Formal Relation of Markov Renewal Theory and Supplementary Variables in the Analysis of Stochastic Petri Nets

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Abstract

Non-Markovian stochastic Petri nets have been investigated mainly by means of Markov renewal theory and by the method of supplementary variables. Both approaches provide different analytic descriptions of the same system. Numerical algorithms based on these descriptions lead to similar results. Parallel research effort resulted from the fact that an exact relationship of the two was not known. In this paper such a formal relationship is established for Markov regenerative stochastic Petri nets with general preemption policies in both the transient and stationary case. As a by-product, a closed form solution in Laplace domain is derived, which is easier to apply than previously known ones. An example from communications is used for illustrations.

Keywords: Markov Regenerative Stochastic Petri Nets, Transient and Stationary Analysis, Laplace Transforms.

1 Introduction

Stochastic Petri nets (SPNs) are employed for the model-based performance and dependability evaluation and are especially useful as a tool for the numerical analysis [17]. Most commonly, the Markovian case is considered, in which all transition firing times are exponentially distributed. However, in many systems other distributions are commonly considered to be adequate: slotted systems such as ATM have a deterministic clock, data transmission lengths often have a heavy-tailed distribution, e.g., the Pareto distribution, contention based medium access protocols have uniformly distributed backoff times, in reliability the Weibull distribution is common, etc. One branch of research therefore concentrates on the analysis of non-Markovian SPNs, in which generally distributed transition firing times are allowed. (In the sequel we refer to the different kinds of transitions as exponential and general ones, respectively.)

In most of these approaches the common *enabling restriction* is imposed that the general transitions are mutually exclusive[1]. Under the enabling restriction both the results of Markov renewal theory (also known as the method of the embedded Markov chain) [2] and the method of supplementary variables [6] has been applied[2]. The first paper in this line was [18] defining the class of *deterministic and stochastic Petri nets* (DSPNs), in which exponential transitions and those with a deterministic firing time can be mixed. A stationary analysis method was derived by means of an embedded Markov chain. Following research concentrated on more efficient subalgorithms [15] and on the generalization of the results: general transitions [4, 3], transient analysis [3], more general preemption policies [1], and marking-dependent distributions [4]. The class of SPNs with these features is commonly referred to as *Markov regenerative SPNs*, due to the underlying theory. In a parallel branch the method of supplementary variables was applied to the analysis of this class of SPNs as well [10, 7, 19, 9].

The two approaches based on Markov renewal theory and on supplementary variables lead to different analytic descriptions: in the first case a system of Volterra integral equations of the second type and in the latter case a system of integro-differential equations is obtained. Furthermore, the first approach leads to backward and the second to forward equations. Backward equations means that a matrix of

[1] The enabling restriction is slightly more complicated in the presence of general preemption policies, as discussed later.

[2] If general transitions may be concurrently enabled, supplementary variables can still be used in principle [10, 16], but lead to higher-dimensional integrals. If concurrent deterministic transitions are synchronized appropriately, cascaded embedding may be applied [8]. Alternatively, phase-type approximations, either continuous or discrete, can be used.

conditional quantities has to be dealt with, whereas forward equations allow to consider just a vector of unconditional quantities for given initial conditions. The numerical time-domain solution algorithms which can be derived from both analytic descriptions have different computational costs, as shown in [11]. In the stationary case, the differences seem less severe. Unfortunately, an exact relationship of the analysis approaches was not known, making it difficult to reason about differences and equalities of them.

In this paper, first a common notation is introduced which allows to formulate the state equations of both approaches in a unified manner. The notation is used to derive closed form solutions in Laplace domain for both approaches. Such an expression was previously not known in the supplementary variable case. Subsequently, both solutions are brought to an identical form. This has two consequences: first, it shows the equivalence of both analytic descriptions and second the obtained form is easier to evaluate than previously known expressions, because it does not require the renumbering of states during the analysis of subordinated processes. Note that despite this equivalence the time-domain solution algorithms are quite different. This is not the case for the stationary analysis. Here it is possible to derive a formal relationship of the approaches such that the resulting numerical solution algorithms can be considered as almost identical. As a consequence, results derived by one approach can be immediately applied to the other, leading to a cross-fertilization.

The rest of the paper is organized as follows. The considered model class and common notation is defined in Sec. 2. The method of supplementary variables and Markov renewal theory are investigated in Sec. 3 and 4, respectively. The relationship of both approaches is addressed in Sec. 5 and illustrations by an example are given in Sec. 6.

2 Considered Model Class and Common Notation

SPNs are considered which consist of places, immediate and timed transitions, arcs, and undistinguishable tokens, as described in, e.g., [17]. Timed transitions are either exponential or general. General transitions can have the preemption policies *preemptive repeat different* (prd) or *preemptive resume* (prs) [1]. When a prd-transition is preempted, its memory is reset (already performed work is lost and the firing time is resampled), while when a prs-transition is preempted, its memory is maintained (both the already performed work and the firing time sample is maintained). The following *enabling restriction* has to be imposed in order to apply the subsequent theory: in each tangible marking of the SPN at most one general transition may have memory. As a consequence, in each state at most one general transition may be enabled or may have a memory of a maintained firing time.

Let $\mathcal{S}$ denote the set of tangible markings which is required to be finite. Refer to a state transition caused by the firing of an exponential and a general transitions as exponential state transition and general state transition, respectively. Let G denote the set of general transitions. For a $g \in G$, denote by $F^g(x)$ its firing time distribution, by $f^g(x) = \frac{d}{dx}F^g(x)$ its density, by $\bar{F}^g = 1 - F^g(x)$ its complementary distribution function, and by $\mu^g(x) = f^g(x)/\bar{F}^g(x)$ its instantaneous firing rate. We assume that $F^g(x)$ has no mass at zero, i.e., $F^g(0) = 0$. The following subsets of $\mathcal{S}$ are defined:

$$\begin{aligned} \mathcal{S}^E &= \{n \in \mathcal{S} \mid \text{no gen. tr. has memory in } n\}, \\ \mathcal{S}^g &= \{n \in \mathcal{S} \mid g \text{ has memory in } n\}, \\ \mathcal{S}^{\bar{g}} &= \{n \in \mathcal{S} \mid g \text{ disabled but has memory in } n\}, \end{aligned}$$

for $g \in G$, and the unions $\mathcal{S}^G = \bigcup_{g \in G} \mathcal{S}^g$ and $\mathcal{S}^{\bar{G}} = \bigcup_{g \in G} \mathcal{S}^{\bar{g}}$. $\mathcal{S}^E$ represents all "exponential" states, $\mathcal{S}^G$ all "general" states, and $\mathcal{S}^{\bar{G}}$ all general states in which a prs-transition is preempted. Note that $\mathcal{S}^{\bar{g}} \subset \mathcal{S}^g$ (as well as $\mathcal{S}^{\bar{G}} \subset \mathcal{S}^G$) and that the sets $\mathcal{S}^E$ and $\mathcal{S}^g$, for all $g \in G$, constitute a partition of $\mathcal{S}$ (as well as $\mathcal{S}^E$ and $\mathcal{S}^G$).

An SPN describes a stochastic process, referred to as the *marking process* $\{N(t), t \geq 0\}$. The aim of the analysis are the *unconditional transient state probabilities* $\pi_n(t) = Pr\{N(t) = n\}$. Additionally, we are interested in the *general firing frequencies* $\varphi_n(t) = E\{\text{firing frequency of gen. tr. } g \text{ in } n \text{ at } t\}$. Note that $\pi_n(t)$ corresponds to rate and $\varphi_n(t)$ to impulse rewards. In the rest of the paper we assume that the marking process is given by the matrices $\mathbf{Q}$, $\bar{\mathbf{Q}}$, and $\mathbf{\Delta}$ of size $|\mathcal{S}| \times |\mathcal{S}|$:

- $\mathbf{Q} = \{q_{ij}\}$, where q_{ij}, $i \neq j$, is the rate of the exponential state transitions from i to j except for those which preempt a prd-transition, q_{ii} is the negative sum of all rates of exponential state transitions out of state i, including those which preempt a prd-transition,

- $\bar{\mathbf{Q}} = \{\bar{q}_{ij}\}$, where $\bar{q}_{ij}$ is the rate of the exponential state transitions from i to j which preempt a prd-transition (note that $\bar{q}_{ij} = 0$ if $i \in \mathcal{S}^E$),

- $\mathbf{\Delta} = \{\delta_{ij}\}$, where δ_{ij} is the *branching probability*, the probability that the firing of a general transition g in state i leads to state j, given that g fires in i (note that $\delta_{ij} = 0$ if $i \in \mathcal{S}^E$).

Separating the matrices $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ has the following motivation: $\bar{\mathbf{Q}}$ allows to represent exponential transitions which preempt a prd-transition and enable it immediately again, i.e., which reset the memory of a prd-transition. Furthermore, the separation turns out to be convenient in the formulation of state equations.

In order to avoid a cumbersome reordering of states *filter matrices* are defined as follows. Let $\mathbf{I}^E$ be the diagonal matrix whose ith element is equal to one if $i \in \mathcal{S}^E$ and equal to zero otherwise, $\mathbf{I}^g$ and $\mathbf{I}^G$ are defined analogously. Filtered vectors and matrices are then $\boldsymbol{\pi}^E(t) = \boldsymbol{\pi}(t)\mathbf{I}^E$, $\mathbf{Q}^E = \mathbf{I}^E\mathbf{Q}$, $\mathbf{Q}^g = \mathbf{I}^g\mathbf{Q}$, $\mathbf{Q}^{E,g} = \mathbf{I}^E\mathbf{Q}\mathbf{I}^g$, etc. Note that multiplication from the left filters out rows and multiplication from the right filters out columns, all other elements are set to zero. Due to the enabling restriction $\mathbf{Q}^{g,h} = \mathbf{0}$ if $g, h \in G$ and $g \neq h$, and $\bar{\mathbf{Q}}^g = \mathbf{0}$ if g is of prs type. As special filter matrices we define $\mathbf{R} = \mathbf{I}^G - \mathbf{I}^{\bar{G}}$ and $\bar{\mathbf{R}} = \mathbf{I} - \mathbf{R} = \mathbf{I}^E + \mathbf{I}^{\bar{G}}$.

If the states would be ordered according to $\mathcal{S}^E, \mathcal{S}^{g_1}, \mathcal{S}^{g_2}, \ldots, \mathcal{S}^{g_n}$, the structure of $\mathbf{Q}$ would be the following:

$$\mathbf{Q} = \begin{array}{|cccc|} \hline Q^E & 0 & \cdots & 0 \\ 0 & Q^{g_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & Q^{g_n} \\ \hline \end{array}$$

(here the italics represent matrices where superfluous rows and columns are removed). Hence the matrices associated with the sets $\mathcal{S}^E, \mathcal{S}^{g_i}, i \in 1,..,n$ are "independent" and we can write, e.g. $\mathbf{Q} = \mathbf{Q}^E + \sum_{i=1}^n \mathbf{Q}^{g_i}$.

3 The Method of Supplementary Variables

In the method of supplementary variables the discrete state description is supplemented by a continuous variable counting for the performed work of a general transition. State equations can be formulated for this hybrid state space [6]. The resulting system of equations is a generalization of the Kolmogorov forward differential equations [5], a system of integro-differential equations.

3.1 State Equations

The discrete marking process is supplemented by a continuous component: $\{(N(t), X(t)), t \geq 0\}$, $X(t) \in \mathbb{R}_0^+ \cup \{\infty\}$. $X(t)$ is the supplementary variable which equals to ∞ if $N(t) \in \mathcal{S}^E$ and to the performed work since $N(t)$ sojourns in $\mathcal{S}^g$ if $N(t) \in \mathcal{S}^g$. Note that $X(t)$ represents the memory mentioned in the previous section. The following quantities are required:

- *age densities* are defined as

$$\pi_n(t,x) = \frac{\partial}{\partial x} Pr\{N(t) = n, X(t) \leq x\}$$

- *age intensities* are defined as

$$p_n(t,x) = \frac{\pi_n(t,x)}{\bar{F}^g(x)}, \quad \text{if } n \in \mathcal{S}^g$$

  and equal to zero if $n \in \mathcal{S}^E$.

- the *state probabilities* are obtained by integration of the age densities for $n \in \mathcal{S}^g$:

$$\pi_n(t) = \int_0^\infty \pi_n(t,x)\,dx = \int_0^\infty p_n(t,x)\bar{F}^g(x)\,dx,$$

- the *firing frequencies* are obtained by integration[3] of the age densities with respect to the instantaneous firing rate:

$$\varphi_n(t) = \int_0^\infty \pi_n(t,x)\mu^g(x)\,dx = \int_0^\infty p_n(t,x)f^g(x)\,dx\,,$$

  if $n \in \mathcal{S}^g$ and g is enabled in state n, otherwise it equals to zero, and

- the probability mass at $X(t) = 0$ is defined as

$$\Pi_n(t,0) = Pr\{N(t) = n, X(t) = 0\}, \quad \text{if } n \in \mathcal{S}^{\bar{g}}\ ,$$

  and equal to zero otherwise (note that this mass may be accumulated in a state of $\mathcal{S}^{\bar{G}}$ if the state is entered by another way than the preemption of the prs-transition).

Denote by $\boldsymbol{\pi}(t,x)$, $\mathbf{p}(t,x)$, $\boldsymbol{\pi}(t)$, and $\boldsymbol{\varphi}(t)$, and $\boldsymbol{\Pi}^{\bar{G}}(t,0)$ the corresponding vectors. Additionally, the following sum is defined:

$$\boldsymbol{\pi}^{E\bar{G}}(t) = \boldsymbol{\pi}^E(t) + \boldsymbol{\Pi}^{\bar{G}}(t,0)\ .$$

Note that if the firing time of g has a finite support on $(0, x^g_{\max})$ then all quantities vanish for $x > x^g_{\max}$.

The transient behavior is described by the following system of equations. A more detailed derivation can

[3] We use a version in which the ordinary integral and not the Stieltjes integral is used. We prefer this version since it allows to deal with impulses directly in the sense of generalized functions, as it is exposed in the appendix of [14]. Later we will also use the Laplace transform and not the Laplace-Stieltjes transform. However, all equations could be derived by using Stieltjes integrals in a similar manner.

be found in [19]. See also [9] for the stationary case and a slightly different notation. Ordinary differential equations (ODEs) describe the change of probabilities $\boldsymbol{\pi}^{E\bar{G}}(t)$ for $t > 0$:

$$\frac{d}{dt}\boldsymbol{\pi}^{E\bar{G}}(t) = \boldsymbol{\pi}^{E\bar{G}}(t)\mathbf{Q}\bar{\mathbf{R}} + \boldsymbol{\varphi}(t)\boldsymbol{\Delta}\bar{\mathbf{R}} + \boldsymbol{\pi}^{G}(t)\bar{\mathbf{Q}}\bar{\mathbf{R}}, \quad (1)$$

partial differential equations (PDEs) describe the evolution of the age intensities for $t, x > 0$:

$$\frac{\partial}{\partial t}\mathbf{p}^{G}(t,x) + \frac{\partial}{\partial x}\mathbf{p}^{G}(t,x)\mathbf{R} = \mathbf{p}^{G}(t,x)\mathbf{Q}, \quad (2)$$

the enabling of general transitions gives boundary conditions for $t > 0$:

$$\mathbf{p}^{G}(t,0) = \boldsymbol{\pi}^{E\bar{G}}(t)\mathbf{Q}\mathbf{R} + \boldsymbol{\varphi}(t)\boldsymbol{\Delta}\mathbf{R} + \boldsymbol{\pi}^{G}(t)\bar{\mathbf{Q}}\mathbf{R}, \quad (3)$$

$\boldsymbol{\pi}(0)$ is given as initial condition together with $\mathbf{p}^{G}(0,x) = \boldsymbol{\pi}^{G}(0)\mathbf{R}\delta(x)$ and $\boldsymbol{\Pi}^{G}(0,0) = \boldsymbol{\pi}^{G}(0)\bar{\mathbf{R}}$, where $\delta(x)$ denotes the Dirac impulse. The firing frequencies and the state probabilities of $\mathcal{S}^{G}$ are given by the integrals:

$$\boldsymbol{\varphi}(t) = \sum_{g\in G}\int_0^\infty \mathbf{p}^{g}(t,x)\mathbf{R}f^{g}(x)\,dx, \quad (4)$$

$$\boldsymbol{\pi}^{G}(t) = \sum_{g\in G}\int_0^\infty \mathbf{p}^{g}(t,x)\bar{F}^{g}(x)\,dx + \boldsymbol{\Pi}^{G}(t,0). \quad (5)$$

See [7, 11, 13, 19] for the analysis of the equations in time domain.

3.2 Transient Analysis in Laplace Domain

An expression for the state probabilities can be derived in Laplace domain. In the following first a solution of the PDE system is obtained and inserted into the integrals. Then the sum of the ODE system and of the boundary conditions is considered. In Laplace domain an explicit solution of the exponential state probabilities and of the age intensities can be obtained from this sum. A final application of the integrals leads to the solution.

Applying the initial condition ($\mathbf{p}^{G}(0,x) = \boldsymbol{\pi}^{G}(0)\mathbf{R}\delta(x)$), the PDE system (2) can be transformed to the double Laplace domain where s and v are the transform pair of t and x, respectively

$$s\mathbf{p}^{G^{**}}(s,v) - \boldsymbol{\pi}^{G}(0)\mathbf{R} + v\mathbf{p}^{G^{**}}(s,v)\mathbf{R} - \mathbf{p}^{G^{*}}(s,0)\mathbf{R} = \mathbf{p}^{G^{**}}(s,v)\mathbf{Q},$$

whose solution is

$$\mathbf{p}^{G^{**}}(s,v) = \left(\boldsymbol{\pi}^{G}(0) + \mathbf{p}^{G^{*}}(s,0)\right)\mathbf{R}\,(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} = \sum_{g\in G}\left(\boldsymbol{\pi}^{g}(0) + \mathbf{p}^{g^{*}}(s,0)\right)\mathbf{R}^{g}\;(s\mathbf{I} + v\mathbf{R} - \mathbf{Q}^{g})^{-1}. \quad (6)$$

For the inversion of this solution with respect to v we introduce $\mathbf{Q}^g_1 = \mathbf{R}\cdot\mathbf{Q}^g\cdot\mathbf{R}$, $\mathbf{Q}^g_2 = \mathbf{R}\cdot\mathbf{Q}^g\cdot\bar{\mathbf{R}}$, $\mathbf{Q}^g_3 = \bar{\mathbf{R}}\cdot\mathbf{Q}^g\cdot\mathbf{R}$, $\mathbf{Q}^g_4 = \bar{\mathbf{R}}\cdot\mathbf{Q}^g\cdot\bar{\mathbf{R}}$ and

$$\hat{\mathbf{Q}}^g(s) = \mathbf{Q}^g_1 + \mathbf{Q}^g_2\cdot(s\mathbf{I} - \mathbf{Q}^g_4)^{-1}\cdot\mathbf{Q}^g_3\,.$$

Note that $\mathbf{Q}^g_4$ is invertible if the CTMC described by $\mathbf{Q}$ contains no closed subsets inside $\mathcal{S}^{\bar{g}}$. Utilizing the structure of matrix $\mathbf{Q}$, a symbolic inverse Laplace transformation gives:

$$\mathbf{p}^{g^*}(s,x) = \left(\boldsymbol{\pi}^g(0) + \mathbf{p}^{g^*}(s,0)\right)\cdot\mathbf{R}^g\cdot e^{\hat{\mathbf{Q}}^g(s)x}\cdot\hat{\mathbf{D}}^g(s)\;e^{-sx}\,, \quad (7)$$

where $\hat{\mathbf{D}}^g(s) = \mathbf{R}^g + \mathbf{Q}^g_2\cdot(s\mathbf{I} - \mathbf{Q}^g_4)^{-1}$. Note that if transition g is persistent (i.e., cannot be preempted) the second term of $\hat{\mathbf{Q}}^g(s)$ and $\hat{\mathbf{D}}^g(s)$ vanishes. Also

$$\mathbf{p}^{g^*}(s,x)\mathbf{R} = \left(\boldsymbol{\pi}^g(0) + \mathbf{p}^{g^*}(s,0)\right)\mathbf{R}^g e^{\hat{\mathbf{Q}}^g(s)x}e^{-sx} \quad (8)$$

since the second term of $\hat{\mathbf{D}}^g(s)$ vanishes by the multiplication from right with $\mathbf{R}$.

Transformation of the integral (4) for the firing frequencies and an insertion of the PDE solution (8) yields

$$\boldsymbol{\varphi}^*(s) = \sum_{g\in G}\int_0^\infty \mathbf{p}^{g^*}(s,x)\mathbf{R}f^g(x)\,dx = \sum_{g\in G}\int_0^\infty \left(\boldsymbol{\pi}^g(0) + \mathbf{p}^{g^*}(s,0)\right)\mathbf{R}e^{\hat{\mathbf{Q}}^g(s)x}e^{-sx}f^g(x)\,dx.$$

With the definitions $\mathbf{A}^*(s) = \sum_{g\in G}\mathbf{A}^{g^*}(s)$ and

$$\mathbf{A}^{g^*}(s) = \mathbf{I}^g\int_0^\infty e^{\hat{\mathbf{Q}}^g(s)x}\;e^{-sx}\;f^g(x)\;dx, \quad (9)$$

we get

$$\begin{aligned}\boldsymbol{\varphi}^*(s) &= \left(\boldsymbol{\pi}^G(0) + \mathbf{p}^{G^*}(s,0)\right)\mathbf{R}\;\mathbf{A}^*(s)\\ &= \left(\boldsymbol{\pi}^G(0) + \mathbf{p}^{G^*}(s,0)\right)\;\mathbf{A}^*(s).\end{aligned} \quad (10)$$

since the $\mathbf{I}^g$ filter in $\mathbf{A}^{g^*}(s)$ makes the multiplication with $\mathbf{R}$ useless. By similar arguments the transforms of the general state probabilities are

$$\boldsymbol{\pi}^{G^*}(s) = \left(\boldsymbol{\pi}^G(0) + \mathbf{p}^{G^*}(s,0)\right)\mathbf{B}^*(s) + \boldsymbol{\Pi}^{G^*}(s,0) \quad (11)$$

with $\mathbf{B}^*(s) = \sum_{g\in G}\mathbf{B}^{g^*}(s)$ and

$$\mathbf{B}^{g^*}(s) = \mathbf{I}^g\int_0^\infty e^{\hat{\mathbf{Q}}^g(s)x}\;e^{-sx}\;\bar{F}^g(x)\;dx\;\hat{\mathbf{D}}^g(s). \quad (12)$$

The Laplace transform of sum of the ODE system (1) and of the boundary conditions (3) is

$$\begin{aligned}&s\boldsymbol{\pi}^{E\bar{G}^*}(s) - \boldsymbol{\pi}^E(0) - \boldsymbol{\Pi}^G(0,0) + \mathbf{p}^{G^*}(s,0) =\\ &\boldsymbol{\pi}^{E\bar{G}^*}(s)\mathbf{Q} + \boldsymbol{\varphi}^*(s)\boldsymbol{\Delta} + \boldsymbol{\pi}^{G^*}(s)\bar{\mathbf{Q}},\end{aligned} \quad (13)$$

insertion of the previous result for the integrals and applying the initial condition for $\mathbf{\Pi}^G(0,0)$ yields

$$s\boldsymbol{\pi}^{E\bar{G}^*}(s) - \boldsymbol{\pi}^E(0) - \boldsymbol{\pi}^G(0)\bar{\mathbf{R}} + \mathbf{p}^{G^*}(s,0) = \boldsymbol{\pi}^{E\bar{G}^*}(s)\mathbf{Q} + \left(\boldsymbol{\pi}^G(0) + \mathbf{p}^{G^*}(s,0)\right) \ (\mathbf{A}^*(s)\boldsymbol{\Delta} + \mathbf{B}^*(s)\bar{\mathbf{Q}}),$$

with solution

$$\boldsymbol{\pi}^{E\bar{G}^*}(s) + \mathbf{p}^{G^*}(s,0) = \left(\boldsymbol{\pi}^E(0) + \boldsymbol{\pi}^G(0)\bar{\mathbf{R}} + \boldsymbol{\pi}^G(0)\left(\mathbf{A}^*(s)\boldsymbol{\Delta} + \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)\right) \cdot \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1}. \tag{14}$$

This is a Laplace-domain solution of the exponential state probabilities and the age intensities with an age variable equal to zero. A solution of all state probabilities in Laplace domain is then obtained by a further application of Eq. (11):

$$\begin{aligned}\boldsymbol{\pi}^*(s) &= \boldsymbol{\pi}^{E^*}(s) + \boldsymbol{\pi}^{G^*}(s)\\ &= \left(\boldsymbol{\pi}^{E\bar{G}^*}(s) + \mathbf{p}^{G^*}(s,0)\right)\left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right) + \boldsymbol{\pi}^G(0)\mathbf{B}^*(s)\\ &= \left(\boldsymbol{\pi}^E(0) + \boldsymbol{\pi}^G(0)\bar{\mathbf{R}} + \boldsymbol{\pi}^G(0)\left(\mathbf{A}^*(s)\boldsymbol{\Delta} + \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)\right) \cdot\\ &\quad \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1} \cdot \left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right)\\ &\quad + \boldsymbol{\pi}^G(0)\ \mathbf{B}^*(s).\end{aligned} \tag{15}$$

This is the desired closed from solution in Laplace domain.

3.3 Stationary Analysis

By taking t to infinity, t can be eliminated from the state equations (Eqns. (1)-(5)). A systematic approach to the analysis of these stationary state equations is, according to [9]:

$$\tilde{\mathbf{Q}}^g = \lim_{s\to 0} \hat{\mathbf{Q}}^g(s) = \mathbf{Q}_1^g - \mathbf{Q}_2^g\left(\mathbf{Q}_4^g\right)^{-1}\mathbf{Q}_3^g \tag{16}$$

$$\tilde{\mathbf{D}}^g = \lim_{s\to 0} \hat{\mathbf{D}}^g(s) = \mathbf{R}^g - \mathbf{Q}_2^g\left(\mathbf{Q}_4^g\right)^{-1} \tag{17}$$

Let $\tilde{\mathbf{D}} = \sum_{g\in G}\tilde{\mathbf{D}}^g$. $\tilde{\mathbf{Q}}^g$ is the generator matrix of the subordinated CTMC of g. (It describes the marking process in the subset of states where g is enabled during a firing cycle of g, if the time interval while g is disabled is not considered.) In the case of a prd-transition $\tilde{\mathbf{Q}}^g = \mathbf{Q}^g$ and $\tilde{\mathbf{D}}^g = \mathbf{R}^g$. Based on $\tilde{\mathbf{Q}}^g$, the matrices $\mathbf{\Omega}^g$ and $\mathbf{\Psi}^g$ can be defined. ω_{ij}^g is the probability of being in state j when g fires or is prd-preempted, given the state in the instant of enabling of g was i. ψ_{ij}^g is the expected sojourn time in state j until g fires or is prd-preempted, given the state in the instant of enabling of g was i. Thus, both quantities represent conditional state probabilities (conditional sojourn times) of the subordinated CTMC when (until) g fires:

$$\mathbf{\Omega}^g = \mathbf{I}^g\int_0^\infty e^{\tilde{\mathbf{Q}}^g x} f^g(x)\,dx,\ \ \mathbf{\Psi}^g = \mathbf{I}^g\int_0^\infty e^{\tilde{\mathbf{Q}}^g x} \bar{F}^g(x)\,dx. \tag{18}$$

Let the sums of the matrices be $\mathbf{\Omega} = \sum_{g\in G}\mathbf{\Omega}^g$ and $\mathbf{\Psi} = \sum_{g\in G}\mathbf{\Psi}^g$. Finally, a matrix $\mathbf{S}$ and a conversion matrix $\mathbf{D}$ can be defined:

$$\mathbf{S} = \bar{\mathbf{R}}\mathbf{Q} + \mathbf{\Omega}\boldsymbol{\Delta} + \mathbf{\Psi}\bar{\mathbf{Q}} - \mathbf{R}^G,\ \ \mathbf{D} = \bar{\mathbf{R}} + \mathbf{\Psi}\tilde{\mathbf{D}}. \tag{19}$$

$\mathbf{\Omega}$ and $\mathbf{\Psi}$ are obtained from the differential Eq. (2) and integral Eqns. (4) and (5), $\mathbf{S}$ and $\mathbf{D}$ are obtained from the differential Eq. (1), boundary condition (3), and normalization condition. It can be shown that $\mathbf{S}$ is the generator matrix of a CTMC [9]. Since it is very closely related to the embedded DTMC which is obtained by Markov renewal theory, we refer to it as the *embedded CTMC*. Defining the solution vector $\mathbf{w} = \boldsymbol{\pi}^E + \mathbf{p}^G(0) + \mathbf{\Pi}^G(0)$, the solution of the marking process can be obtained by first solving the embedded CTMC

$$\mathbf{w}\mathbf{S} = \mathbf{0},\ \text{subject to } \mathbf{w}\mathbf{D}\mathbf{e} = 1, \tag{20}$$

and by the conversion:

$$\boldsymbol{\pi} = \mathbf{w}\mathbf{D},\ \ \boldsymbol{\varphi} = \mathbf{w}\mathbf{\Omega}. \tag{21}$$

The solution is unique if the embedded CTMC contains one recurrent class. Often the linear system can be reduced significantly by eliminating the equations corresponding to the transient class of the embedded CTMC. The remaining numerical issues are to compute the matrices $\mathbf{\Omega}$ and $\mathbf{\Psi}$ and solve the linear system (20).

4 Markov Renewal Theory

Markov renewal theory takes a different point of view: the stochastic process is considered at discrete time instants, the so-called *regeneration points*. The choice is such that the supplementary variable defined in the previous section is set to zero or infinity in these points. State equations can be formulated with respect to the regeneration points. The resulting system of equations is a generalization of the Kolmogorov backward integral equations [5], a system of Volterra integral equations of the second type.

4.1 The State Equations

In the following we give a short informal introduction to Markov renewal theory, see [2] for a rigorous treatment. The regeneration points are defined as $0 = T_0 \le T_1 \le T_2 \le \cdots$. At those instants the process $N(t)$ is memoryless and the future evolution is a replica of itself. An embedded DTMC is defined at the regeneration points as $Y_k = N(T_k^+)$. The dynamics of the process at time t can be expressed by means of the following quantities: the *conditional state probabilities* $\pi_{ij}(t) = Pr\{N(t) = j \mid N(0) = i\}$, the conditional probabilities $e_{ij}(t) = Pr\{N(t) = j, T_1 > t \mid Y_0 = i\}$, and $k_{ij}(t) = Pr\{Y_1 = j, T_1 \leq t \mid Y_0 = i\}$. Using matrix notation, $\mathbf{E}(t)$ is referred to as the *local kernel* and $\mathbf{K}(t)$ as the *global kernel*. The so-called generalized Markov renewal equation is then

$$\mathbf{\Pi}(t) = \mathbf{E}(t) + \mathbf{K}' * \mathbf{\Pi}(t), \tag{22}$$

where $'$ denotes the derivation and $*$ denotes the convolution over matrices. The Laplace transform of Eq. (22) and its solution are

$$\mathbf{\Pi}^*(s) = \mathbf{E}^*(s) + s\mathbf{K}^*(s)\mathbf{\Pi}^*(s) = \left(\mathbf{I} - s\mathbf{K}^*(s)\right)^{-1} \mathbf{E}^*(s). \tag{23}$$

For the marking process of the SPN the next regeneration point is chosen as follows: the instant of the next state transition, if $N(t) \in \mathcal{S}^E$; the instant when the general transition fires or is preempted, if $N(t) \in \mathcal{S}^g$ and g is a prd-transition; and the instant when the general transition fires, if $N(t) \in \mathcal{S}^g$ and g is a prs-transition.

In the following $\mathbf{M}_D$ denotes the diagonal matrix composed by the diagonal elements of a matrix $\mathbf{M}$ and $\mathbf{M}_D{}^{-1}$ denotes its inverse, restricted to the non-zero rows and columns. Unfortunately, a closed form of the kernels in time domain is not available in case of prs-transitions. For only prd-transitions the kernels are given in time domain in [3]. Based on the kernel matrices the system (22) of generalized Markov renewal equations yields the transient state equations of a given SPN. Numerical time-domain solution algorithms are discussed in [11].

4.2 Transient Analysis in Laplace Domain

By fitting the matrix functions introduced in [1] to the matrix structure used in this paper, closed form expressions can be given for both the prd- and prs-case in Laplace domain. For the local kernel we get

$$\mathbf{E}^*(s) = \left(s\mathbf{I} - \bar{\mathbf{R}}\mathbf{Q}_D\right)^{-1} - \frac{1}{s}\mathbf{R} + \mathbf{B}^*(s) \tag{24}$$

and the transform of the global kernel is

$$\begin{aligned} s\mathbf{K}^*(s) &= \left(s\mathbf{I} - \bar{\mathbf{R}}\mathbf{Q}_D\right)^{-1} \bar{\mathbf{R}}\left(\mathbf{Q} - \mathbf{Q}_D\right) \\ &+ \mathbf{A}^*(s)\mathbf{\Delta} + \mathbf{B}^*(s)\bar{\mathbf{Q}}. \end{aligned} \tag{25}$$

Insertion into the solution of the Markov renewal equation yields

$$\mathbf{\Pi}^*(s) = \Big(\mathbf{I} - \left(s\mathbf{I} - \bar{\mathbf{R}}\mathbf{Q}_D\right)^{-1}\left(\bar{\mathbf{R}}\mathbf{Q} - \bar{\mathbf{R}}\mathbf{Q}_D\right) - \mathbf{A}^*(s)\mathbf{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\Big)^{-1} \cdot \Big(\left(s\mathbf{I} - \bar{\mathbf{R}}\mathbf{Q}_D\right)^{-1} - \tfrac{1}{s}\mathbf{R} + \mathbf{B}^*(s)\Big).$$

This is the desired closed form Laplace domain expression for the conditional state probabilities derived by Markov renewal theory. A simplification can be derived as follows: The expression consists of two factors: $\mathbf{\Pi}^*(s) = (\cdots)^{-1}(\cdots)$. It remains the same if we write $(\cdots)^{-1}\mathbf{C}^{-1}\mathbf{C}(\cdots)$ with $\mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$. We choose $\mathbf{C} = \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q}_D + \mathbf{R}\right)$, multiply $\mathbf{C}^{-1}$ with the left factor and $\mathbf{C}$ with the right factor. The result is

$$\begin{aligned} \mathbf{\Pi}^*(s) &= \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\mathbf{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1} \\ &\quad \left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right). \end{aligned} \tag{26}$$

The main advantage of (26) with respect to the previously known approaches based on Markov renewal theory (e.g., [1]) is its compactness. It integrates in a single matrix expression of size $|\mathcal{S}| \times |\mathcal{S}|$ the analysis of all the subordinated processes of the Markov regenerative (marking) process and the result of the Markov renewal equation. The previous approaches were composed by the independent analysis of all subordinated processes over a subset of $\mathcal{S}$ which required the use of matrices associated with the relevant subset of $\mathcal{S}$, and so a complicated renumbering of states which is hard to implement in any computer program.

4.3 Stationary Analysis

By taking t to infinity a system for the stationary case can be derived. It is slightly different from the system given by Eqns. (20) and (21). Two further quantities are required, the conversion factors and the transition probabilities of the embedded DTMC:

$$c_{ij} = E[\text{time in } j \text{ during } (0, T_1) \mid Y_0 = i] = \int_0^\infty e_{ij}(t)\, dt,$$

$$p_{ij} = Pr\{Y_1 = j \mid Y_0 = i\} = \lim_{t \to \infty} k_{ij}(t).$$

c_{ij} is the mean time the marking process spends in state j between two regeneration points, given that it started in state i after the last regeneration. Under certain restrictions which include aperiodicity, irreducibility and that the c_{ij} are finite it can be shown [2] that the limiting state probabilities are given by

$$\mathbf{u} = \mathbf{u}\mathbf{P},\ \mathbf{u}\mathbf{e} = 1,\ \boldsymbol{\pi} = \frac{\mathbf{u}\mathbf{C}}{\mathbf{u}\mathbf{C}\mathbf{e}}, \tag{27}$$

where $\mathbf{u}$ is the stationary solution of the embedded DTMC. Equation (27) expresses the state probabilities as the fraction of time the process spends in each state between two regenerations. The proof is quite complicated. The matrices $\mathbf{C}$ and $\mathbf{P}$ are obtained as:

$$\mathbf{C} = \int_0^\infty \mathbf{E}(t)\, dt = \lim_{s \to 0} \mathbf{E}^*(s) = -\bar{\mathbf{R}}\mathbf{Q}_D{}^{-1} + \mathbf{\Psi}\tilde{\mathbf{D}}, \tag{28}$$

and

$$\mathbf{P} = \mathbf{K}(\infty) = \lim_{s\to 0} s\mathbf{K}^*(s) = \mathbf{R} - \mathbf{Q}_D{}^{-1}\bar{\mathbf{R}}\mathbf{Q} + \boldsymbol{\Omega}\boldsymbol{\Delta} + \boldsymbol{\Psi}\bar{\mathbf{Q}}, \tag{29}$$

The analysis algorithm is analogous to that described in Sec. 3.3.

5 Formal Relationship

A formal relationship of the expressions obtained by the method of supplementary variables and by Markov renewal theory can be given, both in the transient and stationary case.

5.1 Transient Case

The solution (15) derived by the method of supplementary variables can be transformed to the solution (26) derived by Markov renewal theory. Since Eq. (15) gives the unconditional state probabilities, it is first expressed for the conditional state probabilities:

$$\begin{aligned}\boldsymbol{\Pi}^*(s) = {} & \left(\bar{\mathbf{R}} + \mathbf{A}^*(s)\boldsymbol{\Delta} + \mathbf{B}^*(s)\bar{\mathbf{Q}}\right) \cdot \\ & \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1} \\ & \cdot \left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right) + \mathbf{B}^*(s).\end{aligned}$$

For any matrix $\mathbf{M}$ (with proper size) we have $\mathbf{M}\left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right) + \mathbf{B}^*(s) = (\mathbf{M} + \mathbf{R})\left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right)$ since $\mathbf{R}\bar{\mathbf{R}} = \mathbf{0}$ and $\mathbf{R}\mathbf{B}^*(s) = \mathbf{B}^*(s)$. Applying this and

$$\begin{aligned}\mathbf{R} = {} & \left(\mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right) \\ & \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1},\end{aligned}$$

$\boldsymbol{\Pi}^*(s)$ is equal to

$$\begin{aligned}& \left(\bar{\mathbf{R}} + \mathbf{A}^*(s)\boldsymbol{\Delta} + \mathbf{B}^*(s)\bar{\mathbf{Q}} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right) \cdot \\ & \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1}\left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right) \\ = & \left(s\bar{\mathbf{R}} - \bar{\mathbf{R}}\mathbf{Q} + \mathbf{R} - \mathbf{A}^*(s)\boldsymbol{\Delta} - \mathbf{B}^*(s)\bar{\mathbf{Q}}\right)^{-1}\left(\bar{\mathbf{R}} + \mathbf{B}^*(s)\right)\end{aligned}$$

which is the same as derived by Markov renewal theory.

Thus, the same expression can be obtained from both approaches by algebraic manipulations. Since all steps are equivalences it is possible to pass from the transient state equations of one approach to the transient state equations of the other just by algebraic manipulations. Therefore it has been shown that both approaches are formally equivalent. Note however that the time domain solution algorithms motivated by the state equations of both approaches are different and lead to significantly different costs, as investigated in [11]. A major difference is that in the Markov renewal approach unconditional quantities are required leading to matrices of unknown functions rather than vectors like in the supplementary variable approach.

5.2 Stationary Case

In the case of the stationary equations a stronger relationship exists, such that both solution algorithms can be considered as almost the same. The system given by Eq. (27) derived by Markov renewal theory can also be written in the form

$$\mathbf{v} = \mathbf{v}\mathbf{P},\ \mathbf{v}\mathbf{C}\mathbf{e} = 1,\ \boldsymbol{\pi} = \mathbf{v}\mathbf{C}, \tag{30}$$

which is closer to the form of Eqns. (20) and (21) which were derived by the method of supplementary variables.

The two embedded Markov chains, the embedded CTMC in the approach of supplementary variables and the embedded DTMC in the approach based on Markov renewal theory, are formally related. From this duality, the relationship of the stationary equation systems can be derived. The generator matrix $\mathbf{S}$ of the embedded CTMC and the corresponding conversion matrix $\mathbf{D}$ (defined in Eq. (19)) and the stochastic matrix $\mathbf{P}$ of the embedded DTMC and the corresponding conversion matrix $\mathbf{C}$ (defined in Eqns. (29) and (28)) are related by

$$\mathbf{P} = \mathbf{I} + \left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D{}^{-1}\right)\mathbf{S},\ \mathbf{C} = \left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D{}^{-1}\right)\mathbf{D}, \tag{31}$$

$$\mathbf{S} = \left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D\right)(\mathbf{P} - \mathbf{I}),\ \mathbf{D} = \left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D\right)\mathbf{C}. \tag{32}$$

The relationship between the embedded DTMC and the embedded CTMC is therefore as follows: the embedded DTMC is the embedded Markov chain of the embedded CTMC with respect to the states of $\mathcal{S}^E$ and $\mathcal{S}^{\bar{G}}$ and the uniformized Markov chain (with a time step $1/q = 1$) of the embedded CTMC with respect to the other states.

The equation systems described by Eqns. (20) and (21) and by (30) are equivalent. Insertion of $\mathbf{S}$ and $\mathbf{D}$ according to Eq. (32) into Eq. (20) leads to:

$$\mathbf{w}\left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D\right)(\mathbf{P} - \mathbf{I}) = \mathbf{0},\ \mathbf{w}\left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D\right)\mathbf{C}\mathbf{e} = 1,$$
$$\boldsymbol{\pi} = \mathbf{w}\left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D\right)\mathbf{C}.$$

Define $\mathbf{v} = \mathbf{w}\left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D\right)$, then Eq. (30) follows. The reverse direction follows by applying Eq. (31) and by setting

$$\mathbf{w} = \mathbf{v}\left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D{}^{-1}\right). \tag{33}$$

5.3 Consequences of the Relationship

The relationship of the two analysis approaches is illustrated in Table 1. The state equations are equivalent and the stationary equations are obtained from

Table 1: Relationship of analysis approaches

	M.R. Theory		Suppl. Var.
Trans. Ana.	Generalized Markov Renewal Equations: $\mathbf{\Pi}(t) = \mathbf{E}(t) + \mathbf{K}' * \mathbf{\Pi}(t)$	$\Leftrightarrow$	PDEs, ODEs, ...
	$\Downarrow (t \to \infty)$		$\Downarrow (t \to \infty)$
Stat. Ana.	Embedded DTMC: $\mathbf{v} = \mathbf{vP}, \mathbf{vCe} = 1$ $\boldsymbol{\pi} = \mathbf{vC}$	$\Leftrightarrow$	Embedded CTMC: $\mathbf{0} = \mathbf{wS}, \mathbf{wDe} = 1$ $\boldsymbol{\pi} = \mathbf{wD}$

the transient ones by passing t to the limit. It is therefore possible to apply results derived by one of the approaches in the other, leading to a cross-fertilization. This allows to avoid double research efforts in the future. It allows also to circumvent the complicated proof of the limit theorem of Markov renewal theory.

As one example of cross-fertilization the result for the general firing frequencies $\boldsymbol{\varphi}$ derived with the method of supplementary variables in Sec. 3 can be shifted to the analysis based on Markov renewal theory. Applying Eqns. (21) and (33) yields

$$\boldsymbol{\varphi} = \mathbf{w\Omega} = \mathbf{v}\left(\mathbf{R} - \bar{\mathbf{R}}\mathbf{Q}_D{}^{-1}\right)\mathbf{\Omega} = \mathbf{v\Omega}.$$

Another example of cross-fertilization is the analysis of reducible marking processes. This case is simple to analyze by means of supplementary variables. The result can then easily be shifted to Markov renewal theory.

6 An Example

Figure 1 shows an SPN model adapted from [12] which models a mechanism for the management of packet switching in connection-oriented networks, referred to as *on-demand connection with delayed release* (OCDR). The transitions `arrival` and `service` model arrival and service of packets, `connect` and `release` model the setup and release of a connection. The model can be used to study the tradeoff of the mean waiting time and bandwidth utilization for different values of the connection release timer.

For illustration purposes we assume that the packets are generated according to a Poisson process with a rate of $\lambda = 100$ per second and that their length is uniformly distributed from 100 to 2000 Bytes. The bit rate of the connection is 10 Mbps, the time for setup is 10 ms and for release is 20 ms. Taking one second as the underlying time unit, the firing time of `service` is uniformly distributed from 0.00008 to 0.0016 and the firing times of `connect` and `release` are deterministically equal to 0.01 and 0.02. In order to give all data

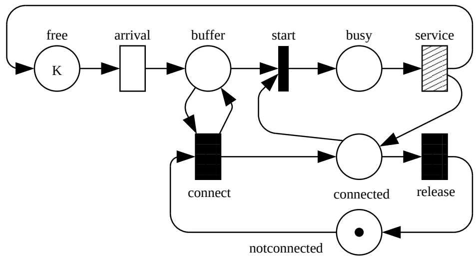

Figure 1: SPN model of the OCDR system

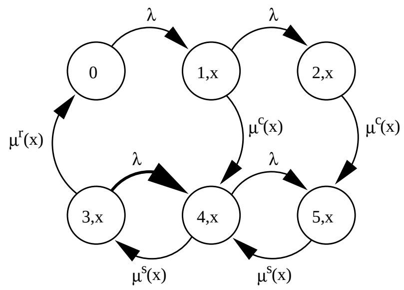

Figure 2: Stochastic process of the OCDR model

structures, we assume a buffer space $K = 2$. General transition `release` has prd-policy and is preempted when the exponential transition `arrival` fires.

For notational convenience, the three general transitions `service`, `connect`, and `release` are in the following abbreviated by their first letters s, c, and r, respectively. The general firing time distributions are denoted as $F^s(x)$, $F^c(x)$, and $F^r(x)$. The set of general transitions is given by $G = \{s, c, r\}$. The marking process $N(t)$ is defined by $\#buffer + \#busy + 1_{\{\#notconnected=0\}}(K+1)$. For $K = 2$ the state space and its subsets are given by $\mathcal{S} = \{0,1,2,3,4,5\}$, $\mathcal{S}^E = \{0\}$, $\mathcal{S}^s = \{4,5\}$, $\mathcal{S}^c = \{1,2\}$, $\mathcal{S}^r = \{3\}$, $\mathcal{S}^G = \{1,2,3,4,5\}$, and $\mathcal{S}^{\bar{s}} = \mathcal{S}^{\bar{c}} = \mathcal{S}^{\bar{r}} = \mathcal{S}^{\bar{G}} = \emptyset$ (since the model contains no prs-transition). The state-transition-rate diagram of the marking process is shown in Figure 2. The preemptive state transition is shown as a thick arc. States with an active general transition are supplemented by the age variable x, general state transitions are labeled with the instantaneous rate which depends on the value of x at the origin of the arc.

The three matrices $\mathbf{Q}$, $\bar{\mathbf{Q}}$, and $\mathbf{\Delta}$ are given by

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

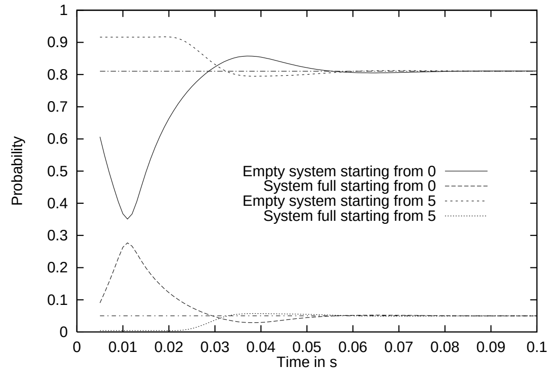

Figure 3: Probability of idle and full system with different initial conditions

$$\bar{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{\Delta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\bar{\mathbf{Q}}$ represents the preemption of the general transition `release`. All entries of the first row of $\mathbf{\Delta}$ are equal to zero since it corresponds to an exponential state.

6.1 Transient Analysis

The transform domain expression of the state transition probability matrix in (26) allows a much easier implementation than the previously published ones. We used a short Matlab program (including numerical inverse Laplace transformation) to evaluate the transient behavior based on the matrices $\mathbf{Q}$, $\bar{\mathbf{Q}}$, $\mathbf{\Delta}$, $\mathbf{I}^G$, and $\mathbf{R}$. The relatively simple expression in (26) does not avoid the problem of high computational complexity of transient analysis methods based on Laplace domain description.

The plot in Figure 3 depicts the transient probabilities of an idle and full system assuming that the system starts from state 0 (i.e., $\pi_{00}(t) + \pi_{03}(t)$ and $\pi_{02}(t) + \pi_{05}(t)$, respectively) and the same probabilities assuming that the system starts from state 5 (i.e., $\pi_{50}(t) + \pi_{53}(t)$ and $\pi_{52}(t) + \pi_{55}(t)$, respectively). The horizontal lines indicate the associated stationary probabilities.

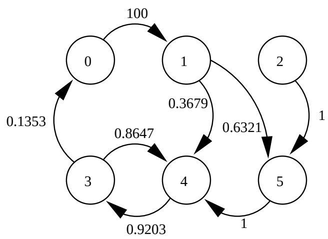

Figure 4: Embedded CTMC of the OCDR model

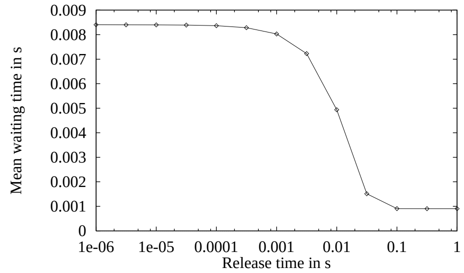

Figure 5: W vs. release time

6.2 Stationary Analysis

The generator matrix $\mathbf{S}$ of the embedded CTMC and the normalization matrix $\mathbf{D}$ evaluate to

$$\mathbf{S} \approx \begin{bmatrix} -100 & 100 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0.3678 & 0.6321 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0.1353 & 0 & 0 & -1 & 0.8647 & 0 \\ 0 & 0 & 0 & 0.9203 & -0.9203 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathbf{D} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.006321 & 0.003679 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.008647 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0007968 & 0.00004317 \\ 0 & 0 & 0 & 0 & 0 & 0.00084 \end{bmatrix}.$$

The state-transition-rate diagram of the embedded CTMC is shown in Figure 4. The embedded CTMC can be restricted to its single recurrent class $\{0, 1, 3, 4, 5\}$ to obtain the unique solution of the linear system (20) $\mathbf{w} \approx (0.1097, 10.969, 0, 81.0508, 88.0684, 6.9337)$, the state probabilities and general firing frequencies are finally obtained by conversion according to Eq. (21): $\boldsymbol{\pi} \approx (0.1097, 0.06934, 0.04035, 0.7008, 0.07018, 0.009626)$, $\boldsymbol{\varphi} \approx (0, 4.0353, 6.9337, 10.969, 81.0508, 13.9513)$. The resulting mean waiting time is $W = (0, 1, 2, 0, 1, 2) \cdot \boldsymbol{\pi} / ((0, 0, 0, 0, 1, 1) \cdot \boldsymbol{\varphi}) \approx 0.002521$ and the utilization $U = (0, 0, 0, 0, 1, 1) \cdot \boldsymbol{\pi} / ((0, 0, 0, 1, 1, 1) \cdot \boldsymbol{\pi}) \approx 0.1022$. As a comparison, $W \approx 0.005006$ and $U = 0.1361$, if all general distributions are replaced with exponential distributions with the same mean. In this case, the exponential assumption leads to results in the same order of magnitude, but with a relative error around 100%. Fig. 5 and 6 show the tradeoff of W and U for a varying value of the release timer and for $K = 1000$.

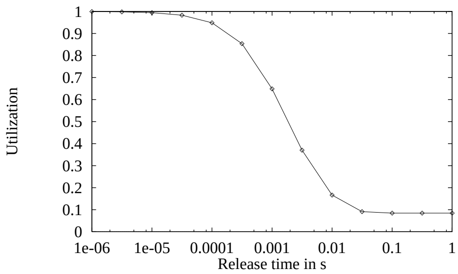

Figure 6: U vs. release time

The stochastic matrix $\mathbf{P}$ of the embedded DTMC (defined in Eq. (29)) and the conversion matrix $\mathbf{C}$ (defined in Eq. (28)) are

$$\mathbf{P} \approx \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3678 & 0.6321 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.1353 & 0 & 0 & 0 & 0.8647 & 0 \\ 0 & 0 & 0 & 0.9203 & 0.07968 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{C} \approx \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.006321 & 0.003679 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.008647 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0007968 & 0.00004317 \\ 0 & 0 & 0 & 0 & 0 & 0.00084 \end{bmatrix}.$$

The only differences to the $\mathbf{S}$ and $\mathbf{D}$ matrices are in the first row and for the diagonal entries of $\mathbf{P}$. The solution of the system (27) is $\mathbf{u} \approx (0.0554, 0.0554, 0, 0.4036, 0.4448, 0.0350)$, leading to the same solution for the state probabilities.

7 Conclusions

A formal relationship of Markov renewal theory and the method of supplementary variables has been derived in the context of stochastic Petri nets. The relationship is valid for Markov regenerative SPNs which may include prd- and prs-transitions and has been shown both in the transient and stationary case. Results derived by one of the approaches are now immediately available for the other. In the transient case, a by-product was a simpler analytical expression to describe the solution in Laplace domain, that allows a simpler implementation of the numerical analysis based on Laplace inversion.

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