

CHAPTER 11

**MARKOV RENEWAL THEORY
APPLIED TO PERFORMABILITY
EVALUATION**

Ricardo Fricks, Miklós Telek,
Antonio Puliafito, Kishor Trivedi

11.1. INTRODUCTION

Computer and communication systems are designed to meet a certain specified behavior. The procurement of metrics to establish how well the system behaves, that is, how closely it follows the specified behavior, is the objective of quantitative analysis. Traditionally, performance and dependability evaluation are used as separate approaches to provide quantitative figures of system behavior. Performance evaluates the quality of service, assuming that the system is failure-free.¹ Dependability focuses on determining deviation of the actual behavior from the specified behavior in the presence of component or subsystem failures.²

Beaudry³ proposed the aggregated measure *computation before failure*, while Meyer⁴ proposed the term *performability*, which has been used since then. Performability analysis aims to capture the interaction between the failure-repair behavior and the performance delivered by the system. Its results are fundamental to the analysis of real-time system performance in the presence of failure.⁵

Performability measures provide better insight into the behavior of fault-tolerant systems. Basic metrics used to evaluate fault-tolerant de-

signs are reliability and availability. The conditional probability that a system survives until some time t , given it is fully operational at $t = 0$, is called the reliability $R(t)$ of the system.⁶ Reliability is used to describe systems which are not allowed to fail; in which the system is serving a critical function and cannot be down. Note that components or subsystems can fail so long as the system does not. The instantaneous availability $A(t)$ of a system is the probability that the system is properly functioning at time t . Availability is typically used as a basis for evaluating systems in which functionality can be delayed or denied for short periods without serious consequences.⁷ Reliability and availability do not consider different levels of system functionality.

Performability analysis of real systems with non-deterministic components and/or environmental characteristics results in stochastic modeling problems. Several techniques for solving them for transient and steady-state measures have been proposed and later combined under the framework of Markov reward models.⁸ The traditional framework allows the solution of stochastic problems enjoying the **Markov property**: *the probability of any particular future behavior of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behavior.*^{6,9} If the past history of the process is completely summarized in the current state and is independent of the current time, then the process is said to be (*time-*) *homogeneous*. Otherwise, the exact characterization of the present state needs the associated time information, and the process is said to be *non-homogeneous*. A wide range of real problems fall in the class of Markov models (both homogeneous and non-homogeneous), but problems in performability analysis have been identified that cannot be adequately described in this traditional framework. The common characteristic these problems share is that the Markov property is not valid (if valid at all) at all time instants. This category of problems is jointly referred to as *non-Markovian* models and can be analyzed using several approaches:

- *Phase-type expansions* - ^{10,11} when the past history of the stochastic process can be described by a discrete variable, an expanded continuous-time homogeneous Markov chain can be used to capture the stochastic behavior of the original system.
- *Supplementary variables* - ¹² when the past history is described by one or more continuous variables, the approach of the supple-

mentary variables can be applied and a set of ordinary or partial differential equations can be defined together with boundary conditions and analyzed.

- *Embedded point-processes* - ^{13,14} when the temporal behavior of the system can be studied by means of some appropriately chosen embedded epochs where the Markov property applies. Several well-known classes of stochastic processes such as regenerative, semi-Markov and Markov regenerative processes are based on the concept of embedded points.

The object of this chapter is to present a theory based on the concept of embedded point-processes that encompass semi-Markov and Markov regenerative processes. This theory, named Markov renewal theory, is reviewed in the first three sections of this chapter and later applied to several non-Markovian performability models.

Our purpose is to provide an up-to-date treatment of the basic analytic models to study non-Markovian systems by means of Markov renewal theory and an accurate description of the solution algorithms. In particular, we develop a general framework which allows us to deal with renewal processes and specifically with semi-Markov and Markov regenerative processes. We hope that this chapter will serve as a reference for practicing engineers, researchers and students in performance and reliability modeling. Other surveys on Markov renewal theory applied to reliability analysis have appeared in the literature,^{15,16} but none of them as complete or as didactic as the present one.

The rest of this chapter is organized as follows. Section 11.2 introduces the basic terminology associated with the theory, including the concepts and distinction between semi-Markov processes and Markov regenerative processes. Section 11.3 presents basic solution techniques for stochastic processes with embedded Markov renewal sequences. Markov regenerative Petri nets, useful as a high-level description language of these kind of stochastic models, are reviewed in Section 11.4 and employed in the analyses of three examples presented in Section 11.5. Examples are selected to illustrate the methodology associated with semi-Markov and Markov regenerative processes. Section 11.6 concludes the chapter.

11.2. MARKOV RENEWAL THEORY

Assume we wish to quantitatively study the behavior of a given non-deterministic system. One possible solution would be to associate a random variable Z_t , taking values in a countable set \mathcal{F} , to describe the state of the system at any time instant t . The family of random variables Z_t constitutes a stochastic process $\mathbf{Z} = \{Z_t; t \in \mathcal{R}_+ = [0, \infty)\}$.

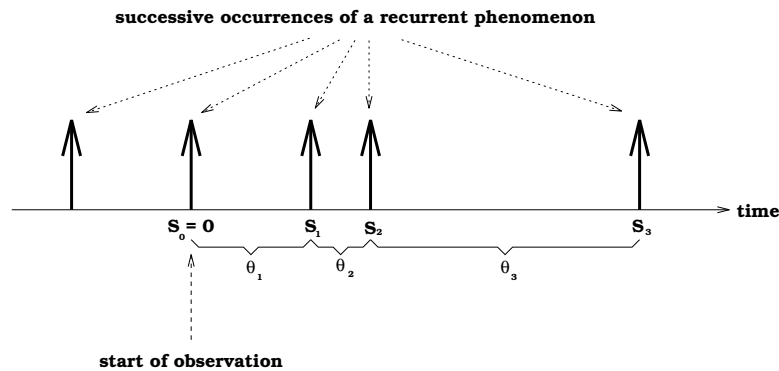


Figure 11.1: A sample realization of a renewal process.

Suppose we are interested in a single event related with the system (e.g., when system components fail). Additionally, assume the times between successive occurrences of this type of event are *independent and identically distributed (iid)* random variables. Let $S_0 < S_1 < S_2 < \dots$ be the time instants of successive events to occur (as shown in Figure 11.1). The sequence of non-negative *iid* random variables, $\mathbf{S} = \{S_n - S_{n-1}; n \in \mathcal{N} = [1, \dots, \infty)\}$ is a *renewal process*. Otherwise, if we do not start observing the system at the exact moment an event has occurred (i.e., $S_0 \neq 0$) the stochastic process \mathbf{S} is a *delayed renewal process*.

Contexts in which renewal processes arise abound in applied probability. For instance, the times between successive electrical impulses or signals impinging on a recording device are often assumed to form a renewal process. Another classical example of renewal process is the item replacement problem explored in ^{9,17} where $S_1 - S_0, S_2 - S_1, \dots$ represent the lifetimes of items (light bulbs, machines, etc.) that are successively placed in service immediately following the failure of the previous one.

However, suppose instead of a single event, we observe that certain transitions between identifiable system states j of a subset \mathcal{E} of \mathcal{F} ,

$\mathcal{E} \subseteq \mathcal{F}$, also resemble the behavior just described, when considered in isolation. Successive times S_n at which a fixed state $j, j \in \mathcal{E}$, is entered form a (possibly delayed) renewal process. In the sample process realization depicted in Figure 11.2 we see that the sequence of time instants $\{S_0, S_4, \dots\}$ forms a renewal process, while $\{S_1, S_5, \dots\}$ and $\{S_2, S_3, \dots\}$ form delayed renewal processes.

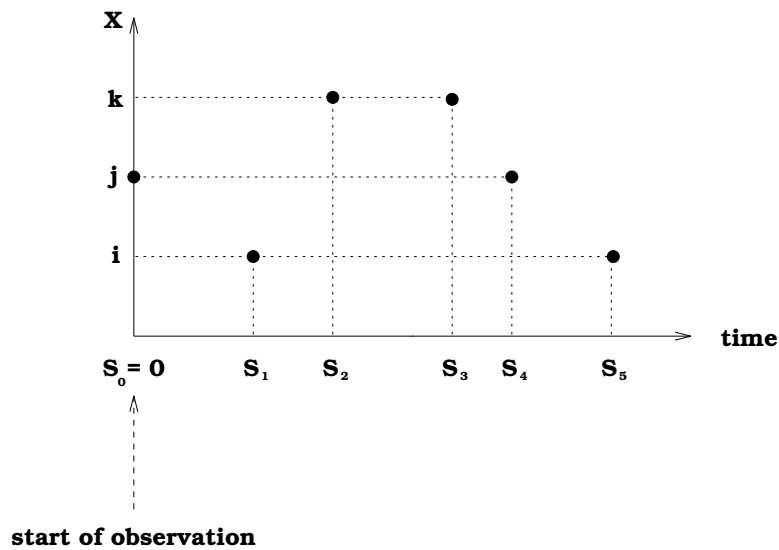


Figure 11.2: A set of renewal processes progressing concurrently.

Additionally, when studying the system evolution we observe that at these particular times the stochastic process \mathbf{Z} exhibits the Markov property, i.e., at any given moment $S_n, n \in \mathcal{N}$, we can forget the past history of the process. In this scenario we are dealing with a countable collection of renewal processes progressing simultaneously such that successive renewals form a discrete-time Markov chain (DTMC). The superposition of all the identified renewal processes gives the points $\{S_n; n \in \mathcal{N}\}$, known as *Markov renewal moments**, and together with the states of the DTMC defines a **Markov renewal sequence** (MRS).

In this section we review the definitions and some of the concepts of **Markov renewal theory**, a collective name that includes MRS's, and two other important classes of stochastic processes with embedded

*Note that these instants S_n are not renewal moments as described in renewal theory, since the distributions of the time interval between consecutive moments are not necessarily iid's.

MRS's, named semi-Markov processes (SMP's) and Markov regenerative processes (MRGP's). Our ultimate aim is to study $\{Z_t; t \in \mathcal{R}_+\}$, however, as a first step we need to study Markov renewal theory. Our emphasis in this chapter is “how to” explore the possibilities of this wealthy theory rather than its technical details (or “why” does it work).

The definitions and terminology mentioned here were influenced by ¹⁸, but the formalism comes from Çinlar.^{19,20} We strongly recommend Çinlar ²⁰ and Kulkarni ²¹ for a more detailed study of Markov renewal theory. Classical references for the other classes of stochastic processes mentioned in this chapter are: renewal processes,^{9,17,22,23} and regenerative processes.^{9,24,25} For the general theory of Markov chains good references are ^{6,9,20}.

11.2.1. Historic Overview of Markov Renewal Theory

Semi-Markov processes were independently introduced by P. Lévy ^{26,27} and W.L. Smith ²⁴ in 1954. Although Smith's work was only published in 1955, its main results were announced in a talk given on the author's behalf by D.V. Lindley at the International Congress of Mathematicians held in Amsterdam in September 1954. At the same congress Lévy announced his results concerning semi-Markovian processes, which were identical with the results given by Smith. Also at the same time, L. Takács ²⁸ introduced and applied the same type of stochastic process to problems in counter theory. Semi-Markov process is a generalization of both continuous and discrete time Markov chains which permit arbitrary sojourn distribution functions, possibly depending both on the current state and on the next state to be entered.

The term Markov renewal sequence is due to R. Pyke, who gave an extensive treatment of many aspects of such processes in 1961.^{29,30} Markov regenerative processes were introduced by R. Pyke and R. Schaufele in 1966 where they were called semi-Markov processes with auxiliary paths.³¹ Most of the theoretical foundations of Markov regenerative processes were laid out in the work of Çinlar in 1975 ²⁰ under the name of semi-regenerative processes. Later, Kulkarni ²¹ suggested the name Markov regenerative process that we use in this chapter.

11.2.2. Markov Renewal Sequence

Define, for each $n \in \mathcal{N}$, a random variable X_n taking values in a countable set \mathcal{E} and a random variable S_n taking values in \mathcal{R}_+ , such that $S_0 \leq S_1 \leq S_2 \leq \dots$, assuming $S_0 \doteq 0$. The bivariate stochastic process $(\mathbf{X}, \mathbf{S}) = \{X_n, S_n; n \in \mathcal{N}\}$ is a **Markov renewal sequence** if it satisfies

$$Pr\{X_{n+1} = j, S_{n+1} - S_n \leq t \mid X_0, \dots, X_n; S_0, \dots, S_n\} =$$

$$Pr\{X_{n+1} = j, S_{n+1} - S_n \leq t \mid X_n\}$$

for all $n \in \mathcal{N}$, $j \in \mathcal{E}$, and $t \in \mathcal{R}_+$. Thus (\mathbf{X}, \mathbf{S}) is a special case of bivariate Markov process in which the increments $S_1 - S_0, S_2 - S_1, \dots$ are all non-negative and are conditionally independent given X_0, X_1, \dots

We will always assume *time-homogeneous* MRS's; that is, the conditional transition probabilities $K_{i,j}(t)$, where

$$K_{i,j}(t) \doteq Pr\{X_{n+1} = j, S_{n+1} - S_n \leq t \mid X_n = i\}$$

are independent of n for any $i, j \in \mathcal{E}$, $t \in \mathcal{R}_+$. Therefore, we can always write

$$K_{i,j}(t) = Pr\{X_1 = j, S_1 \leq t \mid X_0 = i\}, \quad \forall i, j \in \mathcal{E}, t \in \mathcal{R}_+$$

The matrix of transition probabilities $\mathbf{K}(t) = \{K_{i,j}(t) : i, j \in \mathcal{E}, t \in \mathcal{R}_+\}$ is called the **kernel** of the MRS*.

The stochastic sequence $\{X_n; n \in \mathcal{N}\}$ keeps track of the successive states visited at Markov renewal moments and forms a discrete-time Markov chain with state space \mathcal{E} . The one-step transition probabilities of this *embedded Markov chain* (EMC) are

$$\begin{aligned} p_{i,j} &\doteq Pr\{X_{n+1} = j \mid X_n = i\}, \quad \forall i, j \in \mathcal{E} \\ &= \lim_{t \rightarrow \infty} K_{i,j}(t) \end{aligned}$$

*Note that $K_{i,j}(t)$ is a possibly defective distribution function, so that $\lim_{t \rightarrow \infty} K_{i,j}(t) \leq 1$.

There are no restrictions regarding the structure of the EMC on an MRS. There is no imposition that $\{X_n; n \in \mathcal{N}\}$ should be irreducible for instance. Therefore, we can start at time S_0 in a state of \mathcal{E} that will not be reached again at any other Markov renewal moment in the future evolution of the process.

Let the *vector of initial probabilities* of X_0 be described by $\mathbf{a} = (a_0, a_1, \dots)$ where (i) $a_i \doteq Pr\{X_0 = i\} \geq 0, \forall i \in \mathcal{E}$, and (ii) $\sum_{i \in \mathcal{E}} a_i = 1$, then we say that $(\mathbf{X}, \mathbf{S}) = \{X_n, S_n; n \in \mathcal{N}\}$ is an MRS completely determined by $(\mathbf{a}, \mathbf{K}(t))$. Thus, the vector of initial probabilities \mathbf{a} and the kernel matrix $\mathbf{K}(t)$ completely determine all finite-dimensional distributions of the Markov renewal sequence.²¹

Embedded MRS's can be identified associated with semi-Markov and Markov regenerative processes. Hence, we now proceed with a study of such processes.

11.2.3. Semi-Markov Processes

Given an MRS (\mathbf{X}, \mathbf{S}) with state space \mathcal{E} and kernel $\mathbf{K}(t)$, we can introduce the counting process

$$\mathbf{N}(t) \doteq \sup\{n : S_n \leq t\}, \quad \forall t \in \mathcal{R}_+$$

to count the number of Markov renewal moments up to time t , but not considering the one at zero. Using the counting process just defined, we introduce the process $\mathbf{Y} = \{Y_t; t \in \mathcal{R}_+\}$ defined by

$$\begin{aligned} Y_t &\doteq X_{\mathbf{N}(t)} \\ &= X_n, \quad \text{if } S_n \leq t < S_{n+1} \end{aligned}$$

for all $t \in \mathcal{R}_+$, called **semi-Markov process** (SMP) determined by $(\mathbf{a}, \mathbf{K}(t))$. An SMP (for a sample realization see Figure 11.3) is a stochastic process which moves from one state to another within a countable number of states with the successive states visited forming a discrete-time Markov chain, and that the time required for each successive move is a random variable whose distribution function may depend on the two states between which the move is being made. The nomenclature "semi-Markov" comes from the somewhat limited Markov property which \mathbf{Y} has: the future of \mathbf{Y} is independent of its past provided

the present is a Markov renewal moment.²⁰ Note that since we consider $S_0 \doteq 0$, then the initial condition $Y_0 = i$ always means that the SMP has just entered state i at the time origin. Like MRS's, an SMP is specified by its vector of initial probabilities \mathbf{a} and the kernel matrix $\mathbf{K}(t)$.

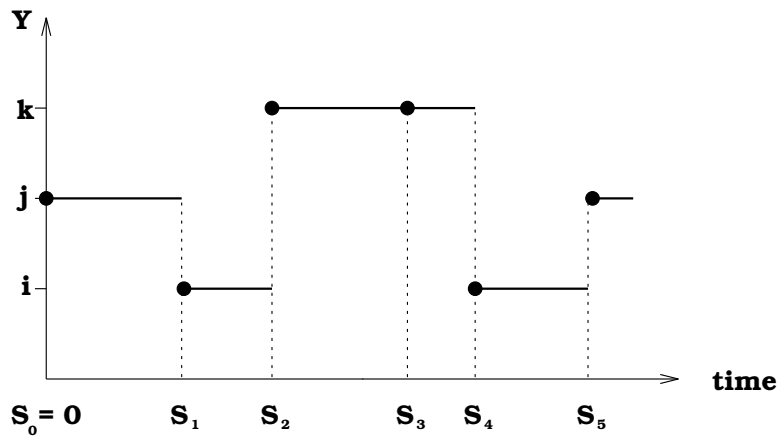


Figure 11.3: A sample realization of a semi-Markov process.

From the SMP definition it should be observed that the process only changes state (possibly back to the same state as shown in Figure 11.3) at the Markov renewal moments S_n . The possibility of transitions not resulting in real state changes can be easily verified by inspecting the kernel matrix $\mathbf{K}(t)$ for non-zero elements on its main diagonal. It follows that, the matrix $\mathbf{K}(t)$ associated with a particular SMP (as well as to an MRS) is not necessarily unique. There are SMP's (as well as MRS's) which can be described by more than one matrix $\mathbf{K}(t)$. A unique kernel matrix can always be defined by its minimal representation, in which all Markov renewal moments represent (real) state transitions. For instance, the minimal representation of the SMP depicted in Figure 11.3 would prevent Markov renewal moment S_3 and, consequently, the time interval between consecutive Markov renewal moments would always represent the sojourn time in each of the EMC states. The derivation of the minimal representation of $\mathbf{K}(t)$ is introduced and explored in ³².

SMP's represent sufficiently general and constructive mathematical models of complex multicomponent systems, whose states are randomly

changed by extreme conditions*. The only essential restriction is the semi-Markov property, that can be interpreted from two different perspectives (actually used to simulate the behavior of SMP's):

- The sequence of system states at times of changes must be described by a homogeneous DTMC, and times to the next Markov renewal moment only depend on the current state and next state of the system.
- The times to the next Markov renewal moment only depend on the current state, and the selection of the destination state only depend on the current one and the time when the state transition happens.

In the subsequent derivations of this paper we consider an extension of SMPs obtained by attaching *reward rates* (r_i) to their states ($i \in \mathcal{E}$). Having introduced these new variables, we can then compute the reward accumulated by \mathbf{Y} over any interval $(0, t)$. The accumulated reward $B(t)$ is defined by the following integral:

$$B(t) = \int_0^t r_{Y(\tau)} d\tau$$

Several important measures of \mathbf{Y} can be defined by this extension (such as the cumulative time \mathbf{Y} spends in a subset of states, etc). These measures are referred to as reward measures of the SMP, and can be characterized by $(\mathbf{a}, \mathbf{K}(t), \mathbf{r})$, where \mathbf{r} is the vector of the reward rates. Details on this and other analytical results associated with the reward concept can be found in ^{5,18,32}.

11.2.4. Markov Regenerative Processes

A stochastic process $\mathbf{Z} = \{Z_t; t \in \mathcal{R}_+\}$ with state space \mathcal{F} is called *regenerative* if there exist time points at which the process probabilistically restarts itself. Such random times when the future of \mathbf{Z} becomes a probabilistic replica of itself are named *times of regeneration* for \mathbf{Z} . This concept may be weakened by letting the future after a time of regeneration depend also on the state of an MRS at that time. We then say that \mathbf{Z} is a Markov regenerative process.

*We call these conditions extreme because we assume that after the condition appears (or is detected) the system state will change instantaneously.

MRGP's are stochastic processes $\{Z_t; t \in \mathcal{R}_+\}$ that exhibit embedded MRS's (\mathbf{X}, \mathbf{S}) with the additional property that all conditional finite distributions of $\{Z_{t+S_n}; t \in \mathcal{R}_+\}$ given $\{Z_u; 0 \leq u \leq S_n, X_n = i\}$ are the same as those of $\{Z_t, t \in \mathcal{R}_+\}$ given $X_0 = i$.²¹ As a special case, the definition implies that

$$Pr\{Z_{t+S_n} = j \mid Z_u, 0 \leq u \leq S_n, X_n = i\} = Pr\{Z_t = j \mid X_0 = i\}$$

It also implies that the future of the process $\{Z_t; t \in \mathcal{R}_+\}$ from $t = S_n$ onwards depends on the past $\{Z_u, 0 \leq u \leq S_n\}$ only through X_n . Observe, that in the regenerative process this future from S_n onwards was completely independent of the past.

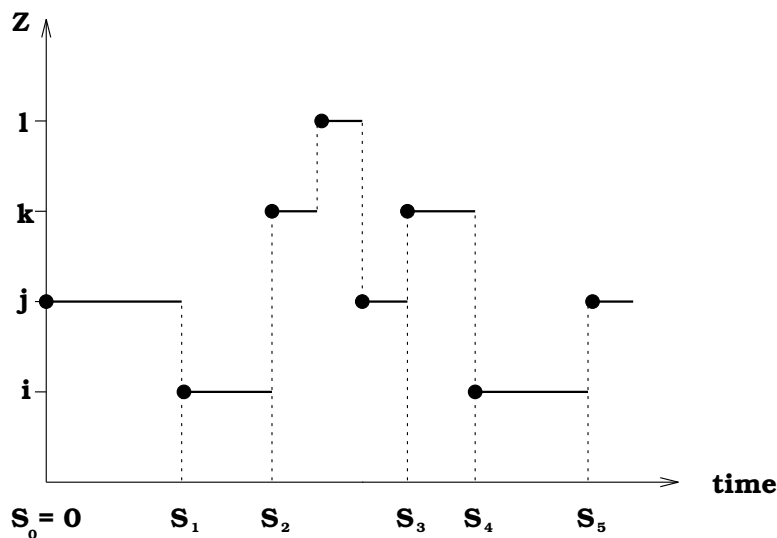


Figure 11.4: A sample realization of a Markov regenerative process.

In contrast to SMP's, state changes (possibly to states outside \mathcal{E}) are allowed between two consecutive Markov renewal moments (see Figure 11.4) in MRGP's. It is possible for the system to return to states in \mathcal{E} without these moments constituting Markov renewal moments. For example, suppose we start observing the system when it has just entered a state j , as shown in Figure 11.4. At that particular instant the Markov property is applicable since there is no past history of the process, but because of system characteristics, we know this property will no longer be valid for that state after the first state transition (not

necessarily to a state in \mathcal{E}). This situation could be understood if we consider that although state j being part of the EMC \mathbf{X} , it does not communicate with other states of \mathbf{X} and, hence, the Markov chain \mathbf{X} is reducible. Although others states of \mathbf{X} are (possibly) accessible from state i , this state cannot be accessed from any other state of \mathbf{X} .

The structural complexity of an MRGP depends on two main constituents:

- the stochastic process between two consecutive Markov renewal moments;
- the cause of the occurrence of a Markov renewal moment.

The stochastic process between the consecutive Markov renewal moments, usually referred to as *subordinated process*, can be any continuous-time discrete-state stochastic process over the same probability space. Recently published examples considered subordinated CTMC's^{18,33}, SMP's^{34,35}, MRGP's³², or a more general stochastic reward process.

The occurrence of a Markov renewal moment can be caused by

- the expiration of a random delay:
 - independent of the subordinated process;
 - some preceding state transition of the subordinated process results in a new Markov renewal moment;
- an accumulated reward measure of the subordinated process reaches a random barrier*:
 - independent of the path of the subordinated process;
 - some preceding state transitions of the subordinated process results in new Markov renewal moment;
- other more complex reason[†].

The subclass of MRGP's with subordinated SMP's (or CTMC's) and with Markov renewal moments caused by one of the first two reasons is considered in^{34,36}.

*While the preemption policy of the reward process (*preemptive repeat with resampling*) can be considered by means of one of the first reason as well, other preemption policies (*preemptive resume*³⁴ or *preemptive repeat without resampling*³⁶) can result in a more complex situation which falls only into reason 2.

[†]This last reason results in the wide class of MRGPs, but we believe that the majority of the practically interesting cases are captured by one of the former reasons.

11.3. PROBLEM SOLVING USING MARKOV RENEWAL THEORY

Let $\mathbf{Z} = \{Z_t; t \in \mathcal{R}_+\}$ be an MRGP with state space \mathcal{F} , whose embedded MRS is $(\mathbf{X}, \mathbf{S}) = \{X_n, S_n; n \in \mathcal{N}\}$ with kernel matrix $\mathbf{K}(t)$ over a countable state space \mathcal{E} , a subset of \mathcal{F} (i.e., $\mathcal{E} \subseteq \mathcal{F}$). For such a process we can define a matrix of conditional transition probabilities as:

$$V_{i,j}(t) \doteq Pr\{Z_t = j \mid Z_0 = i\}, \quad \forall i \in \mathcal{E}, \forall j \in \mathcal{F}, \forall t \in \mathcal{R}_+$$

In many practical problems involving Markov renewal processes, our primary concern is finding ways to effectively compute $V_{i,j}(t)$ since several measures of interest (e.g., reliability and availability) are related to the conditional transition probabilities of the stochastic process.

In this section we review some of the main techniques to determine transition probabilities. The underlying process discussed is an MRGP.

11.3.1. Markov Renewal Equation

At any instant t , the conditional transition probabilities $V_{i,j}(t)$ of \mathbf{Z} can be computed as:^{20,21}

$$\begin{aligned} V_{i,j}(t) &= Pr\{Z_t = j, S_1 > t \mid Z_0 = i\} + Pr\{Z_t = j, S_1 \leq t \mid Z_0 = i\} \\ &= Pr\{Z_t = j, S_1 > t \mid Z_0 = i\} + \\ &\quad \sum_{k \in \mathcal{E}} \int_{u=0}^t Pr\{Z_{t-u} = j \mid Z_0 = k\} d(Pr\{Z_u = k, S_1 < u\}) \\ &= Pr\{Z_t = j, S_1 > t \mid Z_0 = i\} + \sum_{k \in \mathcal{E}} \int_0^t V_{k,j}(t-u) dK_{i,k}(u) \\ &= Pr\{Z_t = j, S_1 > t \mid Z_0 = i\} + \sum_{k \in \mathcal{E}} \int_0^t dK_{i,k}(u) V_{k,j}(t-u) \end{aligned}$$

for all $i \in \mathcal{E}$, $j \in \mathcal{F}$, and $t \in \mathcal{R}_+$. If we define matrix $\mathbf{E}(t)$ by

$$E_{i,j}(t) \doteq Pr\{Z_t = j, S_1 > t \mid Z_0 = i\}, \quad \forall i \in \mathcal{E}, \forall j \in \mathcal{F}, \forall t \in \mathcal{R}_+$$

then, the set of integral equations $V_{i,j}(t)$ defines a **Markov renewal equation**, and can be expressed in matrix form as

$$\mathbf{V}(t) = \mathbf{E}(t) + \int_0^t d\mathbf{K}(u)\mathbf{V}(t-u)$$

where the Lebesgue-Stieltjes integral* is taken term by term.

To better distinguish the roles of matrices $\mathbf{E}(t)$ and $\mathbf{K}(t)$ in the description of the MRGP* we use the following terminology when referring to them:

- We call matrix $\mathbf{E}(t)$ the **local kernel** of the MRGP, since it describes the state probabilities of the process during the interval between successive Markov renewal moments.
- Since matrix $\mathbf{K}(t)$ describes the evolution of the process from the Markov renewal moment perspective, without describing what happens in between these moments we call it the **global kernel** of the MRGP.

The Markov renewal equation represents a set of coupled *Volterra integral equations of the second kind*³⁷ and in general are hard to solve in time-domain. The research for effective numerical solution methods of this equation has only started recently. We believe that the best numerical approach should be based on specific features of MRGP's which are not necessarily captured by the kernel matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$. However, methods based on the kernel matrices are the most general so far, and will be the ones exposed in this chapter.

To summarize, solving problems using Markov renewal theory is a two step process:

- First, we need to construct both kernel matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$ [†].
- We then solve the set of Volterra integral equations for the conditional transition probabilities $V_{i,j}(t)$ or for some measure of interest.

* $\int_0^t dK(u)V(t-u) = \int_0^t k(u)V(t-u)du$ when $K(t)$ possesses a density function $k(t) = \frac{dK(t)}{dt}$.

*Actually, this terminology will be adopted even when discussing issues related with SMP's for consistency in our presentation.

[†]For the case of SMP's, only the global kernel matrix $\mathbf{K}(t)$ is necessary.

11.3.2. Synthesis of the Kernel Matrices

The construction of kernel matrices can proceed by reasoning from particular facts to a general conclusion (*inductive approach*) or reasoning from the general to the specific (*deductive approach*). The inductive approach starts from the analysis of possible state transitions and relies only on basic probability theory to construct both kernel matrices. Conversely, the deductive approach applies general techniques to solve for a particular case. It is hard compare, in general, the two alternative approaches for a particular problem under consideration. Only experience can help in selecting the most suitable technique. Hence, we illustrate both approaches (whenever possible) in the examples in this chapter without discussing their particular merits.

The inductive approach does not have any general formulation. Its application varies from case to case, though the constructive process of the matrices follows approximately a regular pattern. We illustrate this logical pattern step-by-step along the solution of the examples presented in the end of this chapter. Our main interest in this section is to explore the deductive approach because of its algorithmic nature.

The deductive approach provides a closed form expression in transform domain for the elements of matrices $\mathbf{E}(t)$ and $\mathbf{K}(t)$ based on the kernels of the subordinated processes. Due to the generality of this approach it provides a robust and widely applicable procedure to construct the kernel matrices, and that is why we use it in the analysis of subsequent examples as a deductive method with comparison of the inductive one.

Since the first reason of the occurrence of Markov renewal moments can be considered as the special case of the second reason a unified approach was introduced in ³⁴ to analyze the subclass of MRGPs characterized by the first two reasons. Three matrix functions $\mathbf{F}^i(t, w)$, $\mathbf{D}^i(t, w)$ and $\mathbf{P}^i(t, w)$ (where t denotes the time, w denotes the barrier level, and the superscript i refers to the initial (regeneration) state of the subordinated process) were introduced to quantify the different occasions of the completion of the regeneration period and the internal state probabilities with a fixed barrier height. $\mathbf{F}^i(t, w)$ refers to the case when the next regeneration moment is because of the accumulated reward measure reached the fixed value w of the barrier. For the analysis of this case an additional matrix ($\mathbf{\Delta}^i$ referred to as branching probability matrix) is introduced, as well, to describe the state transition

subsequent to the regeneration moment. $\mathbf{D}^i(t, w)$ captures the case when the next regeneration moment is caused by one of the concluding state transitions of the subordinated process. And $\mathbf{P}^i(t, w)$ describes the state transition probabilities inside the regeneration period.

Based on the kernel of the subordinated SMP ($\mathbf{Q}^i(\mathbf{t}) = \{Q_{k,\ell}^i(t)\}$) these functions can be evaluated by the following equations ^{34*}:

$$F_{k,\ell}^{i\sim*}(s, v) = \delta_{k,\ell} \frac{r_k [1 - Q_k^{i\sim}(s + vr_k)]}{s + vr_k} + \sum_{u \in R} Q_{k,u}^{i\sim}(s + vr_k) F_{u,\ell}^{i\sim*}(s, v) \quad (1)$$

$$D_{k,\ell}^{i\sim*}(s, v) = \frac{1}{v} Q_{k,l}^{i\sim}(s + vr_k) + \sum_{u \in R} Q_{k,u}^{i\sim}(s + vr_k) D_{u,\ell}^{i\sim*}(s, v)$$

$$P_{k,\ell}^{i\sim*}(s, v) = \delta_{k,\ell} \frac{s [1 - Q_k^{i\sim}(s + vr_k)]}{v(s + vr_k)} + \sum_{u \in R} Q_{k,u}^{i\sim}(s + vr_k) P_{u,\ell}^{i\sim*}(s, v) \quad (2)$$

where $Q_k^i(t) = \sum_{\ell} Q_{k,\ell}^i(t)$; s is the time variable and v is the barrier level variable in transform domain; r_k is the reward rate associated to state k ; R is the part of the state space from which an exit results in a new regeneration moment; and the superscript \sim ($*$) refers to Laplace-Stieltjes (Laplace) transformation.

Given that $G_g(w)$ is the cumulative distribution function of the random barrier height to reach the next regenerative moment, the elements of the i -th row of matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$ can be expressed as follows, as a function of the matrices $\mathbf{P}^i(t, w)$, $\mathbf{F}^i(t, w)$ and $\mathbf{D}^i(t, w)$:

$$K_{i,j}(t) = \int_0^\infty \left[\sum_{k \in R^i} F_{i,k}^i(t, w) \Delta_{k,j}^i + D_{i,j}^i(t, w) \right] dG_g(w) \quad (3)$$

$$E_{i,j}(t) = \int_0^\infty P_{i,j}^i(t, w) dG_g(w) \quad (4)$$

*This subsection summarizes the results only for the cases when the cumulative or reward measure is accumulated according to *prd* and *prs* models; for *pri* models we refer to ³⁶.

11.3.3. Solution Techniques of Markov Renewal Equations

We can classify the existent solution methods in two categories:

- time domain methods;³⁸
- Laplace-Stieltjes domain method.^{18,34}

One possible time domain solution is based on a discretization approach to numerically evaluate the integrals presented in the Markov renewal equation. The integrals are solved using some approximation rule such as trapezoidal rule, Simpson's rule or other higher order methods:

$$\mathbf{V}(t_n) = \mathbf{E}(t_n) + \sum_{i=0}^n a_i \mathbf{K}'(t_i) \mathbf{V}(t_n - t_i)$$

where $\mathbf{K}'(t_i)$ denotes the derivative $\frac{d\mathbf{K}(x)}{dx}$ evaluated at point t_i *. In these equations h is the discretization step and it is assumed constant, and the coefficients a_i depend on the integration technique used. For example, when the trapezoidal rule is used $a_0 = a_n = \frac{h}{2}$ and $a_i = h, i = 1, 2, \dots, n - 1$. Hence, at any given time $t = t_n = nh$, a linear system of the form:

$$[\mathbf{I} - a_0 \mathbf{K}'(0)] \mathbf{V}(t_n) = \mathbf{E}(t_n) + \sum_{i=1}^n a_i \mathbf{K}'(t_i) \mathbf{V}(t_n - t_i)$$

needs to be solved. Note that if $a_0 \mathbf{K}'(0)$ is a diagonal matrix then the method is explicit, otherwise it is implicit.

A potential problem with this approach is that the right-hand side of the above equation can in general be expensive to compute. Nevertheless, there exist cases where the generalized Markov renewal equation has a simple form and the time-domain solution can be carried out.

Another time domain alternative is to construct a system of partial differential equations (PDEs), using the method of supplementary variables.¹² This method has been considered for steady-state analysis in ³³ and subsequently extended to the transient case in ³⁹. Up to now, this method has been elaborated only for the cases when the occurrence

*When the derivative of matrix $\mathbf{K}(t)$ is difficult to obtain Equation can be approximated as: $\mathbf{V}(t_n) = \mathbf{E}(t_n) + \sum_{i=0}^n [\mathbf{K}'(t_i) - \mathbf{K}'(t_{i-1})] \mathbf{V}(t_n - t_i)$.

of a new Markov renewal moment is due to one of the first two causes discussed in the previous section.

An alternative to the direct solution of the Markov renewal equation in time-domain is the use of transform methods. In particular, if we define $\mathbf{E}^\sim(s) = \int_0^\infty e^{-st} d\mathbf{E}(t)$ and $\mathbf{V}^\sim(s) = \int_0^\infty e^{-st} d\mathbf{V}(t)$, the Markov renewal equation becomes

$$\begin{aligned}\mathbf{V}^\sim(s) &= \mathbf{E}^\sim(s) + \mathbf{K}^\sim(s)\mathbf{V}^\sim(s) \\ &= [\mathbf{I} - \mathbf{K}^\sim(s)]^{-1}\mathbf{E}^\sim(s)\end{aligned}$$

After solving the linear system for $\mathbf{V}^\sim(s)$, transform inversion is required. In very simple cases, a closed-form inversion might be possible but in most cases of interest, numerical inversion will be necessary. The transform inversion however can encounter numerical difficulties especially if $\mathbf{V}^\sim(s)$ has poles in the positive half of the complex plane.

11.4. MARKOV REGENERATIVE STOCHASTIC PETRI NETS

Stochastic Petri nets of various types (SPN ⁴⁰, GSPN ⁴¹, ESPN ⁴², DSPN ⁴³, etc.) have been proposed as model description languages for analyzing the performance and reliability of systems. The analytical/numerical solution of such models proceeds by utilizing mathematical engines based upon the underlying stochastic processes of each Petri net class - CTMC for SPN's and GSPN's, SMP for a subset of ESPN's and MRGP for DSPN's. In this chapter, we use the class of stochastic Petri net named MRSPN to describe and help solve the sample problems explored in the next section.

Markov Regenerative Stochastic Petri Nets (MRSPN's) were introduced in ¹⁸ to overcome limitations on modeling power (notably allowing the solution of non-markovian models) of existing analytical tools. MRSPNs allow transitions with zero firing times, exponentially distributed or generally distributed firing times. The underlying stochastic process of an MRSPN is an MRGP, and it was proved in ¹⁸ that MRSPN's constitutes a true generalization of all the above classes. With a restriction that at most one generally distributed timed transition is enabled in each marking, the transient and steady state analysis

of MRSPN's can be carried out analytically-numerically rather than by discrete-event simulation. We now present some of the basic concepts concerning MRSPN's, but to do that we review some of the classical terminology of Petri nets.

A **Petri net** (PN) ^{44,45} is defined by a set of *places* (drawn as circles), a set of *transitions* (drawn as bars), and a set of *directed arcs*, which connect transitions to places or places to transitions. Places may contain *tokens*. The state of the Petri net, called *marking*, is defined by a vector enumerating the number of tokens in each place.

The states of a PN can be used to represent various entities associated with a system - for example, the number of functioning resources of each type, the number of tasks of each type waiting at a resource, the allocations of resources to tasks, and states of recovery for each failed resource. Transitions represent the changes of states due to the occurrences of simple or compound events such as the failure of one or more resources, the completion of executing tasks, or the arrival of jobs.

A place is an input to a transition if an arc exists from the place to the transition. If an arc exists from the a transition to a place, it is an output place of the transition. A transition is *enabled* when each of its input places contains at least one token. Enabled transitions can fire, by removing one token from each of input place and placing one token in each output place. Thus, the firing of a transition may cause a change of state (producing a different marking) of the PN. The *reachability set* is the set of markings that are reachable from a given initial marking. The reachability set together with arcs joining the marking indicating the transition that cause the change in marking is called the **reachability graph** of the net.

After the original conception, some extensions of PNs were proposed. *Inhibitor arcs* were introduced to increase the fundamental modeling or decision power of ordinary Petri nets. An inhibitor arc from a place to a transition has a circle rather than an arrowhead at the transition. The firing rule for the transition is changed such that the transition is disabled if there is at least one token present in the corresponding inhibiting input place.

In stochastic Petri nets, a random firing time elapses after a transition is enabled until it fires. Transitions which have nonzero firing times are called *timed transitions* and transitions with zero firing times are called *immediate transitions*. In MRSPN a timed transition can fires according to an exponential or any other general distribution function.

Immediate transitions have priority to fire over timed transitions.

The markings of a stochastic Petri net can be classified into *vanishing markings* and *tangible markings*. In a vanishing marking at least one immediate transition is enabled and in a tangible marking no immediate transition is enabled. For stochastic Petri nets classes that avoid immediate transitions the analysis of the embedded stochastic process can start directly from the reachability graph. For instance, the reachability graph of an SPN can be mapped directly into a Markov chain⁴⁰ and then solved for transient and steady-state measures. However, before analysing the underlying stochastic process of an MRSPN we have an extra step after obtaining the reachability graph. The **reduced reachability graph** is obtained from the reachability graph by merging the vanishing markings into their successor tangible markings according some rules.¹⁸ After constructing the reduced reachability graph we can start the analysis of the underlying stochastic process, which is going to be explored in the next section together with the developed examples.

11.5. PERFORMABILITY ANALYSIS APPLYING MARKOV RENEWAL THEORY

The use of Markov renewal theory for performability evaluation will be shown by its application to three examples of computer system architectures:

- series system with repair;
- parallel system with single, shared repair facility; and
- warm standby system with single, shared repair facility.

We start this section by describing each of the sample cases, followed by the identification of underlying stochastic processes, and concluding with the construction of kernel matrices using both approaches discussed (whenever applicable). Finally, the section is closed with the numerical solution of the resulting Markov renewal equations for measures of interest. Our emphasis in this chapter is on *synthesis of the kernel matrices* rather than on solution of the Volterra equations.

11.5.1 Series System with Repair

Consider a series system composed of two machines **a** and **b** with constant failures rates λ_a and λ_b . Upon failure of either machine, the system fails and is repaired with general repair-time distribution functions $G_a(t)$ and $G_b(t)$. We assume that machines cannot fail while the system is down, and that failure of one machine does not affect the operational status of the other.

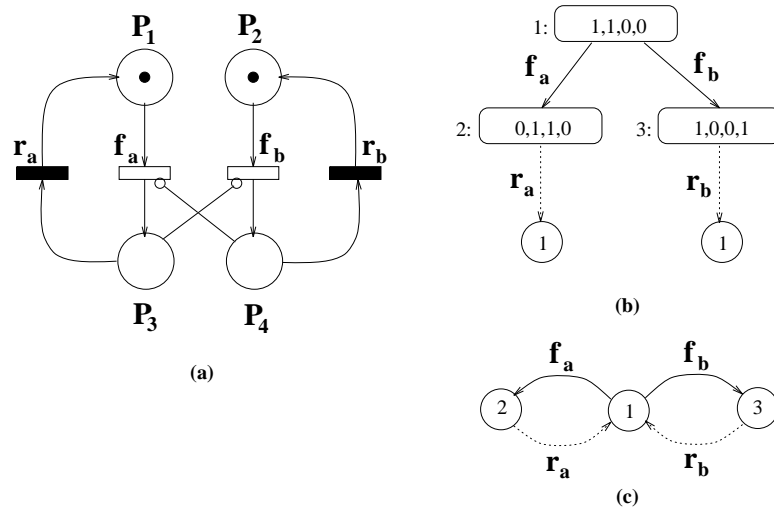


Figure 11.5: (a) Petri net of series system. (b) Reachability graph. (c) State transition diagram.

The overall behavior of the system can be easily understood from the MRSPN illustrated in Figure 11.5(a). Machine **a** is working whenever there is a token in place P_1 . Transition f_a , firing according to an exponential distribution with parameter λ_a , represents the failure process of machine **a**. When machine **a** fails, a token is deposited in place P_3 and repair is immediately started. Transition r_a with a generally distributed firing function $G_a(t)$ represents the random duration of the repair procedure. A symmetrical set of places and transitions describes the behavior of machine **b**. The system has failed whenever a token is deposited in places P_3 or P_4 . The two inhibitor arcs impose the restriction that no machine can fail while the system is undergoing repairs.

The reachability graph corresponding to the Petri net is shown in Figure 11.5(b). Each marking in the graph is a 4-tuple counting the

number of tokens in places P_1 to P_4 . Figure 11.5(b) also corresponds to the reduced reachability graph for the system since there are no vanishing markings. In the graph, solid arcs represent transitions firing according to exponential distribution functions, while dotted arcs denote state transitions firing according to general distributions.

Let a random variable Y_t be defined according to the operational condition of the system at any instant, i.e.,

$$Y_t = \begin{cases} 1 & \text{if the system is working at time } t \\ 2 & \text{if machine } \mathbf{a} \text{ is being repaired at time } t \\ 3 & \text{if machine } \mathbf{b} \text{ is being repaired at time } t \end{cases}$$

Note that possible values of Y_t are the labels corresponding markings in Figure 11.5(b). We are interested in computing performability measures associated with the system. To do so, we need to determine the conditional transition probabilities $V_{i,j}(t)$ of $\{Y_t; t \in \mathcal{R}_+\}$.

We start the solution procedure with the identification of the type of stochastic process underlying the system behavior. From the reachability graph we can conclude that all state transitions correspond to Markov renewal moments $\mathbf{S} = \{S_n; n \in \mathcal{N}\}$, and, consequently, markings labeled 1, 2, and 3, define the embedded Markov chain $\mathbf{X} = \{X_n; n \in \mathcal{N}\}$, such that X_n is the state of the system at time S_{n+} (i.e., $X_n = Y_{S_{n+}}$).

If we assume that at the time origin the system has just entered a new state (i.e., $S_0 = 0$) then the bi-variate stochastic process (\mathbf{X}, \mathbf{S}) is an MRS, and consequently $\{Y_t; t \in \mathcal{R}_+\}$ is an SMP, since whenever the system changes state we identify a Markov renewal moment. Having identified Y_t , we prepare for its solution by constructing the kernel matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$. Note that since we are dealing with an SMP the construction of $\mathbf{E}(t)$ is immediate once $\mathbf{K}(t)$ is determined.

An additional step adopted before starting the construction of the kernel matrices was the construction of a simplified state transition diagram. Figure 11.5(c) shows a simplified version of the (reduced) reachability graph where the markings were replaced by the corresponding state indices. We preserved the convention for the arcs and extended the notation by representing states of the EMC by circles, and (eventually) others states by squares.

The only non-null elements in matrix $\mathbf{K}(t)$ correspond to the possible transitions in a single step. Consequently, we have the following

structure of the matrix:

$$\mathbf{K}(t) = \begin{bmatrix} 0 & K_{1,2}(t) & K_{1,3}(t) \\ K_{2,1}(t) & 0 & 0 \\ K_{3,1}(t) & 0 & 0 \end{bmatrix}$$

The elements of the matrix can be constructed by induction from Figure 11.5(c):

$$\begin{aligned} K_{1,2}(t) &= Pr\{X_1 = 2, S_1 \leq t \mid X_0 = 1\} \\ &= Pr\{\text{the system fail up to time } t \text{ and machine} \\ &\quad \mathbf{a} \text{ is the cause}\} \\ &= \frac{\lambda_a}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] \end{aligned}$$

$$\begin{aligned} K_{1,3}(t) &= Pr\{X_1 = 3, S_1 \leq t \mid X_0 = 1\} \\ &= Pr\{\text{the system fail up to time } t \text{ and machine} \\ &\quad \mathbf{b} \text{ is the cause}\} \\ &= \frac{\lambda_b}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] \end{aligned}$$

$$\begin{aligned} K_{2,1}(t) &= Pr\{X_1 = 1, S_1 \leq t \mid X_0 = 2\} \\ &= Pr\{\text{repair of machine } \mathbf{a} \text{ has completed by time } t\} \\ &= G_a(t) \end{aligned}$$

$$\begin{aligned} K_{3,1}(t) &= Pr\{X_1 = 1, S_1 \leq t \mid X_0 = 3\} \\ &= Pr\{\text{repair of machine } \mathbf{b} \text{ has completed by time } t\} \\ &= G_b(t) \end{aligned}$$

The procedure to construct $K_{1,2}(t)$ and $K_{1,3}(t)$ deserves some further explanation. Since the reasoning is similar for both elements, we only detail the determination of $K_{1,2}(t)$. Let the random variables L_a and L_b be the respective time-to-failure of the two machines, we can compute $K_{1,2}(t)$ in the following way:

$$K_{1,2}(t) = Pr\{X_1 = 2, S_1 \leq t \mid X_0 = 1\}$$

$$\begin{aligned}
&= \Pr\{\text{the system fail up to time } t \text{ and machine } \mathbf{a} \\
&\quad \text{is the cause}\} \\
&= \Pr\{L_a \leq t \wedge L_b > L_a\} \\
&= \int_0^t [1 - (1 - e^{-\lambda_b \tau})] d\{1 - e^{-\lambda_a \tau}\} \\
&= \int_0^t e^{-\lambda_b \tau} \lambda_a e^{-\lambda_a \tau} d\tau \\
&= \frac{\lambda_a}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}]
\end{aligned}$$

We can then write the global kernel matrix as

$$\mathbf{K}(t) = \begin{bmatrix} 0 & \frac{\lambda_a}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] & \frac{\lambda_b}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] \\ G_a(t) & 0 & 0 \\ G_b(t) & 0 & 0 \end{bmatrix}$$

We construct the local kernel matrix $\mathbf{E}(t)$ following a similar inductive procedure. In this case we are looking for the probability that the system will remain in a given state up to the next Markov renewal moment. This happens since $\{Y_t; t \in \mathcal{R}_+\}$ is an SMP, and there are no non-null elements in the main diagonal of matrix $\mathbf{K}(t)$. Therefore, elements of $\mathbf{E}(t)$ are the complementary sojourn distribution functions in each state:

$$\mathbf{E}(t) = \begin{bmatrix} e^{-(\lambda_a + \lambda_b)t} & 0 & 0 \\ 0 & 1 - G_a(t) & 0 \\ 0 & 0 & 1 - G_b(t) \end{bmatrix}$$

We can always verify our answers by summing the elements in each row of both kernel matrices. Corresponding row-sums of the two matrices must add to unity, condition that is easily verified to hold in the example. Having completed the construction of the kernel matrices, we can solve a Markov renewal equation for the transient distribution of Y_t .

11.5.2. Parallel System with Single Repair Facility

Two machines (**a** and **b**) are working in a parallel configuration sharing a single repair facility working on an FCFS schedule. As in the previous

case, we assume that both machines have exponential lifetime distributions with parameters λ_a and λ_b respectively. Whenever one of the machines fails it goes immediately to repair, unless the other machine is still undergoing repair. When the repair facility is busy and a second failure occurs, the second machine to fail waits in a repair queue until the first machine is put back into service.

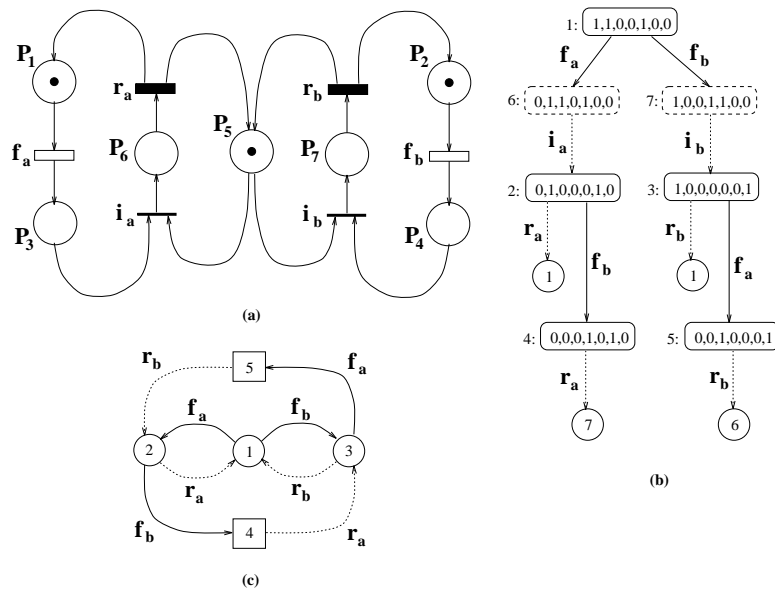


Figure 11.6: (a) Petri net of parallel system. (b) Reachability graph. (c) State transition diagram.

The procedure described in the last section can be repeated to solve this case. Figure 11.6(a) presents an appropriate MRSPN to describe the system behavior, and the corresponding reachability graph is reproduced in Figure 11.6(b). We observe in Figure 11.6(a) the same type of symmetry present in the series system. A token in place P_5 represents the availability of the single repair facility. Other extra places (P_6 and P_7) have been added to model the capture of the repair facility by the first machine to fail. We associate immediate transitions (i_a and i_b) to model the capture of the repair facility since we assume that starting a repair takes no time (if the facility is available). All the other places and transitions preserve the same semantics as in the series system.

Markings in Figure 11.6(b) are 7-tuples due to the addition of the three extra places. Another distinction from the previous system is that

vanishing markings (enclosed by dashed ellipses in the diagram) also occur in the reachability graph. These markings are eliminated when the reduced reachability graph is constructed (not shown), and based on the reduced version we constructed the state transition diagram of Figure 11.6(c).

Define the stochastic process $\mathbf{Z} = \{Z_t; t \in \mathcal{R}_+\}$ to represent the system state at any instant, where

$$Z_t = \begin{cases} 1 & \text{if both machines are working at time } t \\ 2 & \text{if machine } \mathbf{a} \text{ is under repair while machine } \mathbf{b} \text{ is} \\ & \text{working at time } t \\ 3 & \text{if machine } \mathbf{b} \text{ is under repair while machine } \mathbf{a} \text{ is} \\ & \text{working at time } t \\ 4 & \text{if machine } \mathbf{a} \text{ is under repair while machine } \mathbf{b} \text{ is} \\ & \text{waiting for repair at time } t \\ 5 & \text{if machine } \mathbf{b} \text{ is under repair while machine } \mathbf{a} \text{ is} \\ & \text{waiting for repair at time } t \end{cases}$$

Analysis of the resultant (reduced) reachability graph shows that \mathbf{Z} is an MRGP with an embedded Markov chain defined by the states 1, 2, and 3. We can observe that transitions to states 4 and 5 do not correspond to Markov renewal moments because they occur while timed transitions firing according non-exponential distributions are enabled. Furthermore, states 4 and 5 do not belong to the state space of the DTMC embedded in the process (EMC), therefore they are represented by squares in the state transition diagram to show this particular condition.

What makes this example particularly interesting is the fact that it allows us to demonstrate both techniques for the synthesis of the kernel matrices: the inductive and the deductive.

11.5.2.1. Inductive approach.

Once identified as an MRGP, to find the distribution of \mathbf{Z} we need to construct the kernel matrices. Starting with matrix $\mathbf{K}(t)$, we can identify its structure directly from Figure 11.6(c):

$$\mathbf{K}(t) = \begin{bmatrix} 0 & K_{1,2}(t) & K_{1,3}(t) \\ K_{2,1}(t) & 0 & K_{2,3}(t) \\ K_{3,1}(t) & K_{3,2}(t) & 0 \end{bmatrix}$$

The elements $K_{1,2}(t)$ and $K_{1,3}(t)$ are computed in a similar procedure as the correspondent elements in the series system case. Additionally, since determination of elements $K_{2,1}(t)$ and $K_{2,3}(t)$ is quite alike, so we will only show the procedure to construct $K_{2,1}(t)$. The third row is completely symmetrical to the second, so it can be easily understood once $K_{2,1}(t)$ is understood.

We need some auxiliary variables to help in the explanation of the construction process of $K_{2,1}(t)$. Hence, we define the random variables R_a and R_b to represent times necessary to repair machine **a** and **b**. The distribution function of R_a (R_b) is G_a (G_b). Using this new variables we can compute $K_{2,1}(t)$:

$$\begin{aligned} K_{2,1}(t) &= Pr\{X_1 = 1, S_1 \leq t \mid X_0 = 2\} \\ &= Pr\{\text{repair of } \mathbf{a} \text{ is finished up to time } t \text{ and } \mathbf{b} \\ &\quad \text{has not failed during the repair of } \mathbf{a}\} \\ &= Pr\{R_a \leq t \wedge L_b > R_a\} \\ &= \int_0^t Pr\{L_b > \tau\} dG_a(\tau) \\ &= \int_0^t [1 - (1 - e^{-\lambda_b \tau})] dG_a(\tau) \\ &= \int_0^t e^{-\lambda_b \tau} dG_a(\tau) \end{aligned}$$

The global kernel matrix inducted $\mathbf{K}(t)$ is

$$\begin{bmatrix} 0 & \frac{\lambda_a}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] & \frac{\lambda_b}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] \\ \int_0^t e^{-\lambda_b \tau} dG_a(\tau) & 0 & \int_0^t (1 - e^{-\lambda_b \tau}) dG_a(\tau) \\ \int_0^t e^{-\lambda_a \tau} dG_b(\tau) & \int_0^t (1 - e^{-\lambda_a \tau}) dG_b(\tau) & 0 \end{bmatrix}$$

Note that the global matrix is (and always is going to be) a square matrix. In this case with dimensions 3×3 , since we have 3 states in the embedded Markov chain. However, the local kernel matrix not necessary is a square matrix, since the cardinality of the state space of \mathbf{Z} can be larger than the cardinality of the state space of the embedded Markov chain. This can be seen, for instance, in this system since

the embedded Markov chain has only 3 states while the system has 5 possible states.

The construction of the local kernel matrix for an MRGP requires a more elaborate thinking process than for an SMP. This happens because, as explained, an MRGP can change states between two consecutive Markov renewal moments, and we need to capture these changes through the \mathbf{E} matrix. Careful analysis of Figure 11.6(c) reveals the structure of the local kernel matrix:

$$\mathbf{E}(t) = \begin{bmatrix} E_{1,1}(t) & 0 & 0 & 0 & 0 \\ 0 & E_{2,2}(t) & 0 & E_{2,4}(t) & 0 \\ 0 & 0 & E_{3,3}(t) & 0 & E_{3,5}(t) \end{bmatrix}$$

$E_{1,1}$ should be similar to the series systems, since the system can only go from state 1 to the other two states of the EMC exactly as in the previous case. The difficulty comes with the induction of $E_{2,2}(t)$ and $E_{2,4}(t)$ (complement of $E_{2,2}(t)$). Once we solve for these, we have the solution for the remaining components of the matrix due to the symmetry of the problem. Therefore, we explain the induction process that leads to $E_{2,2}(t)$:

$$\begin{aligned} E_{2,2}(t) &= \Pr\{Z_t = 2, S_1 > t \mid X_0 = 2\} \\ &= \Pr\{\text{repair of } \mathbf{a} \text{ is not finished up to time } t \text{ and } \mathbf{b} \\ &\quad \text{has not failed until } t\} \\ &= \Pr\{\text{repair of } \mathbf{a} \text{ is not finished up to time } t\} \times \\ &\quad \Pr\{\mathbf{b} \text{ has not failed until } t\} \\ &= [1 - G_a(t)]e^{-\lambda_b t} \end{aligned}$$

We can now express the local kernel matrix as

$$\mathbf{E}(t) = \begin{bmatrix} \mathbf{E}_1(t) & \mathbf{E}_2(t) \end{bmatrix}$$

where

$$\mathbf{E}_1(t) = \begin{bmatrix} e^{-(\lambda_a + \lambda_b)t} & 0 & 0 \\ 0 & e^{-\lambda_b t} G_a^c(t) & 0 \\ 0 & 0 & e^{-\lambda_a t} G_b^c(t) \end{bmatrix}$$

$$\mathbf{E}_2(t) = \begin{bmatrix} 0 & 0 \\ (1 - e^{-\lambda_b t}) G_a^c(t) & 0 \\ 0 & (1 - e^{-\lambda_a t}) G_b^c(t) \end{bmatrix}$$

$$G_a^c(t) = 1 - G_a(t)$$

$$G_b^c(t) = 1 - G_b(t)$$

Both matrices can be verified using the procedure described in the end of the previous section. Once again, we verify that the answers attained satisfy the necessary requirement.

11.5.2.2. Deductive approach.

With reference to the above discussion here we only evaluate the element of the kernel matrices related to the regenerative period starting from state 2, since there is no state transition during the subordinated process starting from state 1, and the subordinated starting from state 3 is symmetrical to the studied one.

The subordinated process starting from state 2 is a CTMC indeed, but to emphasize the generality of this approach we considered it as a SMP over the reduced state space of the subordinated process (i.e. state $2 \in \mathcal{R}$ and $4 \in \mathcal{R}$) with kernel $\mathbf{Q}(t)$ *:

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & 1 - e^{-\lambda_b t} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}^\sim(\mathbf{s}) = \begin{bmatrix} 0 & \frac{\lambda_b}{s + \lambda_b} \\ 0 & 0 \end{bmatrix}$$

Since states 2 and 4 belongs to \mathcal{R} , i.e. the next regeneration can not be caused by a state transition out of \mathcal{R} , the matrix function $\mathbf{D}(t, w)$ does not play role (i.e. $D_{i,j}(t, w) = 0$, for $i, j \in \mathcal{E}$) in this case.

By applying equations (1) and (2) we have:

*For notational convenience we neglect the superscript 2 refer to the initial regeneration state in the subsequent derivations

$$(s + v)P_{22}^{\sim*}(s, v) = s/v - \lambda_b P_{22}^{\sim*}(s, v)$$

$$(s + v)P_{24}^{\sim*}(s, v) = -\lambda_b P_{24}^{\sim*}(s, v) + \lambda_b P_{44}^{\sim*}(s, v)$$

$$(s + v)P_{44}^{\sim*}(s, v) = s/v$$

Since $P_{44}^{\sim*}(s, v) = \frac{s/v}{s+v}$ we have:

$$P_{22}^{\sim*}(s, v) = \frac{s/v}{s + v + \lambda_b}$$

$$P_{24}^{\sim*}(s, v) = \frac{s/v}{s + v + \lambda_b} \frac{\lambda_b}{s + v}$$

The same steps results:

$$F_{22}^{\sim*}(s, v) = \frac{1}{s + v + \lambda_b}$$

$$F_{24}^{\sim*}(s, v) = \frac{1}{s + v + \lambda_b} \frac{\lambda_b}{s + v}$$

In time domain the sum of a the i^{th} row of \mathbf{F} plus the sum of a the i^{th} row of \mathbf{D} plus the sum of a the i^{th} row of \mathbf{P} has to equal to one for all t and w . In this double Laplace-Stieltjes and Laplace transform domain the same sum has to equal to $1/v$, which holds for $P_{22}^{\sim*}(s, v) + P_{24}^{\sim*}(s, v) + F_{22}^{\sim*}(s, v) + F_{24}^{\sim*}(s, v)$.

To implement equations (3) and (4) a symbolic inverse Laplace transformation is necessary with respect to v . Whenever the subordinated process is a CTMC with binary reward rates this step can be performed symbolically.⁴⁶

The inverse Laplace transformation with respect to v results in:

$$P_{22}^{\sim}(s, w) = \frac{s}{s + \lambda_b} [1 - e^{-(s+\lambda_b)w}]$$

$$P_{24}^{\sim}(s, w) = \frac{\lambda_b}{s + \lambda_b} - e^{-sw} + \frac{s}{s + \lambda_b} e^{-(s+\lambda_b)w}$$

$$F_{22}^{\sim}(s, w) = e^{-(s+\lambda_b)w}$$

$$F_{24}^{\sim}(s, w) = e^{-sw} - e^{-(s+\lambda_b)w}$$

Hence by (3) and (4), the non-zero kernel elements are:

$$\begin{aligned} E_{22}^{\sim}(s) &= \int_{w=0}^{\infty} P_{22}^{\sim}(s, w)dG_a(w) \\ &= \frac{s}{s + \lambda_b}[1 - G_a^{\sim}(s + \lambda_b)] \end{aligned}$$

$$\begin{aligned} E_{24}^{\sim}(s) &= \int_{w=0}^{\infty} P_{24}^{\sim}(s, w)dG_a(w) \\ &= \frac{\lambda_b}{s + \lambda_b} - G_a^{\sim}(s) + \frac{s}{s + \lambda_b}G_a^{\sim}(s + \lambda_b) \end{aligned}$$

$$\begin{aligned} K_{21}^{\sim}(s) &= \int_{w=0}^{\infty} F_{22}^{\sim}(s, w)dG_a(w) \\ &= G_a^{\sim}(s + \lambda_b) \end{aligned}$$

$$\begin{aligned} K_{23}^{\sim}(s) &= \int_{w=0}^{\infty} F_{24}^{\sim}(s, w)dG_a(w) \\ &= G_a^{\sim}(s) - G_a^{\sim}(s + \lambda_b) \end{aligned}$$

The Laplace transform of the relevant entries of $\mathbf{E}(t)$ and $\mathbf{K}(t)$ reached by the inductive method results in the same expressions. Finally the LST domain description of the kernel matrices are:

$$\mathbf{K}^{\sim}(s) = \begin{bmatrix} 0 & \frac{\lambda_a}{s+\lambda_a+\lambda_b} & \frac{\lambda_b}{s+\lambda_a+\lambda_b} \\ G_a^{\sim}(s + \lambda_b) & 0 & G_a^{\sim}(s) - G_a^{\sim}(s + \lambda_b) \\ G_b^{\sim}(s + \lambda_a) & G_b^{\sim}(s) - G_b^{\sim}(s + \lambda_a) & 0 \end{bmatrix}$$

$$\mathbf{E}^{\sim}(s) = [\mathbf{E}_1^{\sim}(s) \quad \mathbf{E}_2^{\sim}(s)]$$

where

$$\mathbf{E}_1^{\sim}(s) = \begin{bmatrix} \frac{s}{s+\lambda_a+\lambda_b} & 0 & 0 \\ 0 & E_{22}^{\sim}(s) & 0 \\ 0 & 0 & \frac{s}{s+\lambda_a}[1 - G_b^{\sim}(s + \lambda_a)] \end{bmatrix}$$

and

$$\mathbf{E}_2 \sim(s) = \begin{bmatrix} 0 & 0 \\ E_{24} \sim(s) & 0 \\ 0 & \frac{\lambda_a}{s+\lambda_a} - G_b \sim(s) + \frac{s}{s+\lambda_a} G_b \sim(s + \lambda_a) \end{bmatrix}$$

11.5.3. Warm Standby with Single Repair Facility

Two statistically identical machines (“X” and “Y”) are working in a warm standby configuration sharing a single repair facility working on an FCFS schedule. If both machines are available, one of the is active on-line, while the other one is active off-line (spare). Machine “X” is the active one and “Y” the spare on the initial condition of the system. An active machine has constant failure rate λ_a , while the machine acting as spare has constant failure rate λ_b (usually $\lambda_a \leq \lambda_b$). If the on-line (off-line) machine fails, its repair begins immediately and completely restores it during a random repair time having an arbitrary distribution function G_a (G_b), and the other component continues to work on-line. We assume that the switchover from the active to the spare machine takes no time. Similarly to the parallel case, when the repair facility is busy and a second failure occurs, the actions on the last machine to fail are postponed until the other machine is put back into service.

The MRSPN on Figure 11.7(a) describes the expected system behavior. We can then construct the correspondent reachability graph, which is reproduced in Figure 11.7(b). A token in place P_1 represents that the machine working as active is in operational condition. When the active machine fails a token is placed in P_3 , and then repairs begin immediately in the failed machine while a switchover to the spare machine is executed. A token in place P_2 indicates that the spare machine is available for switchover, but since we assume that the spare machine can also fail, we included place P_4 to capture this situation. Place P_5 has a token whenever a switchover has just occurred and the failed machine (previous active machine) is undergoing repairs. This place has been included to capture the difference in distribution functions associated with repair of active and spare machines.

Similarly to the previous case, we observe the occurrence of vanishing markings (enclosed by dashed ellipses in the diagram). This

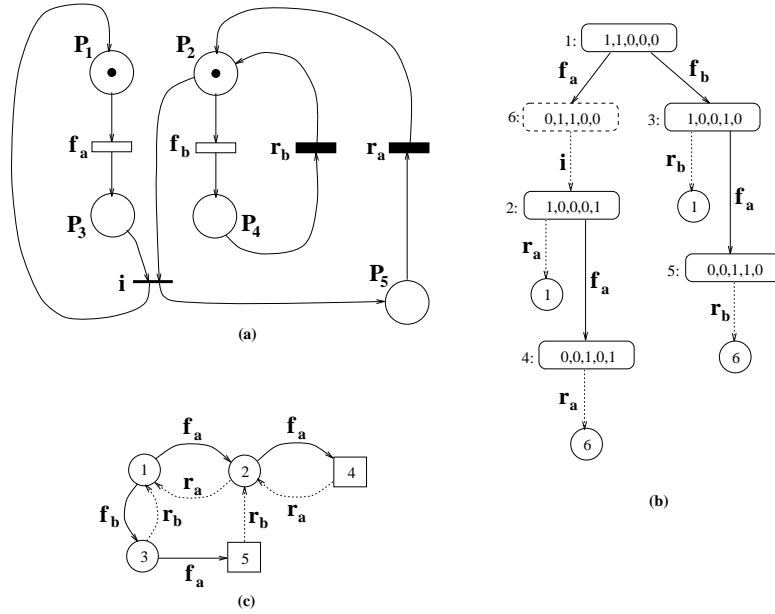


Figure 11.7: (a) Petri net of warm standby system. (b) Reachability graph. (c) State transition diagram.

marking occurs because of the zero time switchover, and is eliminated when the reduced reachability graph is constructed. Based on the reduced reachability graph we constructed the state transition diagram of Figure 11.7(c).

Define the random sequence $\mathbf{Z} = \{Z_t; t \in \mathcal{R}_+\}$ to represent the system state at any instant, where

$$Z_t = \begin{cases} 1 & \text{if active and spare are working at time } t \\ 2 & \text{if previous active is under repair while spare is} \\ & \text{working at time } t \\ 3 & \text{if spare is under repair while active is working} \\ & \text{at time } t \\ 4 & \text{if active has failed while the previous active is} \\ & \text{under repair at time } t \\ 5 & \text{if active has failed while the spare is under repair} \\ & \text{at time } t \end{cases}$$

Analysis of the resultant (reduced) reachability graph shows that \mathbf{Z} is an MRGP with an embedded Markov chain defined by the states 1,

2, and 3. We can observe that transitions to states 4 and 5 do not correspond to Markov renewal moments, therefore they were represented by squares in the state transition diagram. Markings $\langle 0, 0, 1, 0, 1 \rangle$ and $\langle 0, 0, 1, 1, 0 \rangle$ (states 4 and 5 in the transition diagram) represent the failure states of the system: one machine has failed when the other is still undergoing repair.

Once identified as an MRGP, we need to construct the kernel matrices to find the transition probability distributions of \mathbf{Z} . Starting with matrix $\mathbf{K}(t)$, we can identify its structure directly from Figure 11.7(c):

$$\mathbf{K}(t) = \begin{bmatrix} 0 & K_{1,2}(t) & K_{1,3}(t) \\ K_{2,1}(t) & K_{2,2}(t) & 0 \\ K_{3,1}(t) & K_{3,2}(t) & 0 \end{bmatrix}$$

The computation of all the non-zero elements of the global kernel matrix proceeds similarly to the series and parallel system. Therefore, we can induce the following global kernel matrix $\mathbf{K}(t)$:

$$\begin{bmatrix} 0 & \frac{\lambda_a}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] & \frac{\lambda_b}{\lambda_a + \lambda_b} [1 - e^{-(\lambda_a + \lambda_b)t}] \\ \int_0^t e^{-\lambda_a \tau} dG_a(\tau) & \int_0^t (1 - e^{-\lambda_a \tau}) dG_a(\tau) & 0 \\ \int_0^t e^{-\lambda_a \tau} dG_b(\tau) & \int_0^t (1 - e^{-\lambda_a \tau}) dG_b(\tau) & 0 \end{bmatrix}$$

Likewise, we can express the local kernel matrix as

$$\mathbf{E}(t) \begin{bmatrix} \mathbf{E}_1(t) & \mathbf{E}_2(t) \end{bmatrix}$$

where

$$\mathbf{E}_1(t) = \begin{bmatrix} e^{-(\lambda_a + \lambda_b)t} & 0 & 0 \\ 0 & e^{-\lambda_a t} G_a^c(t) & 0 \\ 0 & 0 & e^{-\lambda_a t} G_b^c(t) \end{bmatrix}$$

$$\mathbf{E}_2(t) = \begin{bmatrix} 0 & 0 \\ (1 - e^{-\lambda_a t}) G_a^c(t) & 0 \\ 0 & (1 - e^{-\lambda_a t}) G_b^c(t) \end{bmatrix}$$

$$G_a^c(t) = 1 - G_a(t)$$

$$G_b^c(t) = 1 - G_b(t)$$

The subordinated process starting from state 3 is similar to the parallel system example just discussed. The subordinated process starting from state 2 differs a bit because the failure intensity during the repair of component a is λ_a while it was λ_b in the parallel system, hence we can reach the elements (and all the derivation) of the 2^{nd} row of matrices $\mathbf{E}(t)$ and $\mathbf{K}(t)$ by substituting λ_b by λ_a in the results (derivations) of the previous section.

11.5.4. Numerical Results

For completeness of our analysis, the Markov renewal equations corresponding to the selected examples were solved for numerical values. Deterministic repair-time distribution functions were considered in all three examples, i.e.,

$$G_a(t) = u(t - \mu_a), \quad \mu_a > 0$$

$$G_b(t) = u(t - \mu_b), \quad \mu_b > 0$$

where $u(t)$ is the unitary step function. Table 11. summarizes the numerical values for parameters used in the computations. The units are hours for repair-time (parameters μ_a and μ_b) and hour^{-1} for the failure rates (parameters λ_a and λ_b). In the standby case, the failure rate of the spare unit was made lower than the failure rate of the active unit to characterize the warm standby situation.

A Laplace-Stieltjes transform method was adopted to solve the Markov renewal equation, which in LST domain becomes

$$\mathbf{V}^{\sim}(s) = \mathbf{E}^{\sim}(s) + \mathbf{K}^{\sim}(s)\mathbf{V}^{\sim}(s)$$

The time domain probabilities were calculated by first deriving the matrix $\mathbf{V}^{\sim}(s)$ using a standard package for symbolic analysis (e.g.,

example	λ_a	λ_b	μ_a	μ_b
<i>Series System</i>	.01	.01	5	5
<i>Parallel System</i>	.01	.01	5	5
<i>Standby System</i>	.01	.001	5	5

Table 11.1. Parameters used in the numerical solutions.

MATHEMATICA), and then numerically inverting the resulting LST expressions resorting to the Jagerman's method.⁴⁷

Figures 8, 9, and 10 report availability and performability results for all examples under the parameters established. Following the approach used in ⁴⁸, we also plotted corresponding Markovian system results, where each deterministic firing transition was replaced by an equivalent 25-stage Erlang distribution. The Markovian models were solved using the Stochastic Petri Net Package (SPNP) introduced in ⁴⁹.

As expected the plots reflect better availability for the warm standby system, followed by the parallel system. However, when considered from the interval power available, the situation reverses. The series system with repair provides more available power during a given interval than the parallel or warm standby systems.

11.6. CONCLUSIONS

An overview of Markov renewal theory was presented to introduce a promising alternative for the performability analysis of non-Markovian models. Our emphasis was to clarify the distinction between semi-Markov and Markov regenerative processes, and to establish a methodical approach on how to identify and prepare for the solution of problems involving the mentioned stochastic processes.

Although the development of adequate numerical approaches for the analysis of these models still require further efforts, we studied two approaches for the analytical formulation of systems behavior, and two methods for their analysis.

The major contribution of this chapter is the didactic structure adopted to present and discuss performability analysis in the context of non-Markovian systems. Essential examples of performability anal-

ysis of series, parallel and warm standby systems were elaborated to show the phases of applications of the introduced theoretical results. The whole solution process associated with each of the examples was described in detail aiming to suggest a methodological approach when dealing with Markov renewal theory. Stochastic Petri nets were used to support the analyses of the examples and facilitate their understanding.

REFERENCES

1. J.F. MEYER, "Performability: a retrospective and some pointers to the future," Performance Evaluation, **14**, 139–156 (1992).
2. J. GRAY AND D.P. SIEWIOREK, "High-Availability Computer Systems," Computer, **24**, 39–48 (1991).
3. M. BEAUDRY, "Performance related reliability for computer systems," IEEE Transactions on Computers, **C-27**, 540–547 (1978).
4. J.F. MEYER, "On evaluating the performability of degradable computer systems," IEEE Transactions on Computers, **C-29**, 720–731 (1980).
5. J.K. MUPPALA, S. WOOLET, AND K.S. TRIVEDI, "Real-Time Systems Performance in the Presence of Failures," Computer, **24**, 37–47 (1991).
6. K.S. TRIVEDI, Probability and Statistics with Reliability, Queuing, and Computer Science Applications (Prentice-Hall, Englewood-Cliffs, 1982).
7. D.P. SIEWIOREK, "Architecture of Fault-Tolerant Computers: An Historical Perspective," Proceedings of the IEEE, **79**, 1710–1734 (1991).
8. K.S. TRIVEDI, J.K. MUPPALA, S.P. WOOLET, AND B.R. HAVERKORT, "Composite performance and dependability analysis," Performance Evaluation, **14**, 197–215 (1992).
9. S. KARLIN AND H.M. TAYLOR, A First Course in Stochastic Processes (Academic Press, San Diego, 2nd. ed., 1975).

10. R.F. BOTTA, C.M. HARRIS, AND W.G. MARCHAL, "Characterizations of Generalized Hyperexponential Distribution Functions," Communications in Statistics - Stochastic Models, **3**, 115–148 (1987).
11. M.F. NEUTS, "Renewal Processes of Phase Type," Naval Research Logistics Quarterly, **25**, 445–454 (1978).
12. D.R. COX, "The analysis of non-markovian stochastic processes by the inclusion of supplementary variables," Proceedings of the Cambridge Philosophical Society, **51**, 433–440 (1955).
13. P. FRANKEN, D. KONIG, U. ARNDT, AND V. SCHMIDT, Queues and Point Processes (Akademie-Verlag, Berlin, 1981).
14. T. ROLSKI, "Stationary random processes associated with point processes," Lecture Notes in Statistics (Springer-Verlag, 1981), **5**.
15. V.S. KOROLYUK, "Markov Renewal Sequences in Reliability Analysis," in Semi-Markov Models. Theory and Applications, edited by Jacques Jansen (Plenum Press, New York, 1986), 217–230.
16. P. FRANKEN AND A. STRELLER, "Stochastic Processes with an Embedded Point Process and their Application to System Reliability Analysis," in Semi-Markov Models. Theory and Applications, edited by Jacques Jansen (Plenum Press, New York, 1986) 253–279.
17. D.R. COX, Renewal Theory (Methuen & Co., London, 1970).
18. H. CHOI, V.G. KULKARNI, AND K.S. TRIVEDI, "Markov Regenerative Stochastic Petri Nets," Performance Evaluation, **20**, 337–357 (1994).
19. E. ÇINLAR, "Markov Renewal Theory," Adv. Appl. Prob., **1**, 123–187 (1969).
20. E. ÇINLAR, Introduction to Stochastic Processes (Prentice-Hall, Englewood Cliffs, 1975).

21. V.G. KULKARNI, Modeling and Analysis of Stochastic Systems (Chapman & Hall, London, 1995).
22. W. FELLER, An Introduction to Probability Theory and its Applications. Vol II (John Wiley & Sons, 2nd. ed., 1966).
23. W.L. SIMTH, "Renewal Theory and its Ramifications," J. Roy. Statist. Soc. - Ser B, **20**, 243–302 (1958).
24. W.L. SIMTH, "Regenerative Stochastic Processes," Proc. Roy. Soc. London - Ser A, **232**, 6–31 (1955).
25. W.L. SIMTH, "Remarks on the paper "Regenerative Stochastic Processes," Proc. Roy. Soc. London - Ser A, **256**, 496–501 (1960).
26. P. LÉVY, "Systèmes Semi-Markoviens à au plus une infinité dénombrable d'états possibles," in Proc. Int. Congr. Math. (Amsterdam, 1954), **2**, 294.
27. P. LÉVY, "Processus semi-Markoviens," in Proc. Int. Congr. Math. (Amsterdam, 1954), **3**, 416–426.
28. L. TAKÁCS, "Some Investigations Concerning Recurrent Stochastic Processes of a Certain Type," Magyar Tud. Akad. Mat. Kutató Int. Közl, 115–128 (1954).
29. R. PYKE, "Markov Renewal Sequences: Definitions and Preliminary Properties," Ann. Math. Statist., **32**, 1231–1242 (1961).
30. R. PYKE, "Markov Renewal Sequences with Finitely Many States," Ann. Math. Statist., **32**, 1243–1259 (1961).
31. R. PYKE, AND R. SCHAUFLELE, "The Existence and Uniqueness of Stationary Measures for Markov Renewal Sequences," Ann. Math. Statist., **37**, 1439–1462 (1966).
32. MIKLÓS TELEK, Some advanced reliability modelling techniques, Phd Thesis (Hungarian Academy of Science, 1994).
33. R. GERMAN AND C. LINDEMANN, "Analysis of deterministic and stochastic Petri nets by the method of supplementary variables," Performance Evaluation, **20**, 317–335 (1994).

34. A. BOBBIO AND M. TELEK, "Markov regenerative SPN with non-overlapping activity cycles," in International Computer Performance and Dependability Symposium - IPDS95 (1995).
35. G. CIARDO, R. GERMAN, AND C. LINDEMANN, "A characterization of the stochastic process underlying a stochastic Petri net," IEEE Transactions on Software Engineering, **20**, 506–515 (1994).
36. A. BOBBIO, V.G. KULKARNI, A. PULIAFITO, M. TELEK, AND K.S. TRIVEDI, "Preemptive repeat identical transitions in Markov regenerative stochastic Petri nets," in Petri Net Performance Models '95 (1995).
37. C. FRÖBERG, Introduction to Numerical Analysis (Addison-Wesley, Reading, 2nd. ed., 1969).
38. R. GERMAN, D. LOGOTHETIS, AND K. TRIVEDI, "Transient analysis of Markov Regenerative Stochastic Petri Nets: A Comparison of Approaches," in Petri Net Performance Models '95 (1995).
39. R. GERMAN, "Transient Analysis of deterministic and stochastic Petri nets by the method of supplementary variables," in MASCOT'95 (1995).
40. M.K. MOLLOY, "Performance Analysis using Stochastic Petri Nets," IEEE Trans. Comput., **C-31**, 913–917 (1982).
41. M. AJMONE MARSAN, G. CONTE, AND G. BALBO, "A Class of Generalized Stochastic Petri Net for the Performance Evaluation of Multiprocessor Systems," ACM Trans. Comput. Syst., **2**, 93–122 (1984).
42. J.B. DUNGAN, K.S. TRIVEDI, R.M. GEIST, AND V.F. NICOLA, "Extended Stochastic Petri Nets: Applications and Analysis," in Performance'84, edited by E. Gelembé (Elsevier Science Publishers B.V., Amsterdam, 1985), 507–519.
43. M. AJMONE MARSAN AND G. CHIOLA, "On Petri nets with deterministic and exponentially distributed firing times," Lecture Notes in Computer Science, **266**, 132–145 (1987).

44. J.L. PETERSON, Petri Net Theory and the Modeling of Systems (Prentice-Hall, Englewood Cliffs, 1981).
45. J.L. PETERSON, "Petri Nets," Computing Surveys, **9**, 223–252 (1977).
46. M. TELEK, A. BOBBIO, L. JEREB, A. PULIAFITO, AND K.S. TRIVEDI, "Steady state analysis of Markov Regenerative SPN with age memory policy," in MMB '95 (1995).
47. D.L. JAGERMAN, "An Inversion Technique for the Laplace Transform," The Bell System Technical Journal, **61**, 1995–2002 (1986).
48. D. LOGOTHETIS AND K.S. TRIVEDI, "Time-Dependent Behavior of Redundant Systems with Deterministic Repair," in Computations with Markov Chains, edited by W.J. Stewart (Kluwer Academic Publishers, Norwell, 1995), 135–150.
49. G. CIARDO, J. MUPPALA, AND K.S. TRIVEDI, "SPNP: Stochastic Petri net Package," in Proc. Int. Workshop on Petri Nets and Perf. Models, (IEEE Computer Society Press, Los Alamos, 1989), 142–150.

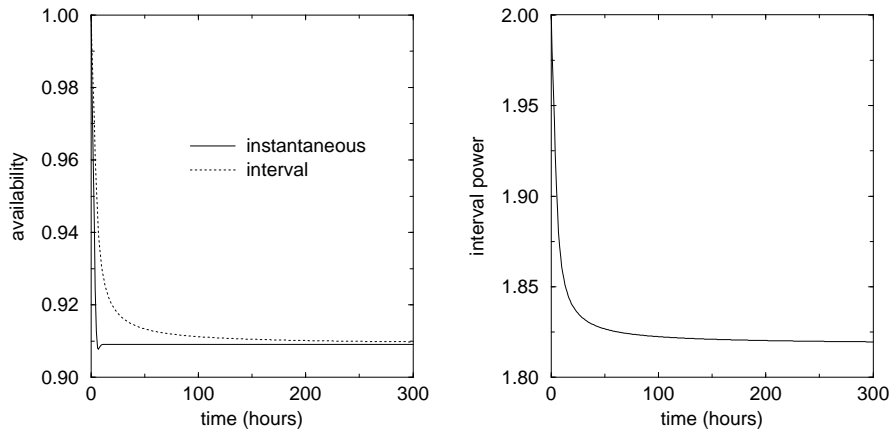
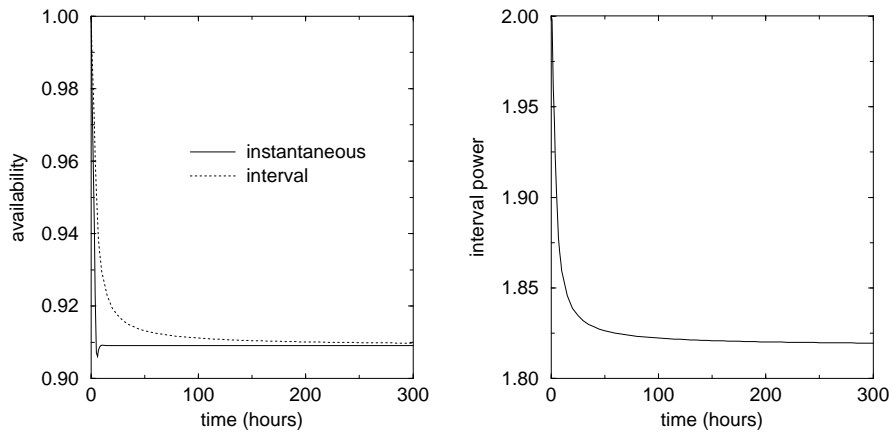
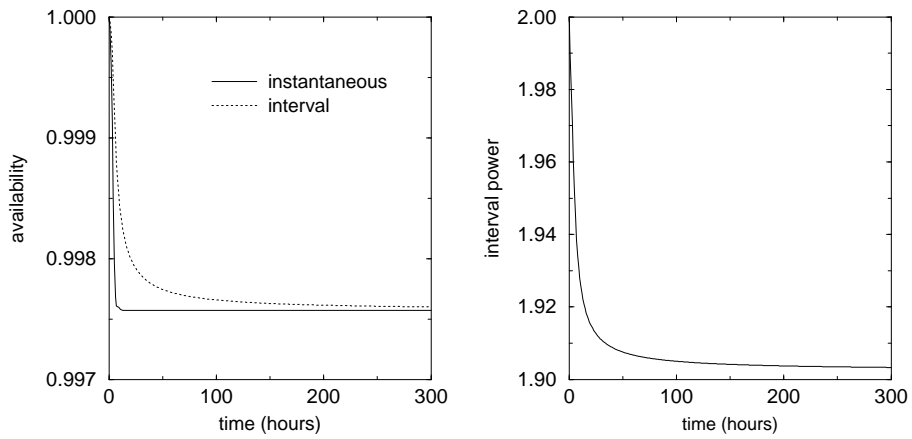
SPNP results:**LST results:**

Figure 8 - Numerical results for the series system.

SPNP results:



LST results:

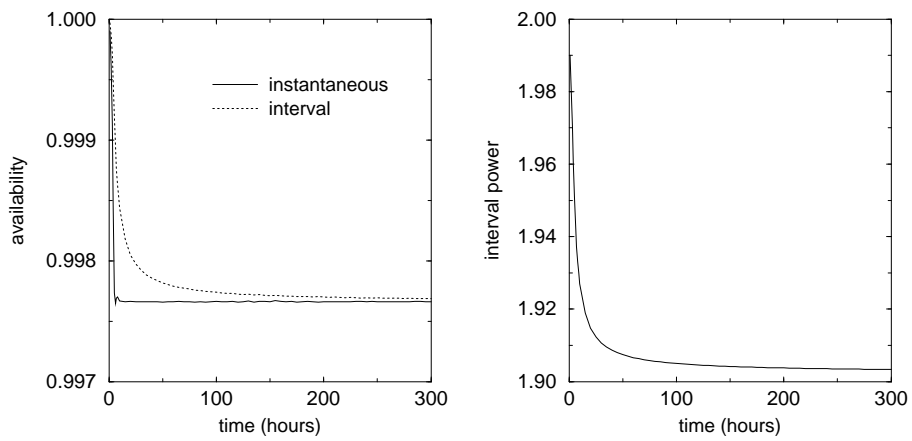


Figure 9 - Numerical results for the parallel system with single repair.

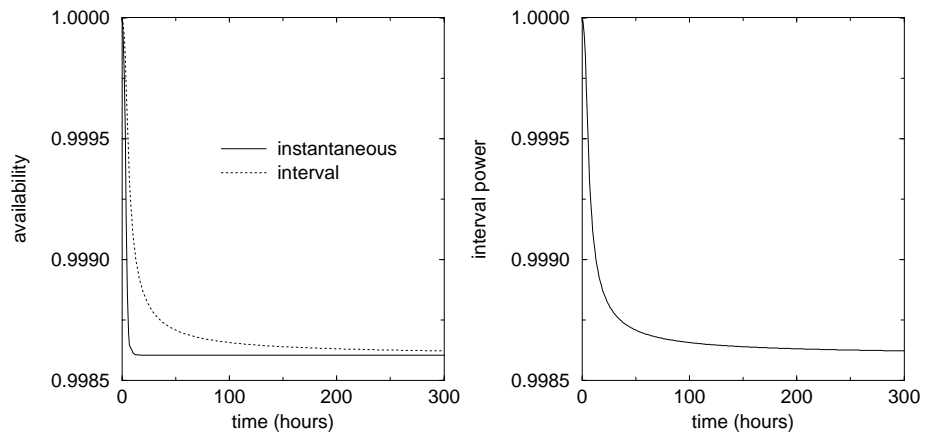
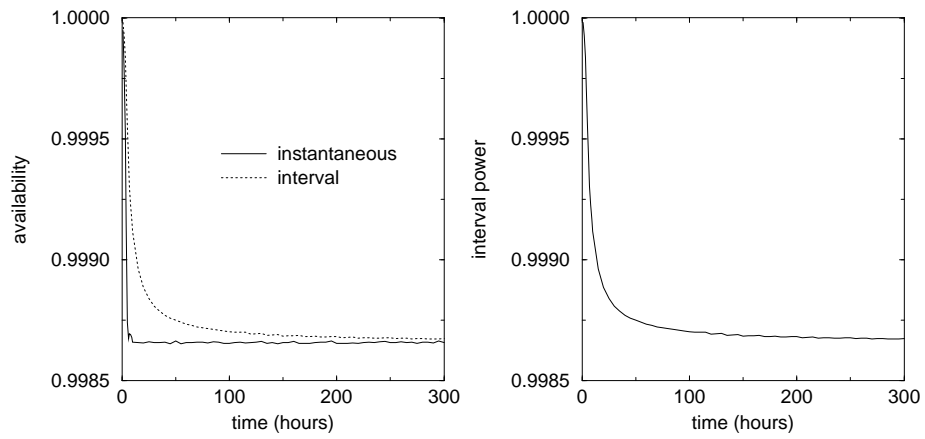
SPNP results:**LST results:**

Figure 10 - Numerical results for the warm standby system with single repair.