Applied Probability Trust (11 December 2018)

MOMENT BOUNDS OF PH DISTRIBUTIONS WITH INFINITE OR FINITE SUPPORT BASED ON THE STEEPEST INCREASE PROPERTY

QI-MING HE,* *Department of Management Sciences, University of Waterloo*

GÁBOR HORVÁTH,** *Dept. of Networked Systems and Services, Budapest University of Technology and Economics*

ILLÉS HORVÁTH,** *MTA-BME Information Systems Research Group*

MIKLÓS TELEK,** *Dept. of Networked Systems and Services, Budapest University of Technology and Economics*

Abstract

The steepest increase property of phase type (PH) distributions was first raised in O'Cinneide [8] and was proved in O'Cinneide [8] and Yao [12], but since then it got little attention in the research community. In this work we demonstrate that the steepest increase property can be applied for proving previously unknown moment bounds of PH distributions with infinite or finite support. Of special interest are moment bounds free of specific PH representations except the size of the representation. For PH distributions with infinite support, it is shown that such a PH distribution is stochastically smaller than or equal to an Erlang distribution of the same size. For PH distributions with finite support, a class of distributions which was introduced and investigated in Ramaswami and Viswanath [9], it is shown that the squared coefficient of variation (SCV) of a PH distribution with finite support is greater than or equal to $1/(m(m+2))$, where m is the size of its PH representation.

Keywords: Phase-type distribution, Infinite support, Finite support, Moment bounds

2010 Mathematics Subject Classification: Primary 60E15

Secondary 62E15

* Postal address: 200 University Avenue West Waterloo, ON, Canada

** Postal address: Magyar Tudosok krt. 2., Budapest, Hungary

1. Introduction

Phase-type (PH) distributions were introduced by Neuts [3] for the study of queueing systems in 1975. Since then, PH distributions have been a subject of research, and have found applications in many areas of applied probability. For example, a serials papers by O'Cinneide ([5], [6], [7], [8]) revealed some fundamental properties of PH distributions. In Aldous and Shepp [1], it was shown that the SCV of a PH distribution with a PH representation of size m is greater than or equal to $1/m$. In Telek [11], the minimal SCV of discrete phase type distributions was found. In He et al. [2], stochastic comparison was utilized in the study of PH distributions, which also led to moment bounds of PH distributions. Those results not only deepen our understanding of PH distributions and PH representations, but also facilitate the applications of PH distributions significantly.

The steepest increase property of PH distribution was first raised in O'Cinneide [8] and was proved in O'Cinneide [8] and Yao [12]. We think that it is a very interesting property of PH distributions, which received little attention in the literature. This paper applies the property to find moment bounds of PH distributions, which demonstrates the usefulness of the property.

PH distributions with finite support were introduced and investigated in Rammaswami and Viswanath [9]. This generalization extends the applications of PH distributions significantly.

This paper finds a number of stochastic and moment bounds for PH distributions with finite and infinite support. Many of the moments bounds depend only on the size of PH representations and the eigenvalue with the largest real part of PH generators. A highlight of this paper is that the SCV of a bounded PH distribution with a PH representation of size m is greater than or equal to $1/(m(m+2))$. The moment bounds of PH distributions reveal fundamental properties of such probability distributions, e.g., they indicate which kind of general distributions can be closely approximated. The results are useful in, not limited to, i) finding moment bounds of performance measures for stochastic models and ii) selecting PH representations in parameter estimation of PH distributions.

The rest of the paper is organized as follows. PH distributions with infinite support are introduced together with the steepest increase property and with some of their moment bounds in Section 2. PH distributions with finite support are introduced and their moment bounds are studied in Section 3. Section 4 concludes the paper.

2. Phase-Type Distributions with Infinite Support

In this section, we first define phase-type distributions and their corresponding PH representations. Then we review the so-called "steepest increase lemma" (Yao [12]) on the density function of (ordinary) PH distributions, since that turned out to be useful to derive the moment bounds. We further extend the "steepest increase lemma" and derive new moment bounds for ordinary PH distributions.

A non-negative random variable $\mathcal{Y}$ has a phase-type distribution if it is the absorption time in a finite state continuous time Markov chain (see Neuts [4]). We assume that $\mathcal{Y}$ has a PH representation $(\boldsymbol{\alpha}, \boldsymbol{A})$ of size m, where $\boldsymbol{\alpha}$ is the initial distribution of the underlying continuous time Markov and $\boldsymbol{A}$ contains the transition rates among the transient states of the underlying continuous time Markov chain (referred to as PH generator or sub-intensity matrix). That is, $\boldsymbol{\alpha}$ is a non-negative probability vector and $\boldsymbol{A}$ has non-negative off diagonal and negative diagonal elements such that the diagonal element dominates each row, $\boldsymbol{A}\mathbb{1} \leq 0$ (element-wise), where $\mathbb{1}$ is a column vector of ones with the appropriate size. Let us denote the density function of $\mathcal{Y}$ by $f(t) = \boldsymbol{\alpha} e^{\boldsymbol{A}t}(-\boldsymbol{A}\mathbb{1})$ and the cumulative distribution function (cdf) by $F(t) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{A}t}\mathbb{1}$ (both for $t \geq 0$). To avoid the trivial case $\mathcal{Y} \equiv 0$, we assume that $0 < \boldsymbol{\alpha}\mathbb{1} \leq 1$. We also note that $f(t) > 0$ for $\forall\ t > 0$ by O'Cinneide [6].

The steepest increase conjecture which was first published and partially proved by O'Cinneide in [8]. Its complete proof is by Yao in [12]. The steepest increase lemma states that $f'(t)/f(t) \leq (m-1)/t$ for $t > 0$, or equivalently, $f(t)/t^{m-1}$ is decreasing in t for $t > 0$. Next, we present and show a "sharp" form of the steepest increase lemma.

Lemma 1. *For a PH distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{A})$ of size m and with density function $f(t)$ we have*

$$\frac{f'(t)}{f(t)} \leq \frac{m-1}{t} - \lambda, \text{ for } \ t > 0, \tag{1}$$

where λ is the absolute value of the eigenvalue of $\boldsymbol{A}$ with the largest real part (which *is real and negative). $-\lambda$ is also referred to as the dominant eigenvalue of matrix $\boldsymbol{A}$. In* (1)*, the equality holds when $\mathcal{Y}$ is Erlang(m,λ) distributed (i.e., $\mathcal{Y}$ is the sum of m independent exponential random variables with the same parameter λ.)*

Proof. The inequality just before equation (12) in O'Cinneide [8], combining with the proof in Yao [12], leads to $((m-1-\lambda)\boldsymbol{I}-\boldsymbol{A})e^{\boldsymbol{A}} \geq 0$, for PH generator $\boldsymbol{A}$, where $\boldsymbol{I}$ is the identity matrix. For PH generator $\boldsymbol{A}$, $\boldsymbol{A}t$ is also a PH generator for $t > 0$. The eigenvalue with the largest real part of $\boldsymbol{A}t$ is $-\lambda t$ for $t > 0$. Setting $\boldsymbol{A} =: \boldsymbol{A}t$ in the inequality, we obtain, for $t > 0$,

$$((m-1-\lambda t)\boldsymbol{I}-\boldsymbol{A}t)e^{\boldsymbol{A}t} \geq 0,$$

which leads to

$$e^{\boldsymbol{A}t}\boldsymbol{A}t \leq (m-1-\lambda t)e^{\boldsymbol{A}t}. \tag{2}$$

Pre-multiplying and post-multiplying both sides of (2) by $\boldsymbol{\alpha}$ and $-\boldsymbol{A}\mathbb{1}$, respectively, we obtain $f'(t)t \leq (m-1-\lambda t)f(t)$, which proves the lemma. □

Similar to Lemma 1, $-\lambda$ stands for the dominant eigenvalue of $\boldsymbol{A}$ in the sequel.

Equation (1) can be written in several alternative forms. For $t \geq 0$,

$$tf'(t) \leq (m-1)f(t) - \lambda t f(t), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) \leq (m-\lambda t)f(t), \tag{4}$$

and utilizing that $\lambda > 0$, we also have, for $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) < mf(t). \tag{5}$$

Lemma 2. *For $\mathcal{Y}$ with PH generator $\boldsymbol{A}$ of size m and dominant eigenvalue $-\lambda$, we have, for $0 < t_1 < t_2$,*

$$\frac{f(t_2)}{f(t_1)} \leq \frac{f_{(m,\lambda)}(t_2)}{f_{(m,\lambda)}(t_1)}, \tag{6}$$

where $f_{(m,\lambda)}(t) = \frac{\lambda^m t^{m-1}}{(m-1)!}e^{-\lambda t}$ is the density function of the Erlang random variable with parameters (m,λ).

Proof. For $t > 0$, Equation (1) can be written as,

$$(m-1)(\ln(t))' - (\lambda t)' \geq (\ln(f(t)))'. \tag{7}$$

Taking integral on both sides of (7) from $t_1 > 0$ to $t_2 > t_1$, we obtain

$$\ln((t_2/t_1)^{m-1} f(t_1)/f(t_2)) \geq \lambda(t_2 - t_1), \tag{8}$$

where $f(t) > 0$ for $\forall\ t > 0$ ensures that the integration can be done properly. Taking the exponent of both sides of (8), we obtain

$$f(t_2) \leq \frac{f(t_1)}{t_1^{m-1}} e^{\lambda t_1} t_2^{m-1} e^{-\lambda t_2} = f(t_1) \frac{f_{(m,\lambda)}(t_2)}{f_{(m,\lambda)}(t_1)}, \quad \text{for } t_2 > t_1 > 0,$$

which leads to the desired result. □

Lemma 2 implies that $f(t)/f_{(m,\lambda)}(t)$ is decreasing in t, which is a generalization of the monotonicity of function $f(t)/t^{m-1}$. Further, Lemma 2 leads to the following stochastic comparison result between $\mathcal{Y}$ and the Erlang random variable $\mathcal{X}_{(m,\lambda)}$ with parameters (m, λ). Random variable Y is stochastically smaller than or equal to random variable X, denoted as $Y \leq_d X$, if $F_Y(t) \geq F_X(t)$ holds for all real t (Stoyan and Daley [10]).

Corollary 1. *The PH distributed random variable $\mathcal{Y}$ (of size m and with dominant eigenvalue $-\lambda$) is stochastically smaller than or equal to $\mathcal{X}_{(m,\lambda)}$. Consequently, we have, for $n \geq 1$,*

$$\mathbb{E}[\mathcal{Y}^n] \leq \mathbb{E}\left[\mathcal{X}^n_{(m,\lambda)}\right] = \frac{(m+n-1)!}{(m-1)!\lambda^n}. \tag{9}$$

Proof. Since both $f(t)$ and $f_{(m,\lambda)}(t)$ are density functions on $[0, \infty)$, there must be at least one intersection in $(0, \infty)$. If t^* is an intersection (i.e., $f(t^*) = f_{(m,\lambda)}(t^*)$), by Lemma 2, we must have $f(t) \leq f_{(m,\lambda)}(t)$ for $t > t^*$ and $f(t) \geq f_{(m,\lambda)}(t)$ for $t < t^*$. Thus, there are only three possible cases: i) $f(t)$ and $f_{(m,\lambda)}(t)$ are identical; ii) $f(t)$ and $f_{(m,\lambda)}(t)$ have exactly two intersections $t = 0, t = t^*$; or iii) $f(t)$ and $f_{(m,\lambda)}(t)$ have exactly one intersection $t = t^*$. Then we must have $f(t) \geq f_{(m,\lambda)}(t)$ for $0 < t \leq t^*$ and $f(t) \leq f_{(m,\lambda)}(t)$ for $t \geq t^*$ (> 0), which leads to $F(t) \geq F_{\mathcal{X}_{(m,\lambda)}}(t)$, where $F_{\mathcal{X}_{(m,\lambda)}}(t)$ is the cdf of $\mathcal{X}_{(m,\lambda)}$, for $0 < t \leq t^*$, and $1 - F(t) \leq 1 - F_{\mathcal{X}_{(m,\lambda)}}(t)$ for $t > t^*$. Consequently, we obtain $F(t) \geq F_{\mathcal{X}_{(m,\lambda)}}(t)$ for $t > 0$, which leads to the first result. All the moment bounds can be obtained from $\mathcal{Y} \leq_d \mathcal{X}_{(m,\lambda)}$ directly. □

A random variable X is smaller in mean residual life than random variable Y, denoted as $X \leq_c Y$, if $\mathbb{E}[\max\{0, X - t\}] \leq \mathbb{E}[\max\{0, Y - t\}]$ holds for all real t (Stoyan and Daley [10]). By O'Cinneide [7], it is known that the Erlang distribution with parameters $(m, m/\mathbb{E}[\mathcal{Y}])$ is smaller in mean residual life than $\mathcal{Y}$, which has the same mean. Then the moments of $\mathcal{Y}$ are bounded from above and below as follows.

$$\frac{(m+n-1)!}{(m-1)!}\left(\frac{\mathbb{E}[\mathcal{Y}]}{m}\right)^n \leq \mathbb{E}[\mathcal{Y}^n] \leq \frac{(m+n-1)!}{(m-1)!}\frac{1}{\lambda^n}. \tag{10}$$

A by-product of the above inequalities is that $\mathbb{E}[\mathcal{Y}] = m/\lambda$ if and only if $\mathcal{Y}$ has an Erlang distribution, which can also be obtained from inequality (4) directly.

Based on Lemma 1, the next lemma refines the upper bound for the $n+1$-st moment based on the n-th one, which gives Corollary 1 an alternative proof.

Lemma 3. *For $n = 0, 1, \ldots,$ the $n+1$-st moment of $\mathcal{Y}$ (of size m and with dominant eigenvalue $-\lambda$) is bounded by*

$$\mathbb{E}\left[\mathcal{Y}^{n+1}\right] \leq \frac{m+n}{\lambda}\mathbb{E}[\mathcal{Y}^n], \tag{11}$$

and the equality holds when $\mathcal{Y} = \mathcal{X}_{(m,\lambda)}$.

Proof. Multiplying both sides of (4) by t^n and integrating from 0 to ∞ gives the following identities for the left-hand side (LHS) and the right-hand side (RHS):

$$\begin{aligned} LHS &= \int_{t=0}^{\infty} t^n \mathrm{d}(tf(t)) = [t^{n+1}f(t)]_0^{\infty} - \int_{t=0}^{\infty} tf(t)\mathrm{d}t^n \\ &= -n\int_{t=0}^{\infty} tf(t)t^{n-1}\mathrm{d}t = -n\mathbb{E}[\mathcal{Y}^n]; \\ RHS &= \int_{t=0}^{\infty} t^n(m-\lambda t)f(t)\mathrm{d}t = m\int_{t=0}^{\infty} t^n f(t)\mathrm{d}t - \lambda\int_{t=0}^{\infty} t^{n+1}f(t)\mathrm{d}t \\ &= m\mathbb{E}[\mathcal{Y}^n] - \lambda\mathbb{E}\left[\mathcal{Y}^{n+1}\right], \end{aligned}$$

from which we have

$$-n\mathbb{E}[\mathcal{Y}^n] \leq m\mathbb{E}[\mathcal{Y}^n] - \lambda\mathbb{E}\left[\mathcal{Y}^{n+1}\right].$$

When $\mathcal{Y}$ is Erlang(λ, m) distributed, the equality in (11) comes from the fact that Lemma 1 gives equality for Erlang distribution for all $t > 0$. □

Applying Lemma 3 for $n = 0$ and $n = 1$ enables us to derive the following upper bounds on the mean $\mathbb{E}[\mathcal{Y}]$ and the squared coefficient of variation $SCV_{\mathcal{Y}} = \frac{\mathbb{E}[\mathcal{Y}^2]}{\mathbb{E}[\mathcal{Y}]^2} - 1$:

$$\mathbb{E}[\mathcal{Y}] \leq \frac{m}{\lambda}, \tag{12}$$

$$SCV_{\mathcal{Y}} \leq \frac{m+1-\lambda\mathbb{E}[\mathcal{Y}]}{\lambda\mathbb{E}[\mathcal{Y}]}. \tag{13}$$

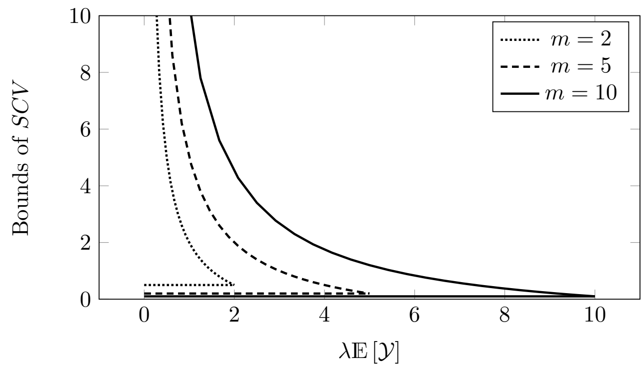


FIGURE 1: Bounds of SCV for ordinary PH distributions. The lower bound is the well known $1/m$ bound provided in [1, 7], while the upper bound is provided based on the steepest increase property.

Interestingly, equation (13) gives an upper bound for $SCV_{\mathcal{Y}}$, while the lower bound for $SCV_{\mathcal{Y}}$ is much more widely known as $SCV_{\mathcal{Y}} \geq 1/m$. Hence we have (see Figure 1)

$$\frac{1}{m} \leq SCV_{\mathcal{Y}} \leq \frac{m+1}{\lambda\mathbb{E}[\mathcal{Y}]} - 1.$$

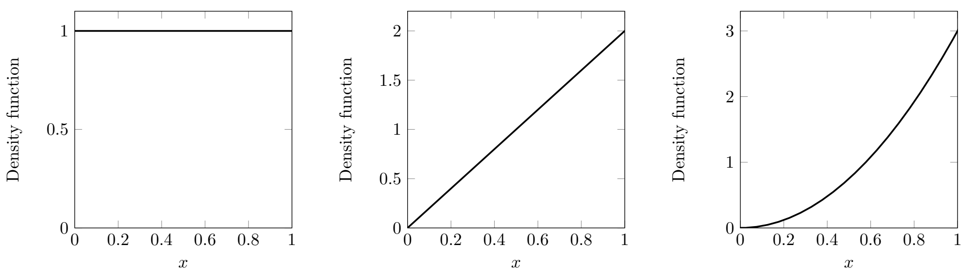
3. Phase-Type Distributions with Finite Support

PH distributions with finite support were introduced in [9], where three classes of finite support distributions were considered (matrix exponential density from lower bound to upper bound, from upper bound to lower bound and convex combination of the two). Instead, we define $\mathcal{Z}$ to have distribution $b + (\mathcal{Y}|\mathcal{Y} < T)$. The support of $\mathcal{Z}$ is thus $[b, B)$ with $b < B$ and $B = b + T$. Recall that $\mathcal{Y}$ is the ordinary (or infinite support) PH distribution with density function $f(t) = \boldsymbol{\alpha} e^{\boldsymbol{A}t}(-\boldsymbol{A})\mathbb{1}$. Then the density function of $\mathcal{Z}$ is given by

$$f_{\mathcal{Z}}(t) = \frac{\boldsymbol{\alpha} e^{\boldsymbol{A}(t-b)}(-\boldsymbol{A})\mathbb{1}}{1 - \boldsymbol{\alpha} e^{\boldsymbol{A}T}\mathbb{1}}, \tag{14}$$

for $t \in [b, B)$, and $f_{\mathcal{Z}}(t) = 0$ for $t \notin [b, B)$.

We denote this class of finite PH distributions by $FTPH_1$ (with some further similar classes $FTPH_2$ and $FTPH_3$ in mind: $FTPH_2$ is defined as $\mathcal{Z}_2 = B - (\mathcal{Y}|\mathcal{Y} < T)$ and $FTPH_3$ as the convex combination of $\mathcal{Z}$ and $\mathcal{Z}_2$; these are subject to future



a. Uniform distribution b. Linear distribution c. Quadratic distribution

FIGURE 2: Some interesting members of the $FTPH_1$ class

investigations). In this paper, we shall primarily focus on the moments of $FTPH_1$ distributions.

Although $FTPH_1$ is obtained by just a simple truncation of an ordinary PH distribution, it has some very interesting members. A truncated exponential distribution with a very small intensity parameter leads to a uniform distribution (Figure 2 a.), since

$$\lim_{\lambda\to 0} f_{\mathcal{Z}}^{Exp}(t) = \lim_{\lambda\to 0} \lambda e^{\lambda t}/(1-e^{-\lambda}) = 1.$$

Similarly, a truncated Erlang-N distribution with very small intensity gives

$$\lim_{\lambda\to 0} f_{\mathcal{Z}}^{Erl-N}(t) = N\, t^{N-1},$$

yielding linear, quadratic, cubic distributions, respectively (see Figure 2 b. and c.).

The importance of these special $FTPH_1$ members is that density functions of such shapes are notoriously difficult to capture by ordinary PH distributions. Ordinary PH distributions with a limited number of phases purely approximate them. In practical computations the $\lambda \to 0$ limit also causes difficulties, but easily computable small positive λ values give reasonably good approximations. Those issues of approximations and parameter estimations of PH distributions with finite support will be addressed in a separate paper.

A general formula for computing the moments for PH distributions with finite support can be obtained as follows.

Theorem 1. *The nth moment of a $FTPH_1$ random variable, $\mathcal{Z}$ with parameters*

$(\boldsymbol{\alpha}, \boldsymbol{A})$ *over interval* $[b, B)$*, is expressed by*

$$\mathbb{E}\left[\mathcal{Z}^n\right] = \frac{b^n(1-\boldsymbol{\alpha}\mathbb{1}) + \sum_{d=0}^{n}\binom{n}{d} d!\boldsymbol{\alpha}(-\boldsymbol{A})^{-d}\left(b^{n-d}\boldsymbol{I} - T^{n-d}e^{\boldsymbol{A}T}\right)\mathbb{1}}{1-\boldsymbol{\alpha}e^{\boldsymbol{A}T}\mathbb{1}}, \tag{15}$$

where $T = B - b$.

Proof. The theorem is proved by routine calculations. □

3.1. Moment bounds for the case with $b = 0$

In this subsection, we assume $b = 0$, which is extended to $b > 0$ in the next subsection. A random variable in $FTPH_1$ is denoted as $\mathcal{W} = \mathcal{Y}|\mathcal{Y} < T$. We derive and prove lower and upper bounds for the moments of $\mathcal{W}$. For $i = 0, 1, \ldots$ we introduce the notation $E_i(T) = \mathbb{E}\left[\mathcal{I}_{\{\mathcal{Y}<T\}}\mathcal{Y}^i\right] = \int_{t=0}^{T} t^i f(t)\mathrm{d}t$, where $\mathcal{I}_{\{a\}}$ denotes the indicator of a. Then $\mathbb{E}\left[\mathcal{W}^i\right]$ can be written as, for $i = 0, 1, ...$,

$$\mathbb{E}\left[\mathcal{W}^i\right] = \frac{E_i(T)}{E_0(T)}. \tag{16}$$

The next lemma is similar to Lemma 3 (for ordinary PH distributions), but it does not depend on the dominant eigenvalue $-\lambda$.

Lemma 4. *The moments a* $FTPH_1$ *random variable,* $\mathcal{W}$*, with support on* $(0, T)$*, satisfies*

$$\mathbb{E}\left[\mathcal{W}^n\right] \leq \frac{(m+n-1)T}{m+n}\mathbb{E}\left[\mathcal{W}^{n-1}\right]. \tag{17}$$

Proof. Multiplying both sides of equation (5) by $t^{n-1}(T-t)$ and integrating from 0 to T gives the following identities for the left-hand side

$$\begin{aligned} LHS &= \int_{t=0}^{T} t^{n-1}(T-t)\frac{\mathrm{d}}{\mathrm{d}t}(tf(t))\mathrm{d}t = -\int_{t=0}^{T} tf(t)\mathrm{d}(t^{n-1}(T-t)) \\ &= \int_{t=0}^{T}(nt^n - (n-1)Tt^{n-1})f(t)\mathrm{d}t = nE_n(T) - (n-1)TE_{n-1}(T), \end{aligned}$$

where we used the rule of Stieltjes integration by part in the first step

$$\int_a^b f(x)\mathrm{d}g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\mathrm{d}f(x),$$

and

$$\int_a^b g(x)\mathrm{d}f(x) = \int_a^b g(x)f'(x)\mathrm{d}x,$$

in the second step. For the right-hand side we have

$$RHS = \int_{t=0}^{T} t^{n-1}(T-t)mf(t)\mathrm{d}t = \int_{t=0}^{T} mTt^{n-1}f(t)\mathrm{d}t - m\int_{t=0}^{T} t^{n}f(t)\mathrm{d}t$$
$$= mTE_{n-1}(T) - mE_n(T),$$

from which

$$nE_n(T) - (n-1)TE_{n-1}(T) \leq mTE_{n-1}(T) - mE_n(T),$$
$$(m+n)E_n(T) \leq (m+n-1)TE_{n-1}(T),$$
$$\frac{E_n(T)}{E_0(T)} \leq \frac{(m+n-1)T}{m+n}\frac{E_{n-1}(T)}{E_0(T)}, \tag{18}$$

which leads to the desired result. □

In the following corollary the upper bound for the nth moment is provided, independent from the lower order moments.

Corollary 2. *The nth moment a $FTPH_1$ random variable, $\mathcal{W}$, with support on $(0,T)$, is bounded by*

$$\mathbb{E}\left[\mathcal{W}^n\right] \leq \frac{mT^n}{m+n} \tag{19}$$

and the upper bound is strict except for $\mathcal{Y}$ is Erlang(λ, m) distributed and λ tends to 0. In particular, we have $\mathbb{E}\left[\mathcal{W}\right] \leq mT/(m+1)$, which indicates that no PH distribution with finite support can have a mean close to the upper bound $T = B$.

Proof. Recursively applying (17) for moments $1,\ldots,n$ gives the upper bound. The statement on the equality comes from the fact that the right part of (19) gives equality only when (18) is equality for $1,\ldots,n$ and it occurs only when $\mathcal{Y}$ is Erlang(λ, m) distributed and λ tends to 0, because (5) gives equality only for Erlang(λ, m) distribution when λ tends to 0. □

After deriving these moment bounds, the next question is how to reach the extreme values, what is the $FTPH_1$ structure which realizes the moment bounds. According to the next lemma, Erlang distributions play an important role in this respect.

Lemma 5. *The upper bound in Lemma 4 is strict when $\mathcal{W}$ equals 0 with probability $p = 1 - \frac{(m+n-1)\mu_{n-1}}{mT^{n-1}}$ and is truncated Erlang(λ, m) distributed with probability $1-p$ such that λ tends to 0, where $\mu_{n-1} = \mathbb{E}\left[\mathcal{W}^{n-1}\right]$.*

Proof. To prove the statement we show that compared to the extreme distribution of Lemma 5 any valid change in the probability of the mass at zero, p, decreases the n-th moment. First we note that increasing p is not a valid change, because to maintain $\mathbb{E}\left[\mathcal{W}^{n-1}\right]$ the $n-1$-st moment of a strictly positive $FTPH_1$ needs to be increased above $\frac{mT^{n-1}}{m+n-1}$ (which is not possible according to Corollary 2).

Let us now try to decrease p rather than increasing it. Consider a distribution whose mass at zero has probability $\hat{p} = p - \Delta\mu_{n-1}$, where Δ is a small positive number. In this case, the $n-1$-st moment of the strictly positive part, μ^+_{n-1}, is

$$\begin{aligned}
\mu^+_{n-1} &= \frac{\mu_{n-1}}{1-\hat{p}} = \frac{\mu_{n-1}}{1-p+\Delta\mu_{n-1}} = \frac{\mu_{n-1}}{\frac{(m+n-1)\mu_{n-1}}{mT^{n-1}} + \Delta\mu_{n-1}} \\
&= \frac{mT^{n-1}}{(m+n-1)+\Delta mT^{n-1}} < \frac{mT^{n-1}}{m+n-1}.
\end{aligned}$$

When the $n-1$-st moment of the strictly positive part, $\mathcal{W}^+$, is μ^+_{n-1} its nth moment is bounded by (17), and using that we can write

$$\begin{aligned}
&(1-\hat{p})\mathbb{E}\left[\mathcal{W}^{+n}\right] \\
&< \underbrace{\mu_{n-1}\left(\frac{(m+n-1)+\Delta mT^{n-1}}{mT^{n-1}}\right)}_{1-\hat{p}} \frac{(m+n-1)T}{m+n} \underbrace{\left(\frac{mT^{n-1}}{(m+n-1)+\Delta mT^{n-1}}\right)}_{\mu^+_{n-1}} \\
&= \frac{(m+n-1)T\mu_{n-1}}{m+n},
\end{aligned}$$

where the inequality is strict, because the distribution of $\mathcal{W}^+$ is different from Erlang(λ, m) such that λ tends to 0, since its $n-1$-st moment is less than $\frac{mT^{n-1}}{m+n-1}$. □

Having the dominant eigenvalue independent moment bounds derived in Lemma 4, the following results provide moment bounds as a function of λ.

Lemma 6. *For $n = 1, 2, \ldots$, the $n+1$-st moment of $\mathcal{W}$ is bounded by*

$$\begin{aligned}
&\frac{m+n+\lambda T}{\lambda}\mathbb{E}\left[\mathcal{W}^n\right] - \frac{(m+n-1)T}{\lambda}\mathbb{E}\left[\mathcal{W}^{n-1}\right] \\
&\leq \mathbb{E}\left[\mathcal{W}^{n+1}\right] \leq \frac{m+n}{\lambda}\mathbb{E}\left[\mathcal{W}^n\right] - \frac{T^{n+1}f(T)}{\lambda E_0(T)}. \qquad (20)
\end{aligned}$$

Proof. Multiplying both sides of (4) by $(T-t)t^{n-1}$ and integrating from 0 to T

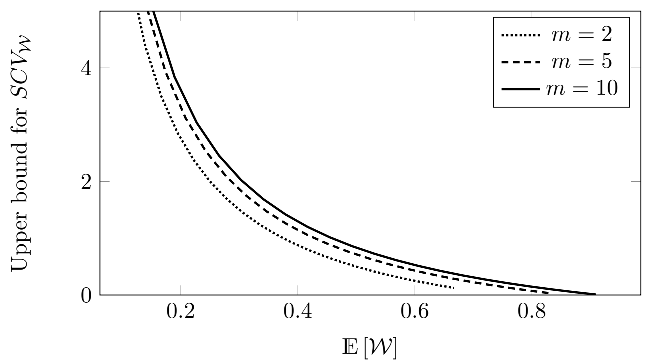


FIGURE 3: Upper bounds of SCV for $FTPH_1$ distributions with parameters $b=0, B=1$

gives

$$\begin{aligned}
LHS &= \int_{t=0}^{T} (T-t)t^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}(tf(t))\mathrm{d}t = -\int_{t=0}^{T} tf(t)\mathrm{d}(Tt^{n-1}-t^n) \\
&= -\int_{t=0}^{T} tf(t)(T(n-1)t^{n-2}-nt^{n-1})\mathrm{d}t \\
&= -T(n-1)E_{n-1}(T)+nE_n(T); \\
RHS &= \int_{t=0}^{T} (Tt^{n-1}-t^n)(m-\lambda t)f(t)\mathrm{d}t \\
&= mTE_{n-1}(T)-mE_n(T)-\lambda TE_n(T)+\lambda E_{n+1}(T),
\end{aligned}$$

from which it follows that

$$-T(n-1)E_{n-1}(T)+nE_n(T) \le mTE_{n-1}(T)-(m+\lambda T)E_n(T)+\lambda E_{n+1}(T),$$

$$\begin{gathered}
(m+n+\lambda T)E_n(T)-(m+n-1)TE_{n-1}(T) \le \lambda E_{n+1}(T). \\
\frac{m+n+\lambda T}{\lambda}\frac{E_n(T)}{E_0(T)} - \frac{(m+n-1)T}{\lambda}\frac{E_{n-1}(T)}{E_0(T)} \le \frac{E_{n+1}(T)}{E_0(T)}.
\end{gathered}$$

On the other hand, multiplying both sides of (4) by t^n and integrating from 0 to T gives

$$\begin{aligned}
LHS &= \int_{t=0}^{T} t^n\frac{\mathrm{d}}{\mathrm{d}t}(tf(t))\mathrm{d}t = T^{n+1}f(T) - \int_{t=0}^{T} tf(t)\mathrm{d}t^n \\
&= T^{n+1}f(T) - \int_{t=0}^{T} tf(t)nt^{n-1}\mathrm{d}t = T^{n+1}f(T) - nE_n(T); \\
RHS &= \int_{t=0}^{T} t^n(m-\lambda t)f(t)\mathrm{d}t = m\int_{t=0}^{T} t^n f(t)\mathrm{d}t - \lambda\int_{t=0}^{T} t^{n+1}f(t)\mathrm{d}t \\
&= mE_n(T) - \lambda E_{n+1}(T),
\end{aligned}$$

from which we obtain

$$T^{n+1}f(T) - nE_n(T) \leq mE_n(T) - \lambda E_{n+1}(T),$$

$$\lambda E_{n+1}(T) \leq (m+n)E_n(T) - T^{n+1}f(T),$$
$$\frac{E_{n+1}(T)}{E_0(T)} \leq \frac{m+n}{\lambda}\frac{E_n(T)}{E_0(T)} - \frac{T^{n+1}f(T)}{\lambda E_0(T)},$$

which leads to the desired results. □

Lemma 6 gives a tight moment bound, for which the upper and the lower limits are identical if $\mathcal{Y}$ is Erlang distributed. To get rid of the density function in the boundary a loose version of Lemma 6 is

$$\frac{m+n+\lambda T}{\lambda}\mathbb{E}\left[\mathcal{W}^n\right] - \frac{(m+n-1)T}{\lambda}\mathbb{E}\left[\mathcal{W}^{n-1}\right]$$
$$\leq \mathbb{E}\left[\mathcal{W}^{n+1}\right] \leq \frac{m+n}{\lambda}\mathbb{E}\left[\mathcal{W}^n\right] - \frac{T^{n+1}f(T)}{\lambda E_0(T)} < \frac{m+n}{\lambda}\mathbb{E}\left[\mathcal{W}^n\right], \tag{21}$$

where the strict inequality indicates the loose boundary.

Now, we look at lower order moments by focusing on moment bounds for the mean and SCV for $FTPH_1$ distributions with $b = 0$.

Corollary 3. *The squared coefficient of variation of $\mathcal{W}$, $SCV_{\mathcal{W}} = \frac{\mathbb{E}[\mathcal{W}^2]}{\mathbb{E}[\mathcal{W}]^2} - 1$, is bounded by*

$$\frac{m+1+\lambda T}{\lambda\mathbb{E}\left[\mathcal{W}\right]} - \frac{mT}{\lambda(\mathbb{E}\left[\mathcal{W}\right])^2} - 1 \leq SCV_{\mathcal{W}} < \frac{m+1}{\lambda\mathbb{E}\left[\mathcal{W}\right]} - 1. \tag{22}$$

Proof. For $n = 1$, Lemma 6 gives

$$\frac{m+1+\lambda T}{\lambda}\mathbb{E}\left[\mathcal{W}\right] - \frac{mT}{\lambda} \leq \mathbb{E}\left[\mathcal{W}^2\right] \leq \frac{m+1}{\lambda}\mathbb{E}\left[\mathcal{W}\right] - \frac{T^2 f(T)}{\lambda E_0(T)},$$

whose right hand side can be upper bounded by $\frac{m+1}{\lambda}\mathbb{E}\left[\mathcal{W}\right]$, from which the corollary comes by dividing with $(\mathbb{E}\left[\mathcal{W}\right])^2$ and subtracting 1. □

We note that the difference between the upper and the lower limit of $SCV_{\mathcal{W}}$ in Corollary 3 is

$$\frac{T}{(\mathbb{E}\left[\mathcal{W}\right])^2}\left(\frac{m}{\lambda} - \mathbb{E}\left[\mathcal{W}\right]\right),$$

for which according to (12) and the definition of $\mathcal{W}$ we have

$$\mathbb{E}\left[\mathcal{W}\right] = \mathbb{E}\left[\mathcal{Y}|\mathcal{Y} < T\right] < \mathbb{E}\left[\mathcal{Y}\right] \leq \frac{m}{\lambda}.$$

That is, when λ tends to zero the upper bound converges to infinity. In this case, the λ independent upper limit $mT/(m+1)$ from Corollary 2 can be applied. Combining the results, we obtain $\mathbb{E}[\mathcal{W}] \leq \min\{m/\lambda,\ mT/(m+1)\}$.

Our final result of this subsection gives a lower bound of $SCV_{\mathcal{W}}$ in terms of m only, which generalizes the result of Aldous and Shepp [1] mentioned in the Introduction.

Theorem 2. *The squared coefficient of variation of the $FTPH_1$ random variable, $\mathcal{W}$ with support on $(0,B)$ is bounded by $SCV_{\mathcal{W}} \geq 1/(m(m+2))$.*

Proof. Recall that $T = B - b = B$. Define

$$g(t) = \frac{(m-1)f(t) - tf'(t)}{mF(T) - Tf(T)}, \quad \text{for } 0 \leq t \leq T.$$

By Lemma 1, $(m-1-\lambda t)f(t) - tf'(t) \geq 0$ for $t > 0$, which leads to $(m-1)f(t) - tf'(t) > 0$ for $t > 0$. By integration from 0 to T, we obtain

$$mF(T) - Tf(T) = \int_0^T ((m-1)f(t) - tf'(t))\mathrm{d}t > 0. \tag{23}$$

Consequently, $g(t)$ is a density function of a random variable, to be called Y_T, with support $[0,T)$. Note that $\int_0^T t^n \mathrm{d}F(t) = \mathbb{E}\left[\mathcal{Y}^n \mathcal{I}_{\{\mathcal{Y}<T\}}\right]$, for $n = 0,1,2,...$. By routine calculations, we obtain

$$\begin{aligned}
\mathbb{E}[Y_T] &= \frac{(m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right] - T^2 f(T)}{mF(T) - Tf(T)}; \\
\mathbb{E}\left[Y_T^2\right] &= \frac{(m+2)\mathbb{E}\left[\mathcal{Y}^2\mathcal{I}_{\{\mathcal{Y}<T\}}\right] - T^3 f(T)}{mF(T) - Tf(T)}.
\end{aligned}$$

It is well-known that $\mathbb{E}\left[Y_T^2\right]/(\mathbb{E}[Y_T])^2 \geq 1$. Using the above expressions, we obtain

$$(m+2)\mathbb{E}\left[\mathcal{Y}^2\mathcal{I}_{\{\mathcal{Y}<T\}}\right] \geq T^3 f(T) + \frac{((m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right] - T^2 f(T))^2}{mF(T) - Tf(T)}.$$

Recall that $\mathcal{W} = \mathcal{Y}|\mathcal{Y} < T$. We also note that $\mathbb{E}\left[\mathcal{W}^2\right] = \mathbb{E}\left[\mathcal{Y}^2\mathcal{I}_{\{\mathcal{Y}<T\}}\right]/F(T)$ and

$\mathbb{E}[\mathcal{W}] = \mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]/F(T)$. The above equation leads to

$$\begin{aligned}
&\frac{\mathbb{E}\left[\mathcal{W}^2\right]}{(\mathbb{E}[\mathcal{W}])^2}\\
&= \frac{\mathbb{E}\left[\mathcal{Y}^2\mathcal{I}_{\{\mathcal{Y}<T\}}\right]F(T)}{\left(\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2}\\
&\geq \frac{F(T)\left(T^3 f(T) + \frac{\left((m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right] - T^2 f(T)\right)^2}{mF(T)-Tf(T)}\right)}{(m+2)\left(\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2}\\
&= \frac{F(T)\left(\left((m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2 - 2(m+1)T^2 f(T)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right] + mT^3 f(T)F(T)\right)}{(m+2)(mF(T)-Tf(T))\left(\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2}\\
&= \frac{(m+1)^2}{m(m+2)}\left(\frac{1 - 2\frac{T^2 f(T)}{(m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]} + \frac{mT^3 f(T)F(T)}{\left((m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2}}{1 - \frac{Tf(T)}{mF(T)}}\right)\\
&= \frac{(m+1)^2}{m(m+2)}\Theta(T).
\end{aligned}$$

We want to show that $\Theta(T) \geq 1$ for all $T > 0$. Since $mF(T) > Tf(T)$ according to (23), $\Theta(T) \geq 1$ is equivalent to

$$\frac{Tf(T)}{mF(T)} + \frac{mT^3 f(T)F(T)}{\left((m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2} \geq 2\frac{T^2 f(T)}{(m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]},$$

which is equivalent to

$$\frac{1}{mF(T)} + \frac{T^2 mF(T)}{\left((m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]\right)^2} \geq 2\frac{T}{(m+1)\mathbb{E}\left[\mathcal{Y}\mathcal{I}_{\{\mathcal{Y}<T\}}\right]}.$$

The last equation holds by applying the well-known inequality $a^2 + b^2 \geq 2ab$ for any real number a and b. Thus, we have shown that $\Theta(T) \geq 1$ for all $T > 0$. Consequently, we have shown

$$\frac{\mathbb{E}\left[\mathcal{W}^2\right]}{(\mathbb{E}[\mathcal{W}])^2} \geq \frac{(m+1)^2}{m(m+2)},$$

which is equivalent to $SCV_{\mathcal{W}} \geq 1/(m(m+2))$. □

By Corollary 2, the lower bound of $SCV_{\mathcal{W}}$ is strict for all PH distributions with finite support. The following lemma and corollary show how the lower bound of $SCV_{\mathcal{W}}$ can be attained approximately by bounded Erlang distributions. Denote by, for $t \geq 0$,

$$\begin{aligned}
F_m(t) &= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} - \dots - \frac{(\lambda t)^{m-1}}{(m-1)!}e^{-\lambda t};\\
f_m(t) &= f_{(m,\lambda)}(t) = \frac{\lambda^m t^{m-1}}{(m-1)!}e^{-\lambda t},
\end{aligned} \tag{24}$$

the distribution function and density function of Erlang random variable $\mathcal{X}_{(m,\lambda)}$, respectively. By routine calculations, we obtain

$$\begin{aligned}
\mathbb{E}\left[\mathcal{X}_{(m,\lambda)}|\mathcal{X}_{(m,\lambda)} < T\right] &= \frac{\int_0^T t f_m(t)\mathrm{d}t}{F_m(T)} = \frac{m}{\lambda}\frac{F_{m+1}(T)}{F_m(T)};\\
\mathbb{E}\left[\mathcal{X}^2_{(m,\lambda)}|\mathcal{X}_{(m,\lambda)} < T\right] &= \frac{\int_0^T t^2 f_m(t)\mathrm{d}t}{F_m(T)} = \frac{m(m+1)}{\lambda^2}\frac{F_{m+2}(T)}{F_m(T)};\\
SCV_{\{\mathcal{X}_{(m,\lambda)}|\mathcal{X}_{(m,\lambda)}<T\}} &= \frac{\mathbb{E}\left[\mathcal{X}^2_{(m,\lambda)}|\mathcal{X}_{(m,\lambda)} < T\right]}{\left(\mathbb{E}\left[\mathcal{X}_{(m,\lambda)}|\mathcal{X}_{(m,\lambda)} < T\right]\right)^2} - 1\\
&= \left(\frac{m+1}{m}\right)\frac{F_m(T)F_{m+2}(T)}{\left(F_{m+1}(T)\right)^2} - 1.
\end{aligned} \tag{25}$$

Lemma 7. *For all $T > 0$, the following bounds apply for the distribution functions of Erlang random variables*

$$\frac{m+1}{m+2} \le \frac{F_m(T)F_{m+2}(T)}{(F_{m+1}(T))^2} \le 1. \tag{26}$$

In addition, we have $\lim_{T\to 0} F_m(T)F_{m+2}(T)/(F_{m+1}(T))^2 = (m+1)/(m+2)$ and $\lim_{T\to\infty} F_m(T)F_{m+2}(T)/(F_{m+1}(T))^2 = 1$.

Proof. For convenience, we denote λT as t in this proof. First, we prove the upper bound. Since $e^{\lambda t} = \sum_{k=0}^{\infty}(\lambda t)^k/k!$, we need to show

$$\left(\sum_{k=m+2}^{\infty}\frac{t^k}{k!}\right)\left(\frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!} + \sum_{k=m+2}^{\infty}\frac{t^k}{k!}\right) \le \left(\frac{t^{m+1}}{(m+1)!} + \sum_{k=m+2}^{\infty}\frac{t^k}{k!}\right)^2,$$

which can be reduced to

$$\left(\sum_{k=m+2}^{\infty}\frac{t^k}{k!}\right)\frac{t^m}{m!} \le \left(\sum_{k=m+2}^{\infty}\frac{t^k}{k!}\right)\frac{t^{m+1}}{(m+1)!} + \left(\frac{t^{m+1}}{(m+1)!}\right)^2.$$

Next, we compare the coefficients of t^k on both sides. For $k = 2m+2$, the left-hand-side is $1/((m!(m+2)!)$ and the right-hand-side is $1/((m+1)!)^2$. From $\frac{1}{m!(m+2)!} = \frac{m+1}{m+2}\left(\frac{1}{(m+1)!}\right)^2$, it follows that the left-hand-side is smaller than the right-hand-side. For $k \ge 2m+3$, we have, for $k = j+m$,

$$\frac{1}{j!m!} \le \frac{1}{(j-1)!(m+1)!}, \tag{27}$$

which is equivalent to $m+1 \le j$, and is true since $j = k-m \ge m+3$. Consequently, we have shown the upper bound.

The proof of the lower bound is similar but tedious. The lower bound expression can be rewritten as $(m+1)F_{m+1}^2(t/\lambda) \le (m+2)F_m(t/\lambda)F_{m+2}(t/\lambda)$, which can be rewritten explicitly as

$$(m+1)\left(\sum_{k=m}^{\infty}\frac{t^k}{k!}-\frac{t^m}{m!}\right)^2 \le (m+2)\left(\sum_{k=m}^{\infty}\frac{t^k}{k!}\right)\left(\sum_{k=m}^{\infty}\frac{t^k}{k!}-\frac{t^m}{m!}-\frac{t^{m+1}}{(m+1)!}\right),$$

which leads to

$$(m+2)\left(\sum_{k=m}^{\infty}\frac{t^k}{k!}\right)\frac{t^{m+1}}{(m+1)!}+(m+1)\left(\frac{t^m}{m!}\right)^2 \le \left(\sum_{k=m}^{\infty}\frac{t^k}{k!}\right)^2+m\left(\sum_{k=m}^{\infty}\frac{t^k}{k!}\right)\frac{t^m}{m!}.$$

To prove the above inequality, we compare the coefficients of t^n on both sides. For $n=2m$, we have

$$\frac{m+1}{m!m!} \le \frac{1}{m!m!}+\frac{m}{m!m!},$$

which is true. For $n \ge 2m+1$, we need to prove

$$\frac{m+2}{(n-m-1)!(m+1)!} \le \frac{m}{(n-m)!m!}+\sum_{i=m}^{n-m}\frac{1}{i!(n-i)!}.$$

Separating the first and last term of the summation and applying $k! = k(k-1)!$, we obtain

$$\frac{(m+2)(n-m)}{(n-m)!(m+1)!} \le \frac{(m+2)(m+1)}{(n-m)!(m+1)!}+\sum_{i=m+1}^{n-m-1}\frac{1}{i!(n-i)!},$$

which leads to

$$\frac{(m+2)(n-2m-1)}{(n-m)!(m+1)!} \le \sum_{i=m+1}^{n-m-1}\frac{1}{i!(n-i)!}. \tag{28}$$

For any $i \in \{m+1,\dots,n-m-1\}$, we have $i \le n-m$ and $m+1 \le n-i$, from which we can write

$$\left(\frac{i}{n-m}\frac{i-1}{n-m-1}\cdots\frac{m+2}{n-i+2}\right)\frac{m+2}{n-i+1} \le 1$$

$$\Big(i(i-1)\dots(m+2)\Big)(m+2) \le (n-m)(n-m-1)\dots(n-i+1)$$

$$\frac{i!}{(m+1)!}(m+2) \le \frac{(n-m)!}{(n-i)!}$$

$$\frac{m+2}{(n-m)!(m+1)!} \le \frac{1}{i!(n-i)!}$$

Considering that $n-m-1-(m+1)+1 = n-2m-1$ terms are summed up on the right hand side of (28) each of which is greater than or equal to $\frac{m+2}{(n-m)!(m+1)!}$, inequality

(28) as well as the lower bound of Lemma 7 are proved. This completes the proof of the lemma. □

Immediate consequences of Lemma 7 are a lower bound and an upper bound of the SCV for bounded $\mathcal{X}_{(m,\lambda)}$.

Corollary 4. *Assume that $\mathcal{X}$ has an Erlang distribution with parameters (m, λ). For all $T > 0$, we have*

$$\frac{1}{m(m+2)} \leq SCV_{\{\mathcal{X}_{(m,\lambda)}|\mathcal{X}_{(m,\lambda)}<T\}} \leq \frac{1}{m}. \tag{29}$$

3.2. Case of $b > 0$

Let $\mathcal{Z} = b + \mathcal{W} = b + (\mathcal{Y}|\mathcal{Y} < T)$, then for $\mathbb{E}[\mathcal{Z}^n]$ we have

$$\mathbb{E}[\mathcal{Z}^n] = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} \frac{E_i(T)}{E_0(T)} = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} \mathbb{E}\left[\mathcal{W}^i\right],$$

where $E_i(T)$ is defined as before. That is, for $n = 1, 2$ we have

$$\mathbb{E}[\mathcal{Z}] = b + \mathbb{E}[\mathcal{W}] \quad \text{and} \quad \mathbb{E}\left[\mathcal{Z}^2\right] = b^2 + 2b\mathbb{E}[\mathcal{W}] + \mathbb{E}\left[\mathcal{W}^2\right].$$

Corollary 5. *The nth moment of a $FTPH_1$ random variable, $\mathcal{Z}$, with support on (b, B) is bounded by*

$$b^n \leq \mathbb{E}[\mathcal{Z}^n] \leq \sum_{i=0}^{n} \binom{n}{i} b^{n-i} \frac{mT^i}{m+i}\ . \tag{30}$$

Proof. Equation (30) directly follows from Corollary 2. □

For lower order moments, according to (30) the mean of $\mathcal{Z}$ is bounded by

$$b \leq \mathbb{E}[\mathcal{Z}] \leq b + \frac{mT}{m+1} = \frac{b+mB}{m+1} < B,$$

where both moment bounds are tight. The lower boundary is reached when $\mathbb{E}[\mathcal{Y}]$ tends to 0 and the upper boundary is reached when $\mathcal{Y}$ is Erlang(λ, m) distributed and λ tends to 0.

For the SCV, we have

Corollary 6. *$SCV_{\mathcal{Z}}$ is bounded by the following λ independent and dependent moment bounds*

$$SCV_{\mathcal{Z}} = \frac{\mathbb{E}\left[\mathcal{W}^2\right] - \mathbb{E}[\mathcal{W}]^2}{(b + \mathbb{E}[\mathcal{W}])^2} \leq \frac{\frac{m+1}{m+2}\mathbb{E}[\mathcal{W}]\,T - \mathbb{E}[\mathcal{W}]^2}{(b + \mathbb{E}[\mathcal{W}])^2} \tag{31}$$

$$\frac{-mT + (m+1+\lambda T)\mathbb{E}[\mathcal{W}] - \lambda\mathbb{E}[\mathcal{W}]^2}{\lambda(b+\mathbb{E}[\mathcal{W}])^2} \le SCV_{\mathcal{Z}} < \frac{(m+1)\mathbb{E}[\mathcal{W}] - \lambda\mathbb{E}[\mathcal{W}]^2}{\lambda(b+\mathbb{E}[\mathcal{W}])^2} \tag{32}$$

Proof. From Lemma 4 and Lemma 6 we have $\mathbb{E}\left[\mathcal{W}^2\right] \le \frac{m+1}{m+2}T\mathbb{E}[\mathcal{W}]$ and

$$\frac{m+1+\lambda T}{\lambda}\mathbb{E}[\mathcal{W}] - \frac{mT}{\lambda} \le \mathbb{E}\left[\mathcal{W}^2\right] < \frac{m+1}{\lambda}\mathbb{E}[\mathcal{W}], \tag{33}$$

respectively. Subtracting $\mathbb{E}[\mathcal{W}]^2$ and then dividing by $(b+\mathbb{E}[\mathcal{W}])^2$ in Equation (33) gives the corollary. □

Different from the case with $b = 0$, the $SCV_{\mathcal{Z}}$ can reach zero for the case with $b > 0$.

4. Discussion and Conclusion

This paper presents new moment bounds on phase-type distributions with infinite and finite support by using the steepest increase property. For PH distributions with infinite support and PH representation $(\boldsymbol{\alpha}, \boldsymbol{A})$ of size m, denoted as $\mathcal{Y}$,

- we have shown that any PH distribution is stochastically smaller than or equal to an Erlang distribution $\mathcal{X}_{(m,\lambda)}$ with λ be the absolute value of the dominant eigenvalue of $\boldsymbol{A}$; and
- we have obtained upper bounds of moments in terms of m and λ (e.g., $\mathbb{E}[\mathcal{Y}] \le m/\lambda$).

For PH distributions with finite support (for the set $FTPH_1$), denoted as $\mathcal{W} = \mathcal{Y}|\mathcal{Y} < T$,

- we have obtained upper bounds of moments in terms of m and T;
- we have obtained lower and upper bounds of moments depending on λ;
- we have shown that $\mathbb{E}[\mathcal{W}] \le \min\{mT/(m+1),\ m/\lambda\}$;
- we have shown that $SCV_{\mathcal{W}} \ge 1/(m(m+2))$.

For the finite support case, we focused on the distribution set $FTPH_1$. Results for the set $FTPH_2$ can be obtained similarly. The set $FTPH_3$ is a convex mixture of $FTPH_1$ and $FTPH_2$. Moment bounds can also be obtained as convex mixture of the moment bounds obtained for $FTPH_1$ and $FTPH_2$, but it is out of the scope of the current work.

Acknowledgements

This work was supported by the Hungarian research project OTKA K123914, and by a NSERC discovery grant (Canada), and by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Authors would like to thank two anonymous referees and an associate editor for their exceptionally careful reviews of the paper.

References

[1] Aldous, D. and Shepp, L. (1987). The least variable phase type distribution is erlang. *Stochastic Models* **3,** 467–473.

[2] He, Q. M., Zhang, H. and Vera, J. (2013). Majorization and extremal ph-distributions. *In Matrix-Analytic Methods in Stochastic Models, Editors: Latouche, G., V. Ramaswami, J. Sethuraman, K. Sigman, M.S. Squillante, and D.D. Yao* 107–121.

[3] Neuts, M. F. (1975). Probability distributions of phase type. *In Liber Amicorum Prof. Emeritus H. Florin* 173–206.

[4] Neuts, M. F. (1981). *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach.* The Johns Hopkins University Press, Boltimore and London.

[5] O'Cinneide, C. A. (1989). On non-uniqueness of representations of phase-type distributions. *Stochastic Models* **5,** 247–259.

[6] O'Cinneide, C. A. (1990). Characterization of phase-type distributions. *Stochastic Models* **6,** 1–57.

[7] O'Cinneide, C. A. (1991). Phase-type distributions and majorization. *Annals of Applied Probability* **1,** 219–227.

[8] O'Cinneide, C. A. (1999). Phase-type distributions: open problems and a few properties. *Stochastic Models* **15,** 731–757.

[9] Ramaswami, V. and Viswanath, N. C. (2014). Phase type distributions with finite support. *Stochastic Models* **30,** 576–597.

[10] Stoyan, D. and Daley, D. J. (1983). *Comparison Methods for Queues and Other Stochastic Models.* John Wiley & Sons, New York.

[11] Telek, M. (2000). The minimal coefficient of variation of discrete phase type distributions. *Proceedings of the First International Conference on Matrix Analytic Methods in Stochastic Models, Editors: G. Latouche and P.G. Taylor*.

[12] Yao, R. (2002). A proof of the steepest increase conjecture of a phase-type density. *Stochastic Models* **18,** 1–6.