Extension of some MAP results to transient MAPs and Markovian binary trees

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Outline

- Questions
- (stationary) MAP results
- Transient MAP (TMAP) results
- Markovian Binary Tree (MBT) results

Questions

Considering (stationary) MAP we have results for the following questions

- how to describe,
- similarity transformation,
- number of parameters,
- characterizing moments set,
- minimal representation,
- canonical representation for order 2.

What are the answers to these questions for Transient MAPs (TMAPs) and Markovian Binary Trees (MBTs)?

Basic results

A pair of square matrices $\boldsymbol{D_0}$, $\boldsymbol{D_1}$ define a (stationary) MAP. Where

- $\boldsymbol{D_0}_{ii} < 0$, $\boldsymbol{D_0}_{ij} \geq 0$ for $i \neq j$,
- $\boldsymbol{D_1}_{ij} \geq 0$,
- $(\boldsymbol{D_0} + \boldsymbol{D_1})\mathbb{1} = 0$,
- $\boldsymbol{D} = \boldsymbol{D_0} + \boldsymbol{D_1}$ is an irreducible generator matrix.

Properties of (stationary) MAPs:

Embedded phase transition: $\boldsymbol{P} = (-\boldsymbol{D_0})^{-1}\boldsymbol{D_1}$

Initial phase distribution: π, where $\pi = \pi\boldsymbol{P}$, $\pi\mathbb{1} = 1$,

Arrival intensity: $\lambda(t) = \pi e^{\boldsymbol{D}t}\boldsymbol{D_1}\mathbb{1}$,

Basic results

The inter-arrival time is PHase type distributed.

Pdf of the inter-arrival time is:

$$f_{X_0}(x) = \pi e^{\boldsymbol{D_0} x} \boldsymbol{D_1} \mathbb{1},$$

where $\int_x f_{X_0}(x)dx = 1$

Joint inter-arrival pdf:

$$f(x_0, x_1, \ldots, x_k) = \pi e^{\boldsymbol{D_0} x_0} \boldsymbol{D_1} e^{\boldsymbol{D_0} x_1} \boldsymbol{D_1} \ldots e^{\boldsymbol{D_0} x_k} \boldsymbol{D_1} \mathbb{1}.$$

Joint moments:

$$E(X_0^{i_0} X_1^{i_1} \ldots X_k^{i_k}) =$$

$$\pi i_0!(-\boldsymbol{D_0})^{-i_0} \boldsymbol{P} i_1!(-\boldsymbol{D_0})^{-i_1} \boldsymbol{P} \ldots\ i_k!(-\boldsymbol{D_0})^{-i_k} \boldsymbol{P} \mathbb{1}.$$

Similarity transformation

Similarity transformation with same size:

If $\boldsymbol{G}$ is a nonsingular matrix and $\boldsymbol{G}\mathbb{1} = \mathbb{1}$ then

$$\text{MAP}(\boldsymbol{D_0}, \boldsymbol{D_1}) \equiv \text{MAP}(\boldsymbol{G}^{-1}\boldsymbol{D_0}\boldsymbol{G}, \boldsymbol{G}^{-1}\boldsymbol{D_1}\boldsymbol{G})$$

and $\text{MAP}(\boldsymbol{G}^{-1}\boldsymbol{D_0}\boldsymbol{G}, \boldsymbol{G}^{-1}\boldsymbol{D_1}\boldsymbol{G})$ is usually non-Markovian.

Similarity transformation with different size:

If $\boldsymbol{V}$ of size $m \times n$ ($n < m$) is such that $\mathbb{1}_m = \boldsymbol{V}\mathbb{1}_n$, $\boldsymbol{D_0}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{C_0}$, $\boldsymbol{D_1}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{C_1}$, then

$$\text{MAP}(\boldsymbol{D_0}, \boldsymbol{D_1})_m \equiv \text{MAP}(\boldsymbol{C_0}, \boldsymbol{C_1})_n$$

If $\boldsymbol{W}$ of size $n \times m$ ($n < m$) is such that $\mathbb{1}_n = \boldsymbol{W}\mathbb{1}_m$, $\boldsymbol{W}\boldsymbol{D_0} = \boldsymbol{C_0}\boldsymbol{W}$, $\boldsymbol{W}\boldsymbol{D_1} = \boldsymbol{C_1}\boldsymbol{W}$, then

$$\text{MAP}(\boldsymbol{D_0}, \boldsymbol{D_1})_m \equiv \text{MAP}(\boldsymbol{C_0}, \boldsymbol{C_1})_n$$

Number of parameters

Number of parameters: n^2 (in case of full rank inter-arrival time distribution)

Basic moments set:

$$\mu_i = E(X_0^i), \quad i = 1, \ldots, 2n-1,$$
$$\eta_{ij} = E(X_0^i X_1^j), \quad i,j = 1, \ldots, n-1,$$

A heuristic:

$\boldsymbol{D_0}, \boldsymbol{D_1}$	+	$2n^2$
$(\boldsymbol{D_0} + \boldsymbol{D_1})\mathbb{1} = \mathbb{1}$	-	n
$\boldsymbol{G}$	-	n^2
$\boldsymbol{G}\mathbb{1} = \mathbb{1}$	+	n
	=	n^2

Minimal representation

Redundancy due to the closing vector:

Find a matrix $\boldsymbol{W}$ such that

$$\boldsymbol{W}^{-1}\boldsymbol{D_0}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{0}} & \star \\ \boldsymbol{0} & \star \end{pmatrix},$$

$$\boldsymbol{W}^{-1}\boldsymbol{D_1}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{1}} & \star \\ \boldsymbol{0} & \star \end{pmatrix}, \boldsymbol{W}^{-1}\mathbb{1}_m = \begin{pmatrix} \mathbb{1}_n \\ \boldsymbol{0} \end{pmatrix},$$

Redundancy due to the initial vector:

Find a matrix $\boldsymbol{W}$ such that

$$\boldsymbol{W}^{-1}\boldsymbol{D_0}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{0}} & \boldsymbol{0} \\ \star & \star \end{pmatrix},$$

$$\boldsymbol{W}^{-1}\boldsymbol{D_1}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{1}} & \boldsymbol{0} \\ \star & \star \end{pmatrix}, \pi\boldsymbol{W} = (\hat{\pi}, \boldsymbol{0}),$$

An implementation is available in the BuTools package.

Canonical representation of MAP(2)

Positive lag-1 correlation:

$$\boldsymbol{D_0} = \begin{pmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}, \quad \boldsymbol{D_1} = \begin{pmatrix} a\lambda_1 & 0 \\ (1-b)\lambda_2 & b\lambda_2 \end{pmatrix}$$

Negative lag-1 correlation:

$$\boldsymbol{D_0} = \begin{pmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}, \quad \boldsymbol{D_1} = \begin{pmatrix} 0 & a\lambda_1 \\ (1-b)\lambda_2 & b\lambda_2 \end{pmatrix}$$

where $\lambda_1 \leq \lambda_2$, $a, b \in (0, 1)$.

Basic results

An initial vector α and a pair of square matrices $\boldsymbol{D_0}$, $\boldsymbol{D_1}$ define a TMAP. Where

- $\alpha_i \geq 0$,
- $\alpha\mathbb{1} = 1$,
- $\boldsymbol{D_0}_{ii} < 0$, $\boldsymbol{D_0}_{ij} \geq 0$ for $i \neq j$,
- $\boldsymbol{D_1}_{ij} \geq 0$,
- $(\boldsymbol{D_0} + \boldsymbol{D_1})\mathbb{1} \leq 0$,
- $\boldsymbol{D} = \boldsymbol{D_0} + \boldsymbol{D_1}$ is a transient generator matrix ($\boldsymbol{D}$ is non singular).

Properties of TMAPs:

Embedded phase transition: $\boldsymbol{P} = (-\boldsymbol{D_0})^{-1}\boldsymbol{D_1}$
where $\boldsymbol{P}\mathbb{1} \leq \mathbb{1}$ (sub-stochastic)

Arrival intensity: $\lambda(t) = \alpha e^{\boldsymbol{D}t}\boldsymbol{D_1}\mathbb{1}$,
where $\lim_{t\to\infty} \lambda(t) = 0$

Mean number of arrivals: $\Lambda = \int_t \lambda(t)dt = \alpha(-\boldsymbol{D})^{-1}\boldsymbol{D_1}\mathbb{1}$

Basic results

The first inter-arrival time has a defective PHase type distribution.

Pdf of the inter-arrival time is:

$$f_{X_0}(x) = \alpha e^{\boldsymbol{D_0} x} \boldsymbol{D_1} \mathbb{1},$$

where $\int_x f_{X_0}(x) dx = Pr(X_0 < \infty) \leq 1$

We define the ith moment of the defective X_0 as follows

$$\mu_i = E(X_0^i I_{\{X_0 < \infty\}}) = \int_{x=0}^{\infty} x^i f_{X_0}(x) dx = i! \alpha (-\boldsymbol{D_0})^{-i-1} \boldsymbol{D_1} \mathbb{1}.$$

where $I_{\{X_0<\infty\}}$ is the indicator of event $X_0 < \infty$.

$\mu_0 = Pr(X_0 < \infty) \leq 1$

Basic results

Joint inter-arrival time also has a defective distribution:

$$f(x_0, x_1, \ldots, x_k) = \pi e^{\boldsymbol{D_0} x_0} \boldsymbol{D_1} e^{\boldsymbol{D_0} x_1} \boldsymbol{D_1} \ldots e^{\boldsymbol{D_0} x_k} \boldsymbol{D_1} \mathbb{1}.$$

such that

$$\int_{x_0} \ldots \int_{x_k} f(x_0, x_1, \ldots, x_k) dx_0 \ldots d_{x_k} = Pr(X_0 < \infty, \ldots, X_k < \infty).$$

Joint moments:

$$E(X_0^{i_0} X_1^{i_1} \ldots X_k^{i_k} I_{\{X_0<\infty,\ldots,X_k<\infty\}}) =$$

$$\int_{x_0} \ldots \int_{x_k} f(x_0, x_1, \ldots, x_k) \prod_{n=0}^{k} x_n^{i_n} dx_0 \ldots d_{x_k} =$$

$$\alpha i_0!(-\boldsymbol{D_0})^{-i_0} \boldsymbol{P} i_1!(-\boldsymbol{D_0})^{-i_1} \boldsymbol{P} \ldots\ i_k!(-\boldsymbol{D_0})^{-i_k} \boldsymbol{P} \mathbb{1}.$$

Similarity transformation

Similarity transformation with same size:

If $\boldsymbol{G}$ is a nonsingular matrix and $\boldsymbol{G}\mathbb{1} = \mathbb{1}$ then

$$\text{TMAP}(\alpha, \boldsymbol{D_0}, \boldsymbol{D_1}) \equiv \text{TMAP}(\alpha\boldsymbol{G}, \boldsymbol{G}^{-1}\boldsymbol{D_0}\boldsymbol{G}, \boldsymbol{G}^{-1}\boldsymbol{D_1}\boldsymbol{G})$$

Similarity transformation with different size:

If $\boldsymbol{V}$ of size $m \times n$ ($n < m$) is such that $\alpha\boldsymbol{V} = \gamma$, $\mathbb{1}_m = \boldsymbol{V}\mathbb{1}_n$, $\boldsymbol{D_0}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{C_0}$, $\boldsymbol{D_1}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{C_1}$, then

$$\text{TMAP}(\alpha, \boldsymbol{D_0}, \boldsymbol{D_1})_m \equiv \text{TMAP}(\gamma, \boldsymbol{C_0}, \boldsymbol{C_1})_n$$

If $\boldsymbol{W}$ of size $n \times m$ ($n < m$) is such that $\alpha = \gamma\boldsymbol{W}$, $\mathbb{1}_n = \boldsymbol{W}\mathbb{1}_m$, $\boldsymbol{W}\boldsymbol{D_0} = \boldsymbol{C_0}\boldsymbol{W}$, $\boldsymbol{W}\boldsymbol{D_1} = \boldsymbol{C_1}\boldsymbol{W}$, then

$$\text{TMAP}(\alpha, \boldsymbol{D_0}, \boldsymbol{D_1})_m \equiv \text{TMAP}(\gamma, \boldsymbol{C_0}, \boldsymbol{C_1})_n$$

Number of parameters

Number of parameters: $n^2 + 2n - 1$ (in case of full rank X_0 distribution)

Basic moments set:

$$\mu_i = E(X_0^i), \quad i = 0, 1, \ldots, 2n-1,$$

$$\eta_{ij} = E(X_0^i X_1^j), \quad i, j = 0, 1, \ldots, n-1 \text{ except } i = j = n-1,$$

A heuristic:

α	+	n
$\alpha\mathbb{1} = 1$	-	1
$\boldsymbol{D_0}, \boldsymbol{D_1}$	+	$2n^2$
$\boldsymbol{G}$	-	n^2
$\boldsymbol{G}\mathbb{1} = \mathbb{1}$	+	n
	=	$n^2 + 2n - 1$

Minimal representation

Redundancy due to the closing vector:

Find a matrix $\boldsymbol{W}$ such that

$$\boldsymbol{W}^{-1}\boldsymbol{D_0}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{0}} & \star \\ \boldsymbol{0} & \star \end{pmatrix},$$

$$\boldsymbol{W}^{-1}\boldsymbol{D_1}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{1}} & \star \\ \boldsymbol{0} & \star \end{pmatrix}, \boldsymbol{W}^{-1}\mathbb{1}_m = \begin{pmatrix} \mathbb{1}_n \\ \boldsymbol{0} \end{pmatrix},$$

Redundancy due to the initial vector:

Find a matrix $\boldsymbol{W}$ such that

$$\boldsymbol{W}^{-1}\boldsymbol{D_0}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{0}} & \boldsymbol{0} \\ \star & \star \end{pmatrix},$$

$$\boldsymbol{W}^{-1}\boldsymbol{D_1}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\boldsymbol{1}} & \boldsymbol{0} \\ \star & \star \end{pmatrix}, \alpha\boldsymbol{W} = (\hat{\alpha}, \boldsymbol{0}),$$

An implementation is available in the BuTools package.

Canonical representation of TMAP(2)

Far more complex than the MAP(2) case

(4 parameters $\rightarrow$ 7 parameters)

A heuristic:

A Markovian representation becomes non-Markovian, when one of its element becomes negative

$$\Downarrow$$

At the border the representation contains 0 elements.

The representation is composed by $2+4+4$ elements and there is 1 constraint ($\alpha\mathbb{1}=1$)

$\Rightarrow$ 9 free elements, 7 parameters

$\Rightarrow$ set 2 elements of the representation to 0

Canonical representation of TMAP(2)

Procedure

- Generate random TMAP(2)
- transform it to all possible representations with 2 zeros.
- choose the smallest set of representations which covers all random TMAP(2)

Result:
apart of state ordering we found 2 possible sets of 4 forms:

$(D_0)_{12} = 0$, $(D_1)_{21} = 0$	$(D_0)_{12} = 0$, $(D_1)_{21} = 0$
$(D_0)_{12} = 0$, $(D_1)_{22} = 0$	$(D_0)_{12} = 0$, $(D_1)_{22} = 0$
$\alpha_1 = 0$, $(D_1)_{12} = 0$	$\alpha_1 = 0$, $(D_1)_{12} = 0$
$\alpha_1 = 0$, $(D_1)_{11} = 0$	$\alpha_1 = 0$, $(D_1)_{22} = 0$

Basic results

An initial vector α and a pair of matrices $\boldsymbol{D_0}$ $(n \times n)$, $\boldsymbol{B}$ $(n \times n^2)$ define a MBT. Where

- $\alpha_i \geq 0$,
- $\alpha \mathbb{1} = 1$,
- $\boldsymbol{D_0}_{ii} < 0$, $\boldsymbol{D_0}_{ij} \geq 0$ for $i \neq j$,
- $\boldsymbol{B}_{ij} \geq 0$,
- $\boldsymbol{D_0} \mathbb{1} + \boldsymbol{B}(\mathbb{1} \otimes \mathbb{1}) \leq 0$,

Properties of MBTs:

Birth intensity: $\lambda(t) = \alpha e^{\boldsymbol{D}t} \boldsymbol{D_1} \mathbb{1}$,
where $\boldsymbol{D_1} = \boldsymbol{B}(\boldsymbol{I} \otimes \mathbb{1})$ and $\boldsymbol{D} = \boldsymbol{D_0} + \boldsymbol{D_1}$.

Mean number of children: $\Lambda = \int_t \lambda(t) dt = \alpha(-\boldsymbol{D})^{-1} \boldsymbol{D_1} \mathbb{1}$

Basic results

The first inter-arrival time has a defective PHase type distribution.

Pdf of the inter-arrival time is:

$$f_{X_0}(x) = \alpha e^{\boldsymbol{D_0} x} \boldsymbol{B}(\mathbb{I} \otimes \mathbb{I}),$$

where $\int_x f_{X_0}(x)dx \leq 1$

We define the ith moment of the defective X_0 as follows

$$\mu_i = E(X_0^i I_{\{X_0<\infty\}}) = \int_{x=0}^{\infty} x^i f_{X_0}(x)dx = i!\alpha(-\boldsymbol{D_0})^{-i-1}\boldsymbol{B}(\mathbb{I}\otimes\mathbb{I}).$$

where $I_{\{X_0<\infty\}}$ is the indicator of event $X_0 < \infty$.

$\mu_0 = Pr(X_0 < \infty) \leq 1$

Basic results

The joint density function of the inter-arrival times X_0, X_1, Y_0 is:

$$f(x_0, x_1, y_0) = \alpha e^{\boldsymbol{D_0} x_0} \boldsymbol{B} \left(e^{\boldsymbol{D_0} x_1} \boldsymbol{B} \mathbb{I} \otimes e^{\boldsymbol{D_0} y_0} \boldsymbol{B} \mathbb{I} \right).$$

where $\mathbb{I} = (\mathbb{I} \otimes \mathbb{I})$

Similarly,

$$\int_{x_0} \int_{x_1} \int_{y_0} f(x_0, x_1, y_0) dx_0 d_{x_1} d_{y_0} = Pr(X_0 < \infty, X_1 < \infty, Y_0 < \infty).$$

The defective joint moments of X_0, X_1, Y_0 are

$$\gamma_{i,j,k} = E(X_0^i X_1^j Y_0^k I_{\{X_0<\infty, X_1<\infty, Y_0<\infty\}}) =$$

$$\int_{x_0} \int_{x_1} \int_{y_0} x_0^i x_1^j y_0^k f(x_0, x_1, y_0) dx_0 d_{x_1} d_{y_0} =$$

$$i!j!k!\ \alpha(-\boldsymbol{D_0})^{-i-1} \boldsymbol{B} \left((-\boldsymbol{D_0})^{-j-1} \boldsymbol{B} \mathbb{I} \otimes (-\boldsymbol{D_0})^{-k-1} \boldsymbol{B} \mathbb{I} \right).$$

Similarity transformation

Similarity transformation with same size:

If $\boldsymbol{G}$ is a nonsingular matrix and $\boldsymbol{G}\mathbb{1} = \mathbb{1}$ then

$$\text{MBT}(\alpha, \boldsymbol{D_0}, \boldsymbol{B}) \equiv \text{MBT}(\alpha\boldsymbol{G}, \boldsymbol{G}^{-1}\boldsymbol{D_0}\boldsymbol{G}, \boldsymbol{G}^{-1}\boldsymbol{B}(\boldsymbol{G} \otimes \boldsymbol{G}))$$

Similarity transformation with different size:

If $\boldsymbol{V}$ of size $m \times n$ ($n < m$) is such that $\alpha\boldsymbol{V} = \gamma$, $\mathbb{1}_m = \boldsymbol{V}\mathbb{1}_n$, $\boldsymbol{D_0}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{C_0}$, $\boldsymbol{B}(\boldsymbol{V} \otimes \boldsymbol{V}) = \boldsymbol{V}\boldsymbol{B_C}$, then

$$\text{MBT}(\alpha, \boldsymbol{D_0}, \boldsymbol{B})_m \equiv \text{MBT}(\gamma, \boldsymbol{C_0}, \boldsymbol{B_C})_n$$

If $\boldsymbol{W}$ of size $n \times m$ ($n < m$) is such that $\alpha = \gamma\boldsymbol{W}$, $\mathbb{1}_n = \boldsymbol{W}\mathbb{1}_m$, $\boldsymbol{W}\boldsymbol{D_0} = \boldsymbol{C_0}\boldsymbol{W}$, $\boldsymbol{W}\boldsymbol{B} = \boldsymbol{B_C}(\boldsymbol{W} \otimes \boldsymbol{W})$, then

$$\text{MBT}(\alpha, \boldsymbol{D_0}, \boldsymbol{B})_m \equiv \text{MBT}(\gamma, \boldsymbol{C_0}, \boldsymbol{B_C})_n$$

Number of parameters

Number of parameters: $n^3 + 2n - 1$ (in case of full rank X_0 distribution)

Basic moments set:

$$\mu_i = E(X_0^i), \quad i = 0, 1, \ldots, 2n-1,$$

$$\gamma_{ijk} = E(X_0^i X_1^j Y_0^k), \quad i, j, k = 0, 1, \ldots, n-1 \text{ except } i = j = k = n-1,$$

A heuristic:

α	+	n
$\alpha \mathbb{1} = 1$	-	1
$\boldsymbol{D_0}$	+	n^2
$\boldsymbol{B}$	+	n^3
$\boldsymbol{G}$	-	n^2
$\boldsymbol{G}\mathbb{1} = \mathbb{1}$	+	n
	=	$n^3 + 2n - 1$

Minimal representation

Redundancy due to the closing vector:

Find a matrix $\boldsymbol{W}$ such that

$$\boldsymbol{W}^{-1}\boldsymbol{B}(\boldsymbol{W}\otimes\boldsymbol{W}) = \left(\underbrace{\left|\begin{matrix} \bullet & \star \\ \mathbf{0} & \star \end{matrix}\right| \cdots \left|\begin{matrix} \bullet & \star \\ \mathbf{0} & \star \end{matrix}\right|}_{n} \Bigg\| \underbrace{\left|\begin{matrix} \star & \star \\ \star & \star \end{matrix}\right| \cdots \left|\begin{matrix} \star & \star \\ \star & \star \end{matrix}\right|}_{m-n} \right),$$

$$\boldsymbol{W}^{-1}\boldsymbol{D_0}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_{\mathbf{0}} & \star \\ \mathbf{0} & \star \end{pmatrix} \text{ and } \boldsymbol{W}^{-1}\mathbb{1}_m = \begin{pmatrix} \mathbb{1}_n \\ \mathbf{0} \end{pmatrix},$$

Minimal representation

Redundancy due to the initial vector:

Find a matrix $\boldsymbol{W}$ such that

$$\boldsymbol{W}^{-1}\boldsymbol{B}(\boldsymbol{W}\otimes\boldsymbol{W}) = \left(\underbrace{\left|\begin{matrix} \bullet & \mathbf{0} \\ \star & \star \end{matrix}\right| \dots \left|\begin{matrix} \bullet & \mathbf{0} \\ \star & \star \end{matrix}\right|}_{n} \underbrace{\left|\begin{matrix} \mathbf{0} & \mathbf{0} \\ \star & \star \end{matrix}\right| \dots \left|\begin{matrix} \mathbf{0} & \mathbf{0} \\ \star & \star \end{matrix}\right|}_{m-n} \right),$$

$$\boldsymbol{W}^{-1}\boldsymbol{D_0}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{D}}_0 & \mathbf{0} \\ \star & \star \end{pmatrix} \text{ and } \boldsymbol{\alpha}\boldsymbol{W} = \begin{pmatrix} \hat{\boldsymbol{\alpha}} & \mathbf{0} \end{pmatrix},$$

An implementation is available in the BuTools package.

Canonical representation of MBT(2)

The representation is composed by $2+4+8$ elements and there is 1 constraint ($\alpha \mathbb{1} = 1$)
$\Rightarrow$ 11 free elements, 9 parameters
$\Rightarrow$ set 2 elements of the representation to 0

Same procedure as for TMAPs:
- Generate random TMAP(2)
- transform it to all possible representations with 2 zeros.
- choose the smallest set of representations which covers all random TMAP(2)

Canonical representation of MBT(2)

Result:
apart of state ordering we found a unique smallest subset of canonical forms covering all random examples

$$d_1 = 0,\ B_{2,11} = 0$$
$$d_1 = 0,\ B_{2,12} = 0$$
$$d_1 = 0,\ B_{2,21} = 0$$
$$d_1 = 0,\ B_{2,22} = 0$$
$$d_1 = 0,\ (D_0)_{21} = 0$$
$$(D_0)_{12} = 0,\ B_{2,11} = 0$$
$$(D_0)_{12} = 0,\ B_{2,12} = 0$$
$$(D_0)_{12} = 0,\ B_{2,21} = 0$$
$$(D_0)_{12} = 0,\ B_{2,22} = 0$$

Conclusions

- Different Markov chain driven processes share essentially similar properties,
- ... but it is not always straight forward to generalize them.

- *BuTools package:*
  mathematica, matlab/octave
  http://webspn.hit.bme.hu/ butools