1}{n}$$

$\longrightarrow$ state space explosion.

Can matrix exponential (ME) distributions help in this respect?

Matrix exponential distributions

$f(t)$ is an order n matrix exponential density function, if

- $f(t) = ve^{At}w'$, where v, w are vectors and A is a square matrix of cardinality n,
- $f(t) \geq 0, \forall t \geq 0$,
- normalized such that $\mu_0 = \int_t f(t)dt = 1$.

See, e.g., the Ph.D. thesis of Mark Fackrell for a recent survey on ME distributions, here we mention only the cv related properties.

In this work, we do not normalize $f(t)$, but minimize

$$cv = \frac{\mu_0 \mu_2}{\mu_1^2} - 1,$$

where $\mu_i = \int_t t^i f(t)dt$.

Properties of ME distributions

The moments of ME distributions are: $\mu_i = i!\; v(-\mathbf{A})^{-k} w'$

ME distributions are known to have rational Laplace transform:

$$f^*(s) = \int_t e^{-st} f(t) dt = \frac{a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + \ldots + b_1 s + b_0}$$

Starting from the Laplace domain representation the moments are $\mu_i = (-1)^i \frac{d^i}{ds^i} f^*(s)\Big|_{s=0}$,

from which

$$cv = \frac{\mu_0 \mu_2}{\mu_1^2} - 1 = \frac{2a_0(a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2 - a_0 b_0 b_2)}{(a_1 b_0 - b_1 a_0)^2}.$$

I.e., the first 6 coefficients determine the cv
$\longrightarrow$ easy to minimize !!!

Main difficulty of ME distributions

The major problem with ME distributions is that

- neither the $\{v, A, w'\}$,
- nor the $f^*(s)$ (or $\{a_n, \ldots, a_0, b_n, \ldots, b_0\}$)

representation indicates the non-negativity of $f(t)$.

$\longrightarrow$ the $\{a_2, a_1, a_0, b_2, b_1, b_0\}$ coefficients which minimize the cv commonly assigned with a negative $f(t)$.

Classes of ME functions

Further definitions:

$f(t) = ve^{At}w'$, is an order n *ME function* if v, w are vectors and A is a square matrix of cardinality n and $\lim_{t\to\infty} f(t) = 0$.

$f(t) = ve^{At}w'$, is an order n *positive ME density function* if v, w are vectors and A is a square matrix of cardinality n, $\lim_{t\to\infty} f(t) = 0$ and $f(t) > 0$ for $\forall t > 0$.

Let

- $\mathcal{ME}(n)$ be the set of order n ME functions,
- $\mathcal{ME}^{\oplus}(n)$ be the set of order n ME density functions,
- $\mathcal{ME}^{+}(n)$ be the set of order n positive ME density functions.

Based on their definition

$$\mathcal{ME}(n) \subset \mathcal{ME}^{\oplus}(n) \subset \mathcal{ME}^{+}(n) \;\Big(\subset \mathcal{PH}(n) \Big).$$

Extreme distributions

Basic observation:

With respect to the minimal coefficient of variation there is local minimum in the $\mathcal{ME}^+(n)$ set.

Crucial consequence:

The order n ME distribution with minimal cv is a member of the $\mathcal{ME}^\oplus(n) \setminus \mathcal{ME}^+(n)$ set.

The statement and the consequence hold also for the subclass of ME distributions with real eigenvalues.

$\longrightarrow$ investigate the $\mathcal{ME}^\oplus(n) \setminus \mathcal{ME}^+(n)$ set.

A set of extreme ME functions

A ME function with real eigenvalues can be represented as

$$f(t) = \sum_{i=1}^{\#\lambda} \sum_{j-1}^{\#\lambda_i} a_{ij} t^{j-1} e^{-\lambda_i t}.$$

But

- this representation does not ensure non-negativity,
- (similar to the PH distributions) the eigenvalues of ME distributions with minimal coefficient of variation are identical.

A subclass of ME distributions satisfying these requirements is

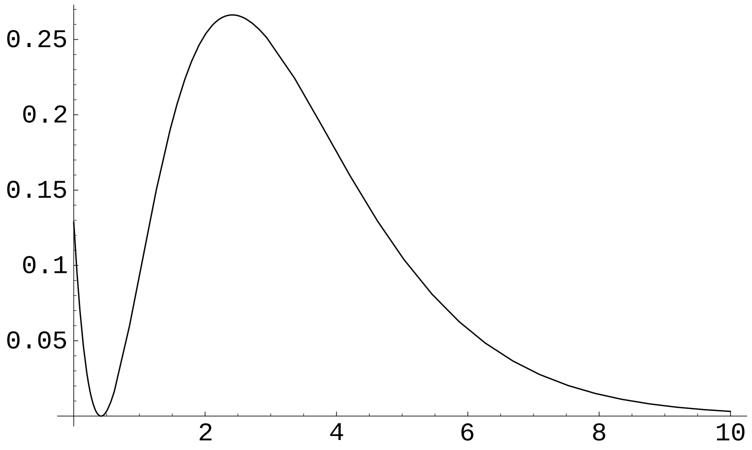
$$f_r^{\odot}(t) = \begin{cases} \prod_{i=1}^{(n-1)/2} (t-\tau_i)^2 \; e^{\lambda t} & \text{if } n \text{ is odd,} \\ \prod_{i=1}^{(n-2)/2} (t-\tau_i)^2 \; t \; e^{\lambda t} & \text{if } n \text{ is even.} \end{cases}$$

- non-negative,
- touches the x access as many times at it can.

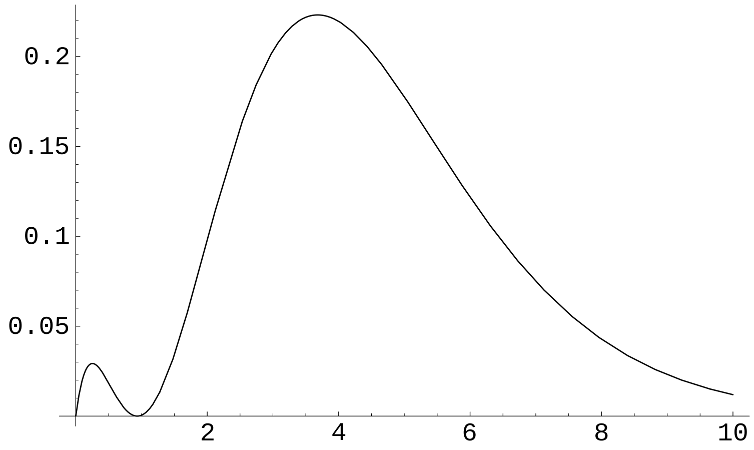
Extreme ME functions with real eigenvalues

The $f_r^{\oplus}(t)$ function with the minimal cv is obtained by optimizing over the $(n-1)/2$ (or $(n-2)/2$) τ_i points.

Symbolic solution for $n=3$ and numeric solutions for higher orders results

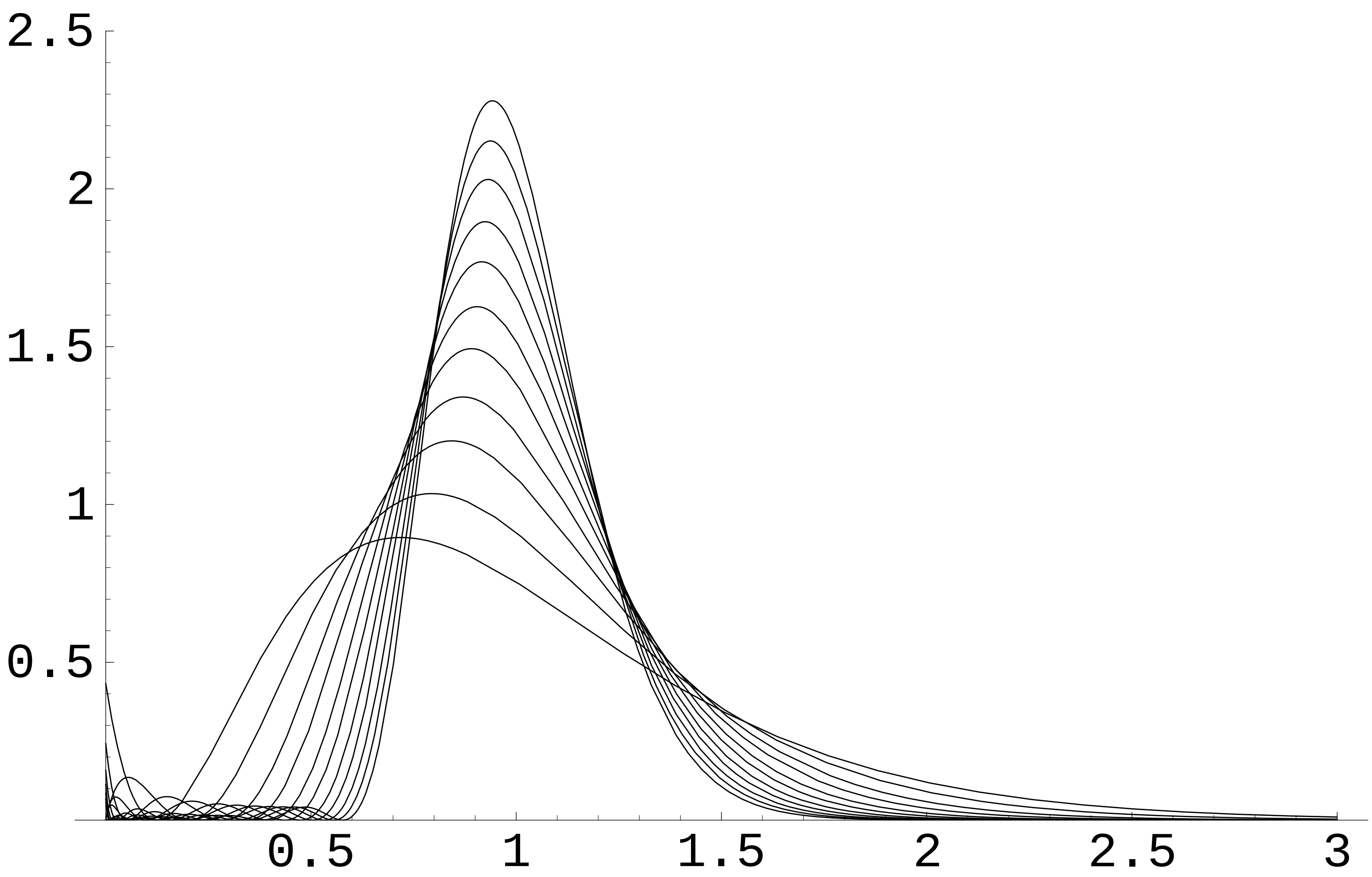


$n=3$



$n=4$

Extreme ME functions with real eigenvalues



Minimal $f_r^{\odot}(t)$ functions with $n = 3, \ldots, 13$

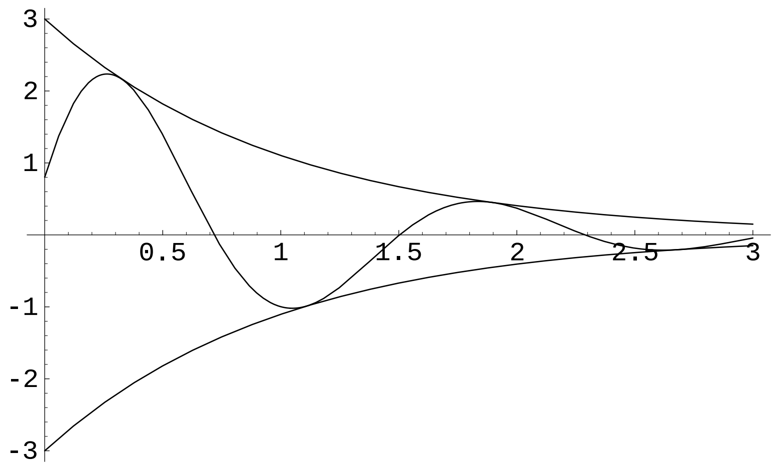
ME functions with complex eigenvalues

Contribution of a pair of complex conjugate eigenvalues

$$
\begin{aligned}
f_c(t) &= a e^{-pt} + \overline{a} e^{-\overline{p}t} \\
&= 2e^{-Re[p]t}\left(Re[a]\mathbf{Cos}(Im[p]t) + Im[a]\mathbf{Sin}(Im[p]t)\right) \\
&= 2ce^{-\lambda t}\mathbf{Cos}(\omega t - \phi)
\end{aligned}
$$

where $Re[p] = \lambda$, $Re[a] = c\ \mathbf{Cos}(\phi)$, $Im[a] = c\ \mathbf{Sin}(\phi)$, $Im[p] = \omega$.

$f_c(t)$ oscillates between $2ce^{-\lambda t}$ and $-2ce^{-\lambda t}$.



It is hard to see if $\sum_i f_{c_i}(t) + \sum_j e^{-\lambda_j t} \geq 0$.

Order 3 ME functions with complex eigenvalu

In case of order 3 ME functions we have one complex conjugate eigenvalue pair and a real one.

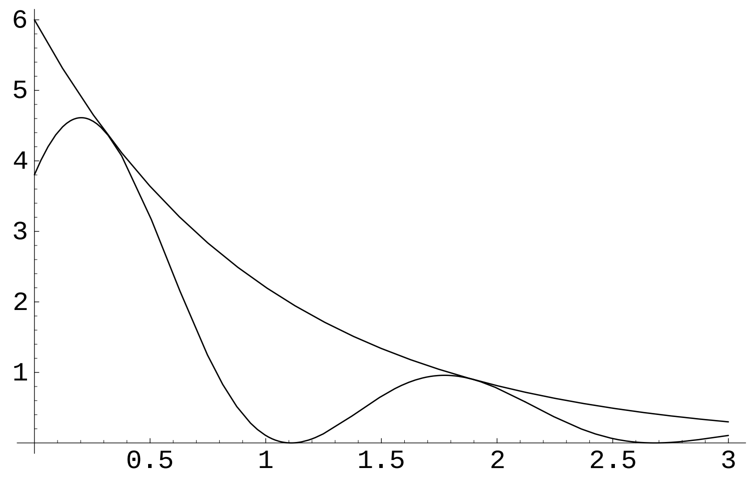
$$f(t) = 2ce^{-\lambda t}\mathbf{Cos}(\omega t - \phi) + a_r e^{-\lambda_r t}$$

If $a_r e^{-\lambda_r t} - 2ce^{-\lambda t} > 0$, then $f(t) \geq 0$. (sufficient condition)

Assuming $a_r e^{-\lambda_r t} = 2ce^{-\lambda t}$, we have:

$$f(t) \quad = \quad 2ce^{-\lambda t}\mathbf{Cos}(\omega t - \phi) + 2ce^{-\lambda t} = 4ce^{-\lambda t}\mathbf{Cos}^2\left(\frac{\omega t - \phi}{2}\right)$$

$f(t)$ oscillates between $4ce^{-\lambda t}$ and 0.

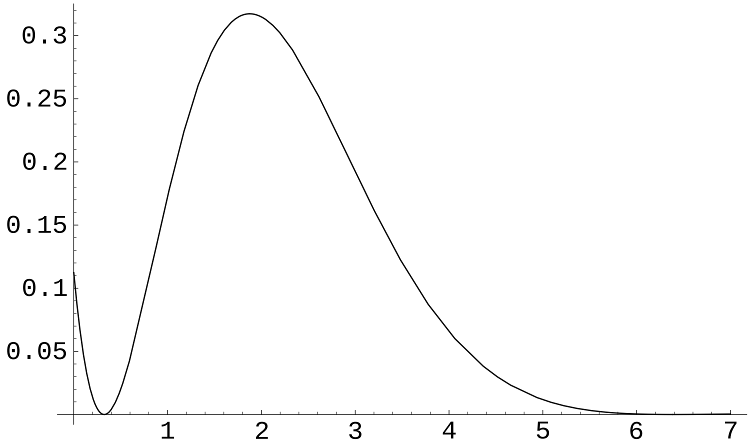


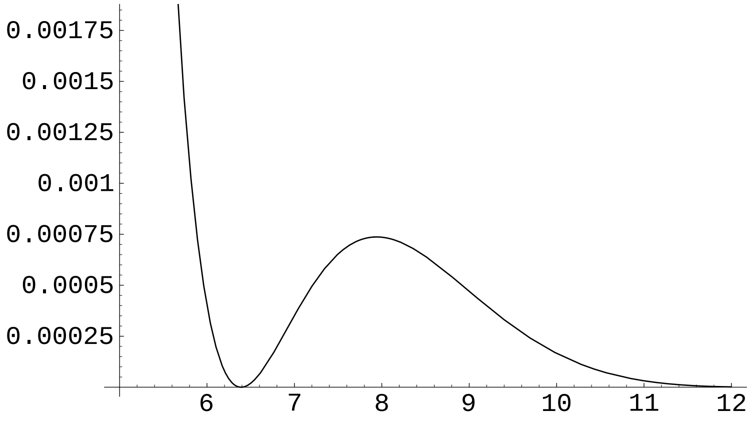
Minimal ME distribution of order 3

We search for the optimum in the subset

$$f_c^{\odot}(t) = 4ce^{-\lambda t}\mathbf{Cos}^2\left(\frac{\omega t - \phi}{2}\right),$$

where c and λ are scaling and normalizing factors. A numerical optimization of cv with respect to ω and ϕ results





- non-negative function,
- touches the x access infinitely many times.

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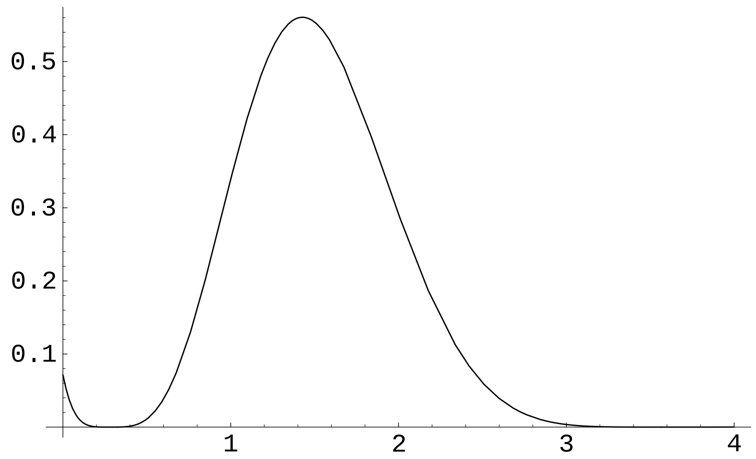
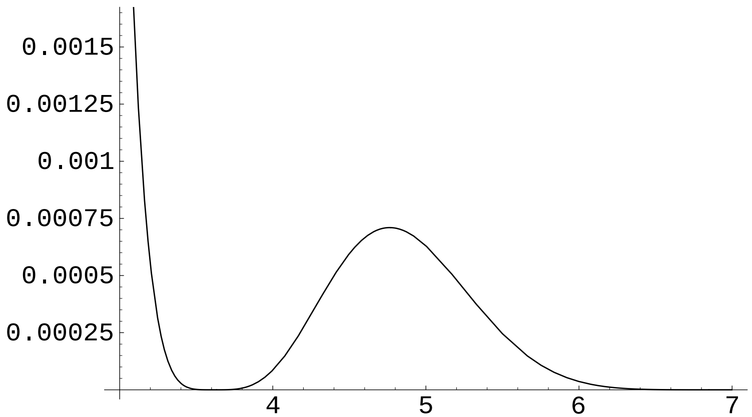
Minimal ME distribution of odd order

When $n = 2k + 1$, we search for the optimum in the subset

$$f_c^{\odot}(t) = e^{-t}\mathbf{Cos}^{2k}\left(\frac{\omega t - \phi}{2}\right),$$

A numerical optimization of cv with respect to ω and ϕ results the distribution with minimal cv.





Similarly, $f_c^{\odot}(t)$

- is a non-negative function,
- touches the x access infinitely many times.

Order 4 ME functions with complex eigenvalu

In case of order 4 ME functions we have one complex conjugate eigenvalue pair and two real ones.

$$f(t) = 2ce^{-\lambda t}\mathbf{Cos}(\omega t - \phi) + a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t}$$

To reduce the general ME class we applied the following constraints:

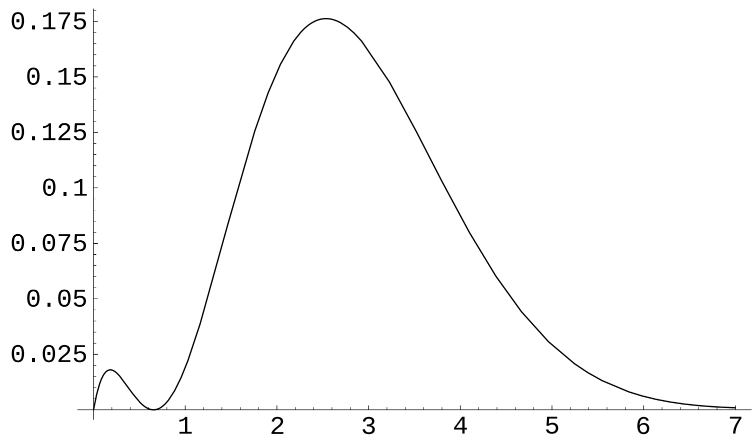
- $\lambda_1 = \lambda$,
- $f(0) = 0$ (similar to the real eigenvalue case),
- $f(t)$ touches the x access at τ, i.e., $f(\tau) = 0$, $f'(\tau) = 0$.

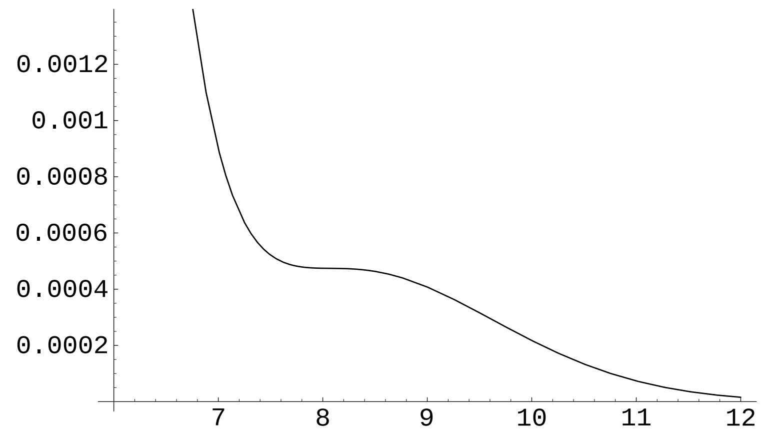
The obtained subclass,

- is non-negative,
- touches the x access once at τ.

Minimal ME distribution of order 4

A numerical optimization with respect to ω, ϕ and τ results the distribution with minimal cv.





Minimal ME distribution of even order

When $n = 2k + 2$, we search for the optimum in the subset

$$f_c^{\odot}(t) = e^{-t}\mathbf{Cos}^{2k}\left(\frac{\omega t - \phi}{2}\right) + ae^{-t} + a_2 e^{-\lambda_2 t},$$

with the constraints

- $f_c^{\odot}(0) = 0$,
- $f_c^{\odot}(t)$ touches the x access at τ, i.e., $f_c^{\odot}(\tau) = 0$, $f'^{\odot}_c(\tau) = 0$.

A numerical optimization with respect to ω, ϕ and τ results the distribution with minimal cv.

Coefficient of variation of ME distributions

Poles	MinVar	1/MinVar	MinVar	1/MinVar
	real poles		complex poles	
3	0.276583	3.61556	0.200902	4.97756
4	0.19333	5.17251	0.149808	6.6752
5	0.138453	7.22266	0.0935254	10.6923
6	0.108623	9.20619	0.0787789	12.6937
7	0.0861277	11.6107	0.0566648	17.6476
8	0.0717026	13.9465	0.0488389	20.4755
9	0.0600486	16.6532	0.0391485	25.5438
10	0.0518365	19.2914	0.0342547	29.1931
11	0.0449173	22.2632	0.0292523	34.1854
12	0.0397335	25.1677	0.0258868	38.6297
13	0.0352403	28.3766	0.0230218	43.4371
14	0.031726	31.5199	0.0205567	48.646
15	0.0286172	34.944	0.0187966	53.201

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Conclusions

It is hard to characterize the borders of ME distributions.

$\longrightarrow$ constrained numerical optimization cannot be applied to find the minimum.

The ME distributions with minimal coefficient of variation is on the border of this set.

We defined non-negative subsets with low coefficient of variation

$\longrightarrow$ unconstrained numerical optimization can be applied to find the minimum.