ANALYSIS OF MARKOV REWARD MODELS WITH PARTIAL REWARD LOSS BASED ON A TIME REVERSE APPROACH

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Abstract. There are effective numerical methods for the analysis of Markov reward models (MRMs) without or with complete reward loss, but the analysis of MRMs with partial reward loss is more complex. This paper presents the analytical description of the distribution and the moments of the accumulated reward of partial increment loss reward models and an effective numerical method to evaluate these measures. The solution method is based on the time reverse process. The time reverse analysis allows to avoid computationally expensive techniques like introducing a supplementary variable or evaluating numerical integrals. Even though, this approach introduces difficulties at the first sight since the reverse process is inhomogeneous, it turns out that with the introduction of a properly chosen performance measure, homogeneous differential equations characterize the desired reward measures. This set of homogeneous differential equations allows one to apply an iterative scheme similar to the randomization (or Jensen's) method with all advantages of that method, i.e., numerical stability, pre-calculated error bound.

Keywords: Time reverse process, Inhomogeneous Markov chain, Reward models, Partial increment loss.

1. Introduction. The extension of discrete state stochastic models with a continuous reward variable results in a very effective modeling tool, the stochastic reward model [9]. Stochastic reward models have been applied for modeling and performance analysis of various engineering systems for a long time. Due to their relative tractability, reward models associated with continuous time Markov chains (CTMC), referred to as Markov reward models (MRM), have gain major attention. The analytical description of Semi-Markov [4] and Markov regenerative reward models [15] are also available, but their numerical analysis is far more complex.

There are two kinds of reward considered in MRMs. During a sojourn of the underlying process in a system state *rate reward* is accumulated continuously. At the state transition of the underlying process *impulse reward* can increase the reward function instantly. The set of reward models with increasing reward function is referred to loss-less or preemptive resume (prs) reward models. Another set of reward models considers a complete or a partial loss of reward at state transitions.

A number of effective numerical methods were developed for the evaluation of MRMs without reward loss [8, 5, 6, 11, 7] or with complete reward loss [3], but the case of partial reward loss was considered only recently. The analytical description of MRM with partial reward loss is provided in [2, 12] and the first numerical procedures for the analysis of larger models in [16]. The most effective numerical method in [16] is restricted to the case when the underlying process is in steady state. We relax this inconvenient restriction here.

This paper focuses on Markov reward models with rate reward accumulation and partial incremental reward loss at state transitions. The rest of the paper is organized as follows. Section 2 defines the set of reward models considered. We summarize the basic properties of the time reverse of the underlying CTMC in section 3 and the analytical description of the accumulated reward based on the time reverse accumulation process in Section 4. Section 5 presents an effective, randomization based numerical method for the analysis of MRM with partial reward loss. Finally, a numerical example demonstrates applicability of the proposed method.

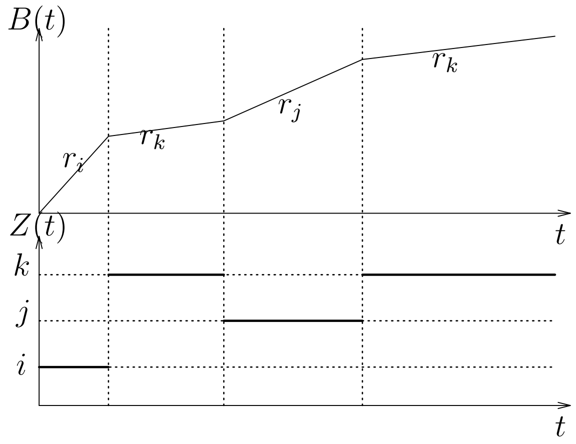

FIGURE 2.1. *Reward accumulation in a preemptive resume model*

2. Model definition.

2.1. Structure state process. Let the (right continuous) structure state process, $\{Z(t), t \geq 0\}$, be an irreducible homogeneous continuous time Markov chain (CTMC) on state space $S = \{1, 2, ..., N\}$ with generator $\mathbf{Q} = \{q_{ij}\}$ and initial probability vector $\underline{\gamma}(0)$. The transient probability vector $\underline{\gamma}(t)$ satisfies

$$\frac{d}{dt}\underline{\gamma}(t) = \underline{\gamma}(t)\mathbf{Q} \tag{2.1}$$

with initial condition $\underline{\gamma}(0)$, whose solution is

$$\underline{\gamma}(t) = \underline{\gamma}(0)e^{\mathbf{Q}t} \; . \tag{2.2}$$

2.2. Reward accumulation without reward loss. In stochastic reward models with rate reward accumulation each state of the structure state process is assigned with a non-negative constant which describes the rate of the reward accumulation during the sojourn of the structure state process in that state. This constant is commonly referred to as *reward rate*. During the sojourn in state $i \in S$ the amount of accumulated reward $B(t)$ increases according to the reward rate of state i, r_i:

$$\frac{dB(t)}{dt} = r_i$$

$\mathbf{R}$ denotes the diagonal matrix composed of the reward rate of the states in S ($\mathbf{R} = diag\!<\! r_i \!>$). In the case when there is no reward loss at the state transitions of the structure state process the amount of reward accumulated during the interval $(0, T)$ is

$$B(T) = \int_0^T r_{Z(t)} dt \; . \tag{2.3}$$

This case is commonly referred to as preemptive resume (prs) case.

Since $Z(t)$ is a CTMC with finite generator ($|q_{ij}| < \infty, \forall i, j \in S$) the $Z(t)$ process has a finite number of state transitions in the interval $(0, T)$ with probability one. Let

θ_i ($i \geq 0$, $\theta_0 = 0$) denote the time of the ith state transition and N the number of state transitions in the interval $(0, T)$. The state of the structure state process after the ith state transition is X_i where $X_i = Z(\theta_i)$. The sequence (θ_i, X_i) for $0 \leq i \leq N$ defines a trajectory of the process on the interval $(0, T)$. Having this trajectory the accumulated reward at time T is

$$B(T) = \sum_{i=1}^{N} r_{X_{i-1}}(\theta_i - \theta_{i-1}) + r_{X_N}(T - \theta_N). \tag{2.4}$$

2.3. Reward accumulation with partial incremental reward loss. During the sojourn of the structure state process in a given state, partial reward loss models accumulate reward in the same way as prs models, except that partial reward loss models might lose a portion of the accumulated reward at the state transitions of the structure state process. When the structure state process undergoes a transition from state i to another state, the fraction $1 - \alpha_i$ of the reward obtained during the last sojourn in state i is lost and only the fraction α_i of the reward (obtained during the last sojourn in i) remains. α_i is a real number, such that $0 \leq \alpha_i \leq 1$. The dynamics of the right continuous reward accumulation process $\{B(t), t \geq 0\}$ ($B(t) = B(t^+)$) is the following (see Figure 2.2):

$$\frac{dB(t)}{dt} = r_{Z(t)} \qquad \text{for} \quad \theta_i < t < \theta_{i+1} \tag{2.5}$$

$$B(\theta_i) = B(\theta_{i-1}) + \alpha_{Z(\theta_i^-)}[B(\theta_i^-) - B(\theta_{i-1})] \tag{2.6}$$

Assuming $Z(t)$ follows the (θ_i, X_i) trajectory the reward accumulated in the interval $(0, T)$ is

$$B(T) = \sum_{i=1}^{N} \alpha_{X_{i-1}} r_{X_{i-1}}(\theta_i - \theta_{i-1}) + r_{X_N}(T - \theta_N), \tag{2.7}$$

since the portion $1 - \alpha_{X_i}$ of the reward accumulated during the first N sojourn is lost, but the reward accumulated in the last state X_n during the (θ_n, T) interval is present at time T.

3. Time reverse of the structure state process. In the subsequent analysis we use the time reverse of the structure state process, $\overleftarrow{Z}(\tau) = Z(T - \tau)$, $0 \leq \tau \leq T$. The time reverse of a CTMC is considered, e.g. in [10, 1]. Here we summarize the main results.

THEOREM 3.1. *If $Z(t)$ is a (homogeneous) CTMC with initial probability $\underline{\gamma}(0)$ and generator $\mathbf{Q}$, its time reverse process defined by $\overleftarrow{Z}(\tau) = Z(T - \tau)$ $(0 \leq \tau \leq T)$ is an inhomogeneous CTMC with initial probability $\overleftarrow{\underline{\gamma}}(0) = \underline{\gamma}(T)$ and generator $\overleftarrow{\mathbf{Q}}(\tau) = \{\overleftarrow{q_{ij}}(\tau)\}$ where*

$$\overleftarrow{q_{ij}}(\tau) = \begin{cases} \dfrac{\gamma_j(T-\tau)}{\gamma_i(T-\tau)}\, q_{ji} & \text{if } i \neq j, \\ -\displaystyle\sum_{k \in S, k \neq i} \dfrac{\gamma_k(T-\tau)}{\gamma_i(T-\tau)}\, q_{ki} & \text{if } i = j. \end{cases}$$

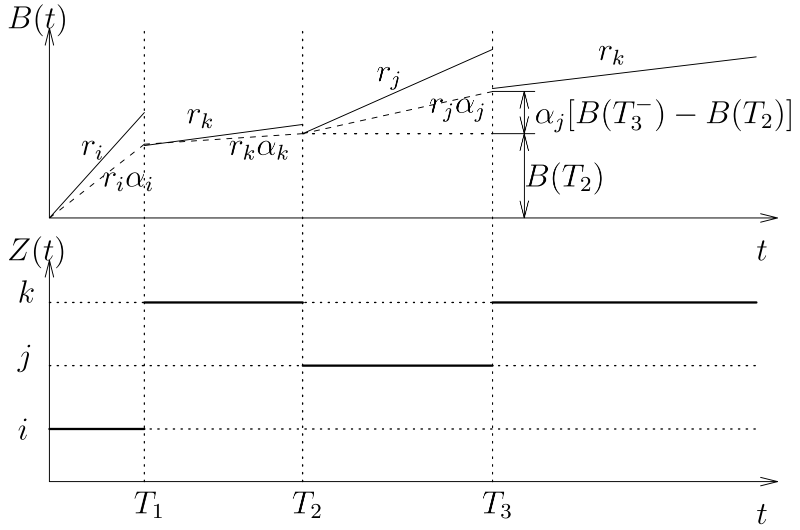

FIGURE 2.2. *Reward accumulation in a partial incremental loss model*

Proof. The generator of a CTMC is defined by the derivatives of the state transition probabilities:

$$\overleftarrow{q_{ij}}(\tau) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr(\overleftarrow{Z}(\tau+\Delta) = j \mid \overleftarrow{Z}(\tau) = i) \qquad \text{if } i \neq j$$

$$\overleftarrow{q_{ii}}(\tau) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Big(Pr(\overleftarrow{Z}(\tau+\Delta) = i \mid \overleftarrow{Z}(\tau) = i) - 1 \Big) = \sum_{k \in S, k \neq i} -\overleftarrow{q_{ik}}(\tau)$$

We focus on the first term:

$$Pr(\overleftarrow{Z}(\tau+\Delta) = j \mid \overleftarrow{Z}(\tau) = i) = Pr(Z(T-\tau-\Delta) = j \mid Z(T-\tau) = i) =$$
$$\frac{Pr(Z(T-\tau-\Delta) = j, Z(T-\tau) = i)}{Pr(Z(T-\tau) = i)} =$$
$$\frac{Pr(Z(T-\tau-\Delta) = j)}{Pr(Z(T-\tau) = i)} \; Pr(Z(T-\tau) = i \mid Z(T-\tau-\Delta) = j) \; .$$

Dividing the last expression by Δ and making the $\Delta \to 0$ limit we obtain the statement of the theorem for $i \neq j$. The $i = j$ case is obviously obtained by its definition. □

COROLLARY 3.2. *The transient probability of the reverse process* $\overleftarrow{\underline{\gamma}}(\tau)$ *is characterized by the ordinary differential equation*

$$\frac{d}{d\tau} \overleftarrow{\underline{\gamma}}(\tau) = \overleftarrow{\underline{\gamma}}(\tau) \overleftarrow{\mathbf{Q}}(\tau) \tag{3.1}$$

with initial condition $\overleftarrow{\underline{\gamma}}(0) = \underline{\gamma}(T)$ *and the solution to this differential equation verifies* $\overleftarrow{\underline{\gamma}}(\tau) = \underline{\gamma}(T-\tau)$.

Proof. The probability of staying in state j at time τ and $\tau+\Delta$ might change due to a departure from state j or an entrance to state j:

$$
\begin{aligned}
&Pr(\overleftarrow{Z}(\tau+\Delta)=j) - Pr(\overleftarrow{Z}(\tau)=j) = \\
&Pr(\overleftarrow{Z}(\tau)\neq j, \overleftarrow{Z}(\tau+\Delta)=j) - Pr(\overleftarrow{Z}(\tau)=j, \overleftarrow{Z}(\tau+\Delta)\neq j) = \\
&\sum_{k\in S,k\neq j} Pr(\overleftarrow{Z}(\tau)=k, \overleftarrow{Z}(\tau+\Delta)=j) - \sum_{k\in S,k\neq j} Pr(\overleftarrow{Z}(\tau)=j, \overleftarrow{Z}(\tau+\Delta)=k) = \\
&\sum_{k\in S,k\neq j} Pr(\overleftarrow{Z}(\tau)=k)\ Pr(\overleftarrow{Z}(\tau+\Delta)=j \mid \overleftarrow{Z}(\tau)=k) - \\
&\quad Pr(\overleftarrow{Z}(\tau)=j) \sum_{k\in S,k\neq j} Pr(\overleftarrow{Z}(\tau+\Delta)=k \mid \overleftarrow{Z}(\tau)=j)
\end{aligned}
$$

Dividing the first and the last expressions by Δ and making the $\Delta \to 0$ limit we obtain eq. (3.1).

The $\overleftarrow{\underline{\gamma}}(\tau) = \underline{\gamma}(T-\tau)$ relation holds for $\tau = 0$ by definition. Assuming $\overleftarrow{\underline{\gamma}}(\tau) = \underline{\gamma}(T-\tau)$ holds for $0 < \tau < T$ eq. (3.1) describes the reverse of the function defined by eq. (2.1):

$$
\begin{aligned}
\frac{d}{d\tau}\overleftarrow{\gamma}_i(\tau) &= \sum_{k\in S}\overleftarrow{\gamma}_k(\tau)\overleftarrow{q}_{ki}(\tau) = \sum_{k\in S,k\neq i}\overleftarrow{\gamma}_k(\tau)\overleftarrow{q}_{ki}(\tau) + \overleftarrow{\gamma}_i(\tau)\overleftarrow{q}_{ii}(\tau) \\
&= \sum_{k\in S,k\neq i}\overleftarrow{\gamma}_k(\tau)\ \frac{\overleftarrow{\gamma}_i(\tau)}{\overleftarrow{\gamma}_k(\tau)} q_{ik} - \overleftarrow{\gamma}_i(\tau) \sum_{k\in S,k\neq i}\frac{\overleftarrow{\gamma}_k(\tau)}{\overleftarrow{\gamma}_i(\tau)} q_{ki} \\
&= \overleftarrow{\gamma}_i(\tau) \sum_{k\in S,k\neq i} q_{ik} - \sum_{k\in S,k\neq i}\overleftarrow{\gamma}_k(\tau)\ q_{ki} \\
&= -\sum_{k\in S}\overleftarrow{\gamma}_k(\tau)\ q_{ki}
\end{aligned}
\tag{3.2}
$$

□

According to Equation (3.2) $\overleftarrow{\underline{\gamma}}(\tau)$ satisfies a homogeneous and an inhomogeneous differential equation at the same time:

$$\frac{d}{d\tau}\overleftarrow{\underline{\gamma}}(\tau) = \overleftarrow{\underline{\gamma}}(\tau)\overleftarrow{\mathbf{Q}}(\tau) = -\overleftarrow{\underline{\gamma}}(\tau)\mathbf{Q}\ ,$$

where $\overleftarrow{\mathbf{Q}}(\tau)$ is a proper generator matrix but $-\mathbf{Q}$ is not. The numerical analysis of $\overleftarrow{\underline{\gamma}}(\tau)$ is much easier based on $-\mathbf{Q}$ although it does not have any probabilistic interpretation (with respect to the reverse process). The solution to the homogeneous differential equation is

$$\overleftarrow{\underline{\gamma}}(\tau) = \overleftarrow{\underline{\gamma}}(0)\ e^{-\mathbf{Q}\tau}\ ,$$

Note that $\overleftarrow{\mathbf{Q}}(\tau) \neq -\mathbf{Q}$ and $e^{\int_0^\tau \overleftarrow{\mathbf{Q}}(u)du} \neq e^{-\mathbf{Q}\tau}$, only $\overleftarrow{\underline{\gamma}}(\tau)\overleftarrow{\mathbf{Q}}(\tau) = -\overleftarrow{\underline{\gamma}}(\tau)\mathbf{Q}$ and $\overleftarrow{\underline{\gamma}}(0)\ e^{-\mathbf{Q}\tau} = \overleftarrow{\underline{\gamma}}(0)\ e^{\int_0^\tau \overleftarrow{\mathbf{Q}}(u)du}$, where $\overleftarrow{\underline{\gamma}}(\tau) = \underline{\gamma}(T-\tau)$ and $\overleftarrow{\underline{\gamma}}(0) = \underline{\gamma}(T)$.

Based on Corollary 3.2 we often apply the following form of the reverse generator in the sequel:

$$
\overleftarrow{q_{ij}}(\tau) = \begin{cases} \dfrac{\overleftarrow{\gamma}_j(\tau)}{\overleftarrow{\gamma}_i(\tau)}\ q_{ji} & \text{if } i \neq j, \\ -\displaystyle\sum_{k\in S,k\neq i}\frac{\overleftarrow{\gamma}_k(\tau)}{\overleftarrow{\gamma}_i(\tau)}\ q_{ki} & \text{if } i = j. \end{cases}
$$

THEOREM 3.3. *The elementary probability that the $Z(t)$ process with generator $\mathbf{Q}$ and initial distribution $\underline{\gamma}(0)$ follows the trajectory (θ_i, X_i) $(0 \leq i \leq N)$ is identical to the elementary probability that the inhomogeneous $\overleftarrow{Z}(\tau)$ process with generator $\overleftarrow{\mathbf{Q}}(\tau)$ and initial distribution $\overleftarrow{\underline{\gamma}}(0)$ follows the trajectory (ω_i, Y_i) $(0 \leq i \leq N)$ where $\overleftarrow{\underline{\gamma}}(0) = \underline{\gamma}(T)$, $\omega_i = T - \theta_{N+1-i}$, $\omega_0 = 0$ and $Y_i = X_{N-i}$.*

Proof. The proof of the theorem is provided in Appendix A ☐

4. Analysis of accumulated reward using the time reverse process. Our goal is to apply the time reverse process for the reward analysis of the original structure state process. The preceding detailed analysis of the reverse process and the following theorems prepare the approach.

4.1. Reward accumulation without reward loss. Let $Y_i(t, w)$ denote the joint distribution of the accumulated reward and the system state, i.e. the probability that the accumulated reward is less than w and the CTMC stays in state i at time t,

$$Y_i(t, w) = Pr(B(t) \leq w, Z(t) = i).$$

Let $\underline{Y}(t, w)$ be the row vector composed of $Y_i(t, w)$, i.e., $\underline{Y}(t, w) = \{Y_i(t, w)\}$. $\underline{Y}(t, w)$ is the solution to the partial differential equation

$$\frac{\partial}{\partial t}\underline{Y}(t, w) + \frac{\partial}{\partial w}\underline{Y}(t, w)\mathbf{R} = \underline{Y}(t, w)\mathbf{Q} \ ,$$

with initial condition $\underline{Y}(0, w) = \underline{\gamma}(0)$ and $\underline{Y}(t, 0) = \underline{0}$ (with $r_i > 0$) [13, 14]. Further more, let $\overleftarrow{Y}_i(\tau, w)$ denote the distribution of reward accumulated by the reverse process, where

$$\overleftarrow{Y}_i(\tau, w) = Pr(\overleftarrow{B}(\tau) \leq w, \overleftarrow{Z}(\tau) = i)$$

and $\overleftarrow{\underline{Y}}(t, w)$ is the row vector composed of $\overleftarrow{Y}_i(\tau, w)$, i.e., $\overleftarrow{\underline{Y}}(\tau, w) = \{\overleftarrow{Y}_i(\tau, w)\}$. $\overleftarrow{\underline{Y}}(\tau, w)$ is the solution to the partial differential equation

$$\frac{\partial}{\partial \tau}\overleftarrow{\underline{Y}}(\tau, w) + \frac{\partial}{\partial w}\overleftarrow{\underline{Y}}(\tau, w)\mathbf{R} = \overleftarrow{\underline{Y}}(\tau, w)\overleftarrow{\mathbf{Q}}(\tau) \ , \tag{4.1}$$

with initial condition $\overleftarrow{\underline{Y}}(0, w) = \underline{\gamma}(T)$ and $\overleftarrow{\underline{Y}}(\tau, 0) = \underline{0}$ (with $r_i > 0$) [14].

THEOREM 4.1. *In a prs reward model, the distribution of the reward accumulated by the $Z(t)$ CTMC with generator $\mathbf{Q}$, reward rate matrix $\mathbf{R}$ and initial distribution $\underline{\gamma}(0)$ during the interval $(0, T)$ is identical to the distribution of reward accumulated by the $\overleftarrow{Z}(\tau)$ inhomogeneous CTMC with generator $\overleftarrow{\mathbf{Q}}(\tau)$ (defined in Theorem 3.1), reward rate matrix $\mathbf{R}$ and initial probability $\overleftarrow{\underline{\gamma}}(0) = \underline{\gamma}(T) = \underline{\gamma}(0)e^{\mathbf{Q}T}$ over the interval $(0, T)$. I.e., $\overleftarrow{\underline{Y}}(T, w) \stackrel{d}{=} \underline{Y}(T, w)$.*

Proof. The accumulated reward of the $Z(t)$ process is characterized by its trajectory via eq. (2.4), which is independent of the order of visited states. The $\overleftarrow{Z}(\tau)$ process follows the inverse trajectory of $Z(t)$ with the same elementary probability (Theorem 3.3), hence the $Z(t)$ and the $\overleftarrow{Z}(\tau)$ processes accumulate the same amount of reward with the same probability. ☐

To obtain a homogeneous partial differential equation we introduce $\overleftarrow{V}_i(\tau, w)$, the conditional distribution of reward accumulated by the reverse process

$$\overleftarrow{V}_i(\tau, w) = Pr(\overleftarrow{B}(\tau) \leq w \mid \overleftarrow{Z}(\tau) = i)$$

and the row vector $\underline{\overleftarrow{V}}(\tau, w) = \{\overleftarrow{V}_i(\tau, w)\}$.

THEOREM 4.2. *The distribution of reward accumulated over the interval* $(0, T)$ *is*

$$Pr(B(T) \le w) = \sum_{i \in S} \overleftarrow{V}_i(T, w)\gamma_i(0) \ ,$$

where $\underline{\overleftarrow{V}}(\tau, w)$ *is the solution to the partial differential equation*

$$\frac{\partial}{\partial \tau}\underline{\overleftarrow{V}}(\tau, w) + \frac{\partial}{\partial w}\underline{\overleftarrow{V}}(\tau, w)\mathbf{R} = \underline{\overleftarrow{V}}(\tau, w)\mathbf{Q}^T \ , \tag{4.2}$$

with initial condition $\underline{\overleftarrow{V}}(0, w) = \underline{e}$ *(where* $\underline{e}$ *is the row vector of ones) and* $\underline{\overleftarrow{V}}(\tau, 0) = \underline{0}$ *(with* $r_i > 0$*).* $\mathbf{Q}^T$ *is the transpose of* $\mathbf{Q}$.

Proof. Since $\overleftarrow{Y}_i(\tau, w) = \overleftarrow{V}_i(\tau, w)\overleftarrow{\gamma}_i(\tau)$, from (4.1) we have

$$\begin{aligned}
&\frac{\partial \overleftarrow{V}_i(\tau, w)}{\partial \tau}\overleftarrow{\gamma}_i(\tau) + \frac{\partial \overleftarrow{\gamma}_i(\tau)}{\partial \tau}\overleftarrow{V}_i(\tau, w) + \frac{\partial \overleftarrow{V}_i(\tau, w)}{\partial w}\overleftarrow{\gamma}_i(\tau) r_i = \\
&\sum_{k \in S} \overleftarrow{V}_k(\tau, w)\overleftarrow{\gamma}_k(\tau)\overleftarrow{q}_{ki}(\tau) = \\
&\sum_{k \in S, k \ne i} \overleftarrow{V}_k(\tau, w)\overleftarrow{\gamma}_k(\tau)\frac{\overleftarrow{\gamma}_i(\tau)}{\overleftarrow{\gamma}_k(\tau)}\ q_{ik} - \overleftarrow{V}_i(\tau, w)\overleftarrow{\gamma}_i(\tau)\sum_{k \in S, k \ne i}\frac{\overleftarrow{\gamma}_k(\tau)}{\overleftarrow{\gamma}_i(\tau)}\ q_{ki},
\end{aligned}$$

where $\overleftarrow{q}_{ki}(\tau)$ is replaced in the second step. Dividing both sides by $\overleftarrow{\gamma}_i(\tau)$ and substituting equation (3.2) result eq. (4.2) after some algebra. The distribution of the accumulated reward is obtained as

$$\begin{aligned}
&Pr(B(T) \le w) = \sum_{i \in S} Y_i(T, w) = \sum_{i \in S} \overleftarrow{Y}_i(T, w) = \\
&\sum_{i \in S} \overleftarrow{V}_i(T, w)\overleftarrow{\gamma}_i(T) = \sum_{i \in S} \overleftarrow{V}_i(T, w)\gamma_i(0) \ .
\end{aligned}$$

□

4.2. Reward accumulation with partial incremental reward loss. Based on eq. (2.7) we can interpret the reward accumulation of a reward model with partial incremental loss as it accumulates reward at rate $\alpha_i r_i$ in the first N visited states and it accumulates reward at rate r_i in the last visited state before time T and there is no reward loss. Using the time reverse of the structure state process we need to evaluate the accumulated reward when the reward rate in the first visited state (of the time reverse process) is r_i and the reward rate in later visited states are $\alpha_i r_i$. We summarize the result of this approach in the following theorem.

THEOREM 4.3. *The distribution of reward accumulated over the interval* $(0, T)$ *by the partial incremental loss Markov reward model with initial distribution* $\underline{\gamma}(0)$, *generator matrix* $\mathbf{Q}$, *reward rate matrix* $\mathbf{R}$ *and reduced reward rate matrix* $\mathbf{R}_\alpha = diag < \alpha_i r_i >$ *is identical to the distribution of reward accumulated over the same interval by the inhomogeneous prs Markov reward model of* $2|S|$ *states with initial distribution* $\pi^*(0)$, *generator* $\overleftarrow{\mathbf{Q}^*}(\tau)$ *and reward rate matrix* $\mathbf{R}^*$, *where*

$$\pi^*(0) = [\underline{\gamma}(T), \underline{0}], \quad \overleftarrow{\mathbf{Q}^*}(\tau) = \begin{array}{|c|c|} \hline \overleftarrow{\mathbf{Q}_D}(\tau) & \overleftarrow{\mathbf{Q}}(\tau) - \overleftarrow{\mathbf{Q}_D}(\tau) \\ \hline \mathbf{0} & \overleftarrow{\mathbf{Q}}(\tau) \\ \hline \end{array}\ , \quad \mathbf{R}^* = \begin{array}{|c|c|} \hline \mathbf{R} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}_\alpha \\ \hline \end{array}\ , \tag{4.3}$$

$\overleftarrow{\mathbf{Q}}(\tau)$ *is defined by Theorem 3.1 and* $\overleftarrow{\mathbf{Q}_D}(\tau) = diag\langle\overleftarrow{q_{ii}}(\tau)\rangle$ *is the diagonal matrix composed of the diagonal elements of* $\overleftarrow{\mathbf{Q}}(\tau)$.

Proof. The structure of the matrices in eq. (4.3) are such that the first $|S|$ states of the inhomogeneous prs Markov reward model represent the reward accumulation in the first visited state (of the time reverse process) and the last $|S|$ states represents the reward accumulation in the later visited states. The theorem is given by the equivalence of the trajectories of the forward and its time reverse process presented in Theorem 3.3 and by the fact that in both cases the accumulated reward is related with the trajectory according to eq. (2.7). □

Theorem 4.3 and eq. (4.1) already provides a computational method. By this method the distribution of the accumulated reward can be calculated using enlarged, possibly inhomogeneous, matrices and vectors with proper initial probability vector.

The numerical solution based on eq. (4.1) could be expensive since $\overleftarrow{\mathbf{Q}^*}(\tau)$ needs to be calculated in each step of the solution method. To overcome this difficulty we look for a partial differential equation description with constant coefficients similar to the one in eq. (4.2).

THEOREM 4.4. *The distribution of reward accumulated over the interval* $(0, T)$ *is*

$$Pr(B(T) \le w) = \sum_{i\in S} \left(\overleftarrow{X1}_i(T, w) + \overleftarrow{X2}_i(T, w))\right)\gamma_i(0) ,$$

where vectors $\overleftarrow{\underline{X1}}(\tau, w)$ *and* $\overleftarrow{\underline{X2}}(\tau, w)$ *are the solution to the partial differential equations*

$$\frac{\partial}{\partial\tau}\overleftarrow{\underline{X1}}(\tau, w) + \frac{\partial}{\partial w}\overleftarrow{\underline{X1}}(\tau, w)\mathbf{R} = \overleftarrow{\underline{X1}}(\tau, w)\mathbf{Q}_D , \tag{4.4}$$

and

$$\frac{\partial}{\partial\tau}\overleftarrow{\underline{X2}}(\tau, w) + \frac{\partial}{\partial w}\overleftarrow{\underline{X2}}(\tau, w)\mathbf{R}_\alpha = \overleftarrow{\underline{X1}}(\tau, w)(\mathbf{Q} - \mathbf{Q}_D)^T + \overleftarrow{\underline{X2}}(\tau, w)\mathbf{Q}^T , \tag{4.5}$$

with initial conditions: $\overleftarrow{\underline{X1}}(0, w) = \underline{e}$, $\overleftarrow{\underline{X2}}(0, w) = \underline{0}$, $\overleftarrow{\underline{X1}}(\tau, 0) = \underline{0}$ *and* $\overleftarrow{\underline{X2}}(\tau, 0) = \underline{0}$ *(when* $r_i > 0$*). In the expressions above* $\mathbf{Q}_D$ *is the diagonal matrix composed of the diagonal elements of* $\mathbf{Q}$ *and* $\mathbf{Q}_D = diag\langle q_{ii}\rangle$.

Proof. The proof of the theorem is provided in Appendix B. □

Since $\mathbf{Q}_D$ is a diagonal matrix the solution to (4.4) is readable

$$\overleftarrow{X1}_i(\tau, w) = \begin{cases} e^{q_{ii}\tau} & \text{if } w \ge r_i\tau , \\ 0 & \text{if } w < r_i\tau . \end{cases} \tag{4.6}$$

4.3. Moments of accumulated reward with partial incremental reward loss. The moments of the accumulated reward are given by the following theorem.

THEOREM 4.5. *The* n*th moment* $(n \ge 0)$ *of reward accumulated over the interval* $(0, T)$ *is*

$$E(B(T)^n) = \sum_{i\in S} \left(\overleftarrow{M1}_i^{(n)}(T) + \overleftarrow{M2}_i^{(n)}(T)\right)\gamma_i(0) ,$$

where $\overleftarrow{\underline{M1}}^{(n)}(\tau)$ *and* $\overleftarrow{\underline{M2}}^{(n)}(\tau)$ *are the solution to the ordinary differential equations*

$$\frac{d}{d\tau}\overleftarrow{\underline{M1}}^{(n)}(\tau) = n\overleftarrow{\underline{M1}}^{(n-1)}(\tau)\mathbf{R} + \overleftarrow{\underline{M1}}^{(n)}(\tau)\mathbf{Q}_D , \tag{4.7}$$

and

$$(4.8)\quad \frac{d}{d\tau}\overleftarrow{\underline{M2}}^{(n)}(\tau) = n\overleftarrow{\underline{M2}}^{(n-1)}(\tau)\mathbf{R}_\alpha + \overleftarrow{\underline{M1}}^{(n)}(\tau)(\mathbf{Q}-\mathbf{Q}_D)^T + \overleftarrow{\underline{M2}}^{(n)}(\tau)\mathbf{Q}^T \ .$$

with initial conditions $\overleftarrow{\underline{M1}}^{(0)}(0) = \underline{e}$, $\overleftarrow{\underline{M1}}^{(i)}(0) = \underline{0}$ *for* $i \ge 1$, *and* $\overleftarrow{\underline{M2}}^{(i)}(0) = \underline{0}$ *for* $i \ge 0$.

Proof. The proof of the theorem is provided in Appendix C. ☐

The solution to (4.7) is

$$(4.9)\qquad \overleftarrow{\underline{M1}}^{(n)}(\tau) = \tau^n \ \underline{e} \ \mathbf{R}^n \mathbf{E}_D(\tau) \ ,$$

where $\mathbf{E}_D(\tau)$ is the diagonal matrix $\mathbf{E}_D(\tau) = diag\langle e^{q_{ii}\tau}\rangle$.

5. A randomization based numerical method. The fact that (4.7) and (4.8) are differential equations with constant coefficients suggests that a stable, computationally effective, randomization based numerical algorithm can be constructed to calculate the moments of the accumulated reward. $\overleftarrow{\underline{M1}}^{(n)}(\tau)$ is explicitly given by (4.9). The solution to $\overleftarrow{\underline{M2}}^{(n)}(\tau)$ is provided by the following theorem.

THEOREM 5.1. $\overleftarrow{\underline{M2}}^{(n)}(\tau)$ *can be calculated as*

$$(5.1)\qquad \overleftarrow{\underline{M2}}^{(n)}(\tau) = n! d^n \sum_{k=0}^{\infty} e^{-\lambda\tau} \ \frac{(\lambda\tau)^k}{k!} \ \underline{D}^{(n)}(k)$$

where $\lambda = \max_{j\in S} \sum_{i\in S, i\neq j} q_{ij}$ *(maximal aggregated transition rate to a state),* $d = \max_{i\in S} r_i/\lambda$, $\mathbf{A}_D = \mathbf{Q}_D/\lambda + \mathbf{I}$, $\mathbf{A} = \mathbf{Q}^T/\lambda + \mathbf{I}$, $\mathbf{S} = \mathbf{R}/(\lambda d)$, $\mathbf{S}_\alpha = \mathbf{R}_\alpha/(\lambda d)$ *and*

$$(5.2)\qquad \underline{D}^{(n)}(k) = \begin{cases} \underline{e}\ (\mathbf{I} - \mathbf{A}_D^k) & n = 0 \\ 0 & k \le n, n \ge 1 \\ \underline{D}^{(n-1)}(k-1)\mathbf{S}_\alpha + \underline{D}^{(n)}(k-1)\mathbf{A} + & \\ \qquad \binom{k-1}{n} \ \underline{e}\ \mathbf{S}^n \mathbf{A}_D^{k-1-n}(\mathbf{A} - \mathbf{A}_D) & k > n, n \ge 1 \end{cases}$$

Proof. Substituting (4.9) into (4.8) results

$$(5.3)\quad \frac{d}{d\tau}\overleftarrow{\underline{M2}}^{(n)}(\tau) = n\overleftarrow{\underline{M2}}^{(n-1)}(\tau)\mathbf{R}_\alpha + \tau^n \underline{e}\mathbf{R}^n\mathbf{E}_D(\tau)(\mathbf{Q}-\mathbf{Q}_D)^T + \overleftarrow{\underline{M2}}^{(n)}(\tau)\mathbf{Q}^T$$

with initial conditions $\overleftarrow{\underline{M2}}^{(n)}(0) = \underline{0}$ and (from the definition of $\overleftarrow{\underline{M2}}^{(n)}(\tau)$) $\overleftarrow{\underline{M2}}^{(0)}(\tau) = \underline{e}(\mathbf{I} - \mathbf{E}_D(\tau))$. Substituting $\mathbf{A}$, $\mathbf{A}_D$, $\mathbf{S}$, $\mathbf{S}_\alpha$, $\mathbf{E}_D(\tau) = \sum_{k=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} \mathbf{A}_D^k$, $\mathbf{I} - \mathbf{E}_D(\tau) =$

$\sum_{k=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} (\mathbf{I} - \mathbf{A}_D^k)$ and (5.1) into (5.3) results

$$n!d^n\lambda \sum_{i=0}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^i}{i!}\left(\underline{D}^{(n)}(i+1) - \underline{D}^{(n)}(i)\right) =$$
$$n(n-1)!d^{n-1}\sum_{i=0}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^i}{i!}\underline{D}^{(n-1)}(i)\lambda d\, \mathbf{S}_\alpha +$$
$$\frac{1}{\lambda^n}\underline{e}\lambda^n d^n \mathbf{S}^n \sum_{i=n}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^i}{i!}\mathbf{A}_D^i \lambda(\mathbf{A} - \mathbf{A}_D) +$$
$$n!d^n \sum_{i=0}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^i}{i!}\underline{D}^{(n)}(i)\lambda(\mathbf{A}_D - \mathbf{I})$$

The recursive formulae of $\underline{D}^{(n)}(i)$ ensures the equality for each $e^{-\lambda\tau}(\lambda\tau)^i$ terms.

□

The error caused by truncating the infinite sum in (5.1) can be bounded as follows.

THEOREM 5.2. *$\overleftarrow{\underline{M2}}^{(n)}(\tau)$ can be calculated as a finite sum and an error vector, $\underline{\xi}(G)$, such that all elements of the error vector is less than ε:*

$$\overleftarrow{\underline{M2}}^{(n)}(\tau) = n!d^n \sum_{k=0}^{G-1} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} \underline{D}^{(n)}(k) + \underline{\xi}(G) \tag{5.4}$$

where $\underline{D}^{(n)}(k)$ is given by (5.2) and

$$G = \min_{g>n}\left((\lambda\tau)^{n+1}d^n \sum_{k=g-n-1}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!} < \varepsilon\right)$$

Proof. The error term, $\underline{\xi}(G)$ satisfies

$$\begin{aligned}
\underline{\xi}(G) &= n!d^n \sum_{k=G}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}\underline{D}^{(n)}(k) \\
&\le n!d^n \sum_{k=G}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}\frac{k!}{n!\,(k-n-1)!}\underline{e} && \text{due to Corollary D.2} \\
&= (\lambda\tau)^{n+1}d^n \sum_{k=G-n-1}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}\underline{e} \\
&< \varepsilon\, \underline{e} && \text{due to the definition of } G
\end{aligned}$$

□

6. Numerical behaviour: complexity and stability. The two main sources of numerical errors in computations with floating point numbers are the summation of numbers with different orders of magnitude and the presence of substractions. The methods based on multiplications and summations of positive numbers between 0 and 1 usually have nice numerical properties.

In our case, the introduction of matrices $\mathbf{A}, \mathbf{A}_D, \mathbf{S}$ and $\mathbf{S}_\alpha$ ensures that the general iteration step ((5.2), case $k > n$) involves only summations of vectors and vector-matrix multiplications where the elements of the vectors and the matrices are between 0 and 1 (e.g., matrix $\mathbf{A} - \mathbf{A}_D$ has this property as well).

The only step where a real substraction takes place is the initialization step at $n = 0$ (5.2). In this step we calculate the diagonal matrix $\mathbf{I} - \mathbf{A}_D^k$, whose elements are non-negative between 0 and 1. Hence this substraction does not affect the numerical stability of the proposed algorithm.

The core iteration step, the computation of $\underline{D}^{(n)}(k)$ based on (5.2), is implemented using vector-matrix multiplications. The complexity of multiplications of vectors with full matrices is $o\left(N^2\right)$ and vectors with diagonal matrices is $o\left(N\right)$. There are only 2 real vector-matrix multiplications in an iteration step (computing $\underline{e}\mathbf{S}^n\mathbf{A}_D^{k-1-n}(\mathbf{A} - \mathbf{A}_D)$ and $\underline{D}^{(n)}(k-1)\mathbf{A}$). The computational complexity of the algorithm is dominated by these vector-matrix multiplications. The core iteration step requires $o\left(N^2\right) \sim 2N^2$ floating point multiplications and summations. To compute the nth moment, we have to compute moments $0 \ldots n-1$ as well. The overall computational complexity of one iteration step is $o\left(n \cdot N^2\right) \sim 2(n+1)N^2$ and G iterations are required to reach the prescribed precision.

7. Numerical example. We implemented (in C) a numerical method that provides the moments of the accumulated reward in reward models with partial incremental loss. Matrices are stored in a sparse manner. With this implementation it is possible to handle very large models, with 10^6 states.

As an example, let consider the following system. The Markov chain describing the model is shown in Figure 7.1. The state numbering reflects the number of working servers. State M is the maintenance state. The reward rates r_i and α_i corresponding to the states are indicated on the dashed line. If there are k working servers, the reward rate is $r_k = k \cdot r$.

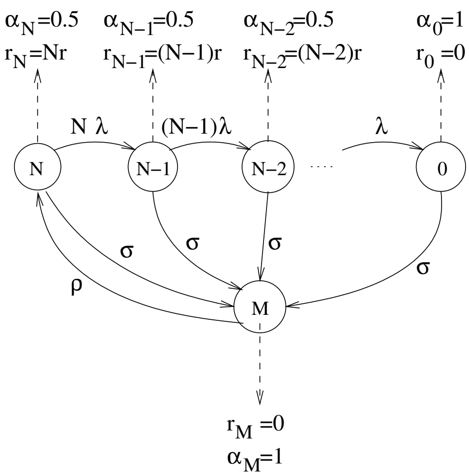

FIGURE 7.1. *Structure of the Markov chain*

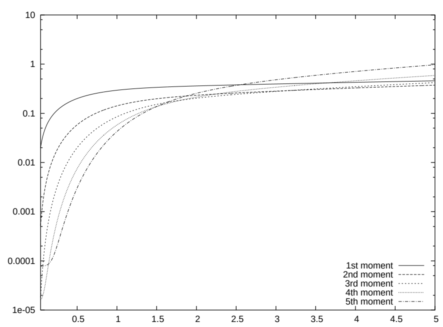

FIGURE 7.2. *Moments of the accumulated reward*

Figure 7.2 shows the first 5 moments of the accumulated reward as the function of time. Figure 7.3 and 7.4 compares the moments of the accumulated reward while changing parameter α to 0.25, 0.5, 0.75, accordingly.

Appendix A. Proof of Theorem 3.3.

To prove the theorem we need the following corollary.

Number of servers:	$N = 500000$
Server break down rate:	$\lambda = 0.000004$
System inter-maintenance rate:	$\sigma = 1.5$
Inverse of system maintenance time:	$\rho = 0.1$
Reward accumulation of one server:	$r = 0.000002$
Incremental reward loss at failures:	$\alpha = 0.5$

TABLE 7.1
Parameters used in the example

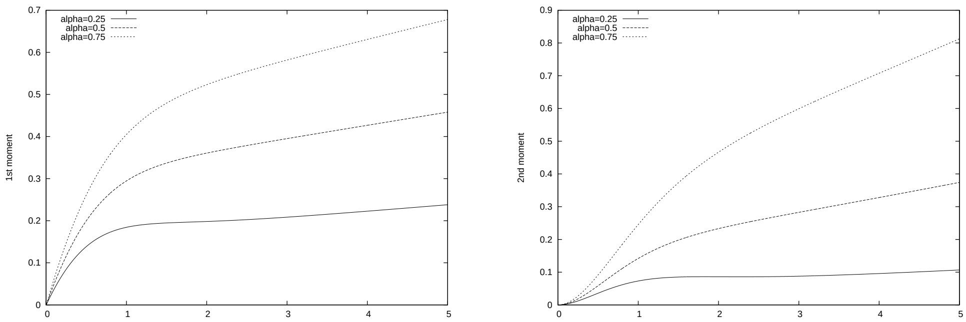

FIGURE 7.3. *The effect of α on the 1st and 2nd moments*

COROLLARY A.1. *The short term behaviour of $Pr(\overleftarrow{Z}(\tau) = j)$ satisfies*

$$Pr(\overleftarrow{Z}(\tau) = j)\Big(1 + \overleftarrow{q}_{jj}(\tau)\Delta + \sigma(\Delta)\Big) = Pr(\overleftarrow{Z}(\tau + \Delta) = j)\Big(1 + q_{jj}\Delta + \sigma(\Delta)\Big) .$$

Proof. We start from the same relation as in the proof of Corollary 3.2:

$$\begin{aligned}
&Pr(\overleftarrow{Z}(\tau + \Delta) = j) - Pr(\overleftarrow{Z}(\tau) = j) = \\
&Pr(\overleftarrow{Z}(\tau) \neq j, \overleftarrow{Z}(\tau + \Delta) = j) - Pr(\overleftarrow{Z}(\tau) = j, \overleftarrow{Z}(\tau + \Delta) \neq j) = \\
&\sum_{k \in S, k \neq j} Pr(\overleftarrow{Z}(\tau) = k, \overleftarrow{Z}(\tau + \Delta) = j) - \sum_{k \in S, k \neq j} Pr(\overleftarrow{Z}(\tau) = j, \overleftarrow{Z}(\tau + \Delta) = k) = \\
&Pr(\overleftarrow{Z}(\tau + \Delta) = j) \sum_{k \in S, k \neq j} Pr(\overleftarrow{Z}(\tau) = k \mid \overleftarrow{Z}(\tau + \Delta) = j) - \\
&\quad Pr(\overleftarrow{Z}(\tau) = j) \sum_{k \in S, k \neq j} Pr(\overleftarrow{Z}(\tau + \Delta) = k \mid \overleftarrow{Z}(\tau) = j) = \\
&Pr(\overleftarrow{Z}(\tau + \Delta) = j)\Big(-q_{jj}\Delta + \sigma(\Delta)\Big) - Pr(\overleftarrow{Z}(\tau) = j)\Big(-\overleftarrow{q}_{jj}(\tau)\Delta + \sigma(\Delta)\Big) .
\end{aligned}$$

Rearranging the first and the last expressions provides the corollary. □

Based on Corollary A.1 the proof of Theorem 3.3 proceeds as follow.

The elementary probability that the $Z(t)$ process follows the trajectory (θ_i, X_i)

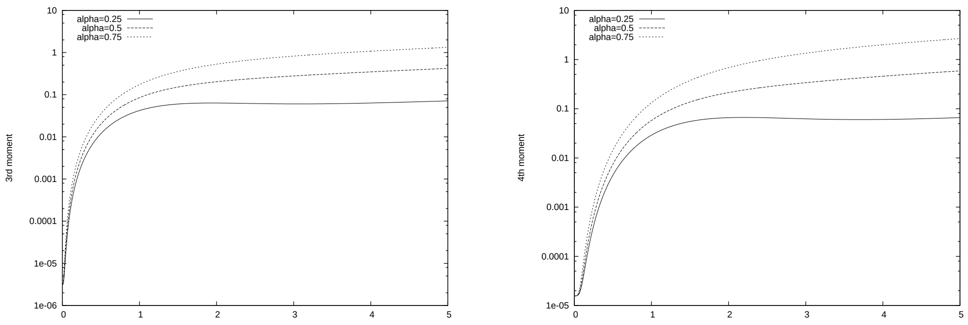

FIGURE 7.4. *The effect of α on the 3rd and 4th moments*

is

$$\underbrace{\gamma_{X_0}(0)}_{\text{init. prob.}} \prod_{i=0}^{N-1} \left(\underbrace{-q_{X_iX_i}\ e^{q_{X_iX_i}(\theta_{i+1}-\theta_i)}}_{\text{sojourn time}} \ \underbrace{\frac{q_{X_iX_{i+1}}}{-q_{X_iX_i}}}_{\text{state tr. prob.}} \right) \underbrace{e^{q_{X_NX_N}(T-\theta_N)}}_{\text{last sojourn}} =$$

$$\gamma_{X_0}(0) \prod_{i=0}^{N-1} \left(e^{q_{X_iX_i}(\theta_{i+1}-\theta_i)} \ q_{X_iX_{i+1}} \right) e^{q_{X_NX_N}(T-\theta_N)} ,$$

and the elementary probability that the $\overleftarrow{Z}(\tau)$ process follows the trajectory (ω_i, Y_i) is

$$\underbrace{\overleftarrow{\gamma}_{Y_0}(0)}_{\text{init. prob.}} \prod_{i=0}^{N-1} \left(\underbrace{-\overleftarrow{q}_{Y_iY_i}(\omega_{i+1})\ e^{\int_{\omega_i}^{\omega_{i+1}} \overleftarrow{q}_{Y_iY_i}(\tau)d\tau}}_{\text{sojourn time}} \ \underbrace{\frac{\overleftarrow{q}_{Y_iY_{i+1}}(\omega_{i+1})}{-\overleftarrow{q}_{Y_iY_i}(\omega_{i+1})}}_{\text{state tr. prob.}} \right) \underbrace{e^{\int_{\omega_N}^{T} \overleftarrow{q}_{Y_NY_N}(\tau)d\tau}}_{\text{last sojourn}} =$$

$$\overleftarrow{\gamma}_{Y_0}(0) \prod_{i=0}^{N-1} \left(e^{\int_{\omega_i}^{\omega_{i+1}} \overleftarrow{q}_{Y_iY_i}(\tau)d\tau} \ \overleftarrow{q}_{Y_iY_{i+1}}(\omega_{i+1}) \right) e^{\int_{\omega_N}^{T} \overleftarrow{q}_{Y_NY_N}(\tau)d\tau} .$$

Let us consider the $\overleftarrow{\gamma}_{Y_i}(\omega_i)\ e^{\int_{\omega_i}^{\omega_{i+1}} \overleftarrow{q}_{Y_iY_i}(\tau)d\tau}$ term first:

$$\overleftarrow{\gamma}_{Y_i}(\omega_i)\ e^{\int_{\omega_i}^{\omega_{i+1}} \overleftarrow{q}_{Y_iY_i}(\tau)d\tau} = \overleftarrow{\gamma}_{Y_i}(\omega_i) \lim_{M\to\infty} e^{\sum_{k=0}^{M-1} h\overleftarrow{q}_{Y_iY_i}(\omega_i+kh)} =$$

$$\lim_{M\to\infty} \overleftarrow{\gamma}_{Y_i}(\omega_i) \prod_{k=0}^{M-1} e^{h\overleftarrow{q}_{Y_iY_i}(\omega_i+kh)} = \lim_{M\to\infty} \overleftarrow{\gamma}_{Y_i}(\omega_i) \prod_{k=0}^{M-1} \left(1 + h\overleftarrow{q}_{Y_iY_i}(\omega_i+kh) + \sigma(h)\right) =$$

$$\lim_{M\to\infty} \overleftarrow{\gamma}_{Y_i}(\omega_i) \left(1 + h\overleftarrow{q}_{Y_iY_i}(\omega_i) + \sigma(h)\right) \prod_{k=1}^{M-1} \left(1 + h\overleftarrow{q}_{Y_iY_i}(\omega_i+kh) + \sigma(h)\right) =$$

$$\lim_{M\to\infty} \left(1 + hq_{Y_iY_i} + \sigma(h)\right) \overleftarrow{\gamma}_{Y_i}(\omega_i+h) \prod_{k=1}^{M-1} \left(1 + h\overleftarrow{q}_{Y_iY_i}(\omega_i+kh) + \sigma(h)\right) = \dots$$

$$\lim_{M\to\infty} \prod_{k=0}^{M-1} \left(1 + hq_{Y_iY_i} + \sigma(h)\right) \overleftarrow{\gamma}_{Y_i}(\omega_{i+1}) = \overleftarrow{\gamma}_{Y_i}(\omega_{i+1})\ e^{q_{Y_iY_i}(\omega_{i+1}-\omega_i)} ,$$

where $h = \frac{\omega_{i+1}-\omega_i}{M}$ and Corollary A.1 is applied in the fifth step. Using this relation the elementary probability of the reverse trajectory is

$$\overleftarrow{\gamma}_{Y_0}(\omega_0) \prod_{i=0}^{N-1} \left(e^{\int_{\omega_i}^{\omega_{i+1}} \overleftarrow{q}_{Y_iY_i}(\tau)d\tau} \overleftarrow{q}_{Y_iY_{i+1}}(\omega_{i+1}) \right) e^{\int_{\omega_N}^{T} \overleftarrow{q}_{Y_NY_N}(\tau)d\tau} =$$
$$\overleftarrow{\gamma}_{Y_0}(\omega_0)\, e^{\int_{\omega_0}^{\omega_1} \overleftarrow{q}_{Y_0Y_0}(\tau)d\tau} \overleftarrow{q}_{Y_0Y_1}(\omega_1) \;\ldots\; e^{\int_{\omega_N}^{T} \overleftarrow{q}_{Y_NY_N}(\tau)d\tau} =$$
$$e^{q_{Y_0Y_0}(\omega_1-\omega_0)} \overleftarrow{\gamma}_{Y_0}(\omega_1) \frac{\overleftarrow{\gamma}_{Y_1}(\omega_1)}{\overleftarrow{\gamma}_{Y_0}(\omega_1)} q_{Y_1Y_0} \;\ldots\; e^{\int_{\omega_N}^{T} \overleftarrow{q}_{Y_NY_N}(\tau)d\tau} =$$
$$e^{q_{Y_0Y_0}(\omega_1-\omega_0)} q_{Y_1Y_0}\, \overleftarrow{\gamma}_{Y_1}(\omega_1) \;\ldots\; e^{\int_{\omega_N}^{T} \overleftarrow{q}_{Y_NY_N}(\tau)d\tau} = \ldots =$$
$$\prod_{i=0}^{N-1} \left(e^{q_{Y_iY_i}(\omega_{i+1}-\omega_i)} q_{Y_{i+1}Y_i} \right) e^{q_{Y_NY_N}(T-\omega_N)} \overleftarrow{\gamma}_{Y_N}(T)$$

Substituting $Y_i = X_{N-i}$, $\overleftarrow{\gamma}_{Y_N}(T) = \gamma_{X_0}(0)$, $\omega_{i+1} - \omega_i = \theta_{N+1-i} - \theta_{N-i}$ and $T - \omega_N = \theta_1$ results the elementary probability of the forward trajectory, which concludes the proof of the theorem.

Appendix B. Proof of Theorem 4.4.

The proof follows the same pattern as the proof of Theorem 4.2, but here we exploit the block structure of $\pi^*(0)$, $\overleftarrow{\mathbf{Q}^*}(\tau)$ and $\mathbf{R}^*$. $\pi^*(\tau)$ is decomposed as $\pi^*(\tau) = [\overleftarrow{\gamma 1}(\tau), \overleftarrow{\gamma 2}(\tau)]$ where

$$\overleftarrow{\gamma 1}_i(\tau) = Pr(\overleftarrow{Z}(T) = i, \text{no state transition in } (0,T)) = \overleftarrow{\gamma}_i(\tau)e^{q_{ii}\tau}, \tag{B.1}$$

$$\overleftarrow{\gamma 2}_i(\tau) = Pr(\overleftarrow{Z}(T) = i, \text{state transition in } (0,T)) = \overleftarrow{\gamma}_i(\tau)(1 - e^{q_{ii}\tau}) \; . \tag{B.2}$$

Substituting $\pi^*(0)$, $\overleftarrow{\mathbf{Q}^*}(\tau)$ and $\mathbf{R}^*$ into eq. (4.1), utilizing the specific block structure provided in (4.3) and completing similar steps as in Theorem 4.2 results:

$$\frac{\partial}{\partial \tau} \overleftarrow{V1}_i(\tau, w) + \frac{\partial}{\partial w} \overleftarrow{V1}_i(\tau, w) r_i = 0 \; , \tag{B.3}$$

and

$$\frac{\partial}{\partial \tau} \overleftarrow{V2}_i(\tau, w) + \frac{\partial}{\partial w} \overleftarrow{V2}_i(\tau, w) r_i \alpha_i =$$
$$\overleftarrow{V2}_i(\tau, w) \frac{\overleftarrow{\gamma}_i(\tau)}{\overleftarrow{\gamma 2}_i(\tau)}\; q_{ii} + \sum_{k \in S, k \neq i} \left(\overleftarrow{V1}_k(\tau, w) \frac{\overleftarrow{\gamma 1}_k(\tau)}{\overleftarrow{\gamma 2}_i(\tau)} + \overleftarrow{V2}_k(\tau, w) \frac{\overleftarrow{\gamma 2}_k(\tau)}{\overleftarrow{\gamma 2}_i(\tau)} \right) \frac{\overleftarrow{\gamma}_i(\tau)}{\overleftarrow{\gamma}_k(\tau)}\; q_{ik} =$$
$$\overleftarrow{V2}_i(\tau, w) \frac{\overleftarrow{\gamma}_i(\tau)}{\overleftarrow{\gamma 2}_i(\tau)}\; q_{ii} + \sum_{k \in S, k \neq i} \left(\overleftarrow{V1}_k(\tau, w) \frac{\overleftarrow{\gamma 1}_k(\tau)}{\overleftarrow{\gamma}_k(\tau)} + \overleftarrow{V2}_k(\tau, w) \frac{\overleftarrow{\gamma 2}_k(\tau)}{\overleftarrow{\gamma}_k(\tau)} \right) \frac{\overleftarrow{\gamma}_i(\tau)}{\overleftarrow{\gamma 2}_i(\tau)}\; q_{ik}$$
(B.4)

Using (B.1) and (B.2) we have

$$\frac{\partial}{\partial \tau}\overleftarrow{V2}_i(\tau,w) + \frac{\partial}{\partial w}\overleftarrow{V2}_i(\tau,w) r_i \alpha_i =$$

$$\overleftarrow{V2}_i(\tau,w)\ \frac{q_{ii}}{1-e^{q_{ii}\tau}} + \sum_{k\in S, k\neq i}\left(\overleftarrow{V1}_k(\tau,w)e^{q_{kk}\tau} + \overleftarrow{V2}_k(\tau,w)\left(1-e^{q_{kk}\tau}\right)\right)\frac{q_{ik}}{1-e^{q_{ii}\tau}} =$$

$$\sum_{k\in S}\overleftarrow{V2}_k(\tau,w)\frac{q_{ik}}{1-e^{q_{ii}\tau}} + \sum_{k\in S, k\neq i}\left(\overleftarrow{V1}_k(\tau,w) - \overleftarrow{V2}_k(\tau,w)\right)\frac{e^{q_{kk}\tau}\ q_{ik}}{1-e^{q_{ii}\tau}} \tag{B.5}$$

The vector-matrix versions of eq. (B.3) and (B.5) are

$$\frac{\partial}{\partial \tau}\underline{\overleftarrow{V1}}(\tau,w) + \frac{\partial}{\partial w}\underline{\overleftarrow{V1}}(\tau,w)\mathbf{R} = \underline{0}\ , \tag{B.6}$$

and

$$\frac{\partial}{\partial \tau}\underline{\overleftarrow{V2}}(\tau,w) + \frac{\partial}{\partial w}\underline{\overleftarrow{V2}}(\tau,w)\mathbf{R}_\alpha =$$

$$\left(\underline{\overleftarrow{V2}}(\tau,w)\mathbf{Q}^T\ + \left(\underline{\overleftarrow{V1}}(\tau,w) - \underline{\overleftarrow{V2}}(\tau,w)\right)\mathbf{E}_D(\tau)(\mathbf{Q}-\mathbf{Q}_D)^T\right)\left(\mathbf{I}-\mathbf{E}_D(\tau)\right)^{-1}, \tag{B.7}$$

with initial conditions: $\underline{\overleftarrow{V1}}(0,w) = \underline{e}$, $\underline{\overleftarrow{V2}}(0,w) = \underline{e}$, $\underline{\overleftarrow{V1}}(\tau,0) = \underline{0}$ and $\underline{\overleftarrow{V2}}(\tau,0) = \underline{0}$ (when $r_i > 0$).

The stochastic meaning of $\overleftarrow{V1}_i(T,w)$ and $\overleftarrow{V2}_i(T,w)$ are

$$\overleftarrow{V1}_i(T,w) = Pr(\overleftarrow{B}(T) < w \mid \overleftarrow{Z}(T) = i, \text{no state transition in } (0,T))\ ,$$

$$\overleftarrow{V2}_i(T,w) = Pr(\overleftarrow{B}(T) < w \mid \overleftarrow{Z}(T) = i, \text{state transition in } (0,T))\ ,$$

hence

$$\begin{aligned} Pr(B(T)\le w) \quad = \quad & \sum_{i\in S}\overleftarrow{V1}_i(T,w)\ Pr(\overleftarrow{Z}(T) = i, \text{no state transition in } (0,T)) + \\ & \sum_{i\in S}\overleftarrow{V2}_i(T,w)\ Pr(\overleftarrow{Z}(T) = i, \text{state transition in } (0,T))\ , \end{aligned}$$

To simplify (B.6) and (B.7) we introduce

$$\overleftarrow{X1}_i(\tau,w) = \overleftarrow{V1}_i(\tau,w)e^{q_{ii}\tau} \text{ and } \overleftarrow{X2}_i(\tau,w) = \overleftarrow{V2}_i(\tau,w)(1-e^{q_{ii}\tau})\ .$$

The stochastic interpretation of $\overleftarrow{X1}_i(\tau,w)$ and $\overleftarrow{X2}_i(\tau,w)$ are

$$\overleftarrow{X1}_i(\tau,w) = Pr(\overleftarrow{B}(\tau) < w,\ \text{no state transition in } (0,\tau) \mid \overleftarrow{Z}(\tau) = i)\ ,$$

and

$$\overleftarrow{X2}_i(\tau,w) = Pr(\overleftarrow{B}(\tau) < w,\ \text{state transition in } (0,\tau) \mid \overleftarrow{Z}(\tau) = i)\ ,$$

from which

$$Pr(B(T)\le w) \quad = \quad \sum_{i\in S}\left(\overleftarrow{X1}_i(T,w) + \overleftarrow{X2}_i(T,w)\right)\gamma_i(0)\ .$$

From (B.3) and (B.5) we have

$$\text{(B.8)} \qquad \frac{\partial}{\partial \tau}\overleftarrow{X1}_i(\tau, w) + \frac{\partial}{\partial w}\overleftarrow{X1}_i(\tau, w) r_i = \overleftarrow{X1}_i(\tau, w) q_{ii} \ ,$$

and

$$\text{(B.9)} \frac{\partial}{\partial \tau}\overleftarrow{X2}_i(\tau, w) + \frac{\partial}{\partial w}\overleftarrow{X2}_i(\tau, w) r_i \alpha_i = \sum_{k \in S, k \neq i} \overleftarrow{X1}_k(\tau, w) q_{ik} + \sum_{k \in S} \overleftarrow{X2}_k(\tau, w) q_{ik} \ .$$

Eq. (4.4) and (4.5) are the vector-matrix versions of eq. (B.8) and (B.9), respectively.

Appendix C. Proof of Theorem 4.5.

The Laplace-Stieltjes transform of $\overleftarrow{X1}_i(T, w)$ and $\overleftarrow{X2}_i(X, w)$ with respect to w is denoted by $\overleftarrow{X1^*}_i(T, v)$ and $\overleftarrow{X2^*}_i(T, v)$, where v is the transform variable.

The nth derivative of $\overleftarrow{X1^*}_i(T, v)$ and $\overleftarrow{X2^*}_i(T, v)$, at $v = 0$ are

$$\overleftarrow{M1}_i^{(n)}(T) = (-1)^n \frac{d^n}{dv^n} \overleftarrow{X1^*}_i(T, v)|_{v=0}$$

and

$$\overleftarrow{M2}_i^{(n)}(T) = (-1)^n \frac{d^n}{dv^n} \overleftarrow{X2^*}_i(T, v)|_{v=0} \ .$$

The stochastic meaning of $\overleftarrow{M1}_i^{(n)}(T)$ and $\overleftarrow{M2}_i^{(n)}(T)$ are

$$\overleftarrow{M1}_i^{(n)}(T) = E(\overleftarrow{B}(T)^n, \text{no state transition in } (0, T) \mid \overleftarrow{Z}(T) = i) \ ,$$

$$\overleftarrow{M2}_i^{(n)}(T) = E(\overleftarrow{B}(T)^n, \text{state transition in } (0, T) \mid \overleftarrow{Z}(T) = i) \ ,$$

and based on these quantities the nth moment ($n \geq 0$) of the reward accumulated over the interval $(0, T)$ is

$$E(B(T)^n) = \sum_{i \in S} \left(\overleftarrow{M1}_i^{(n)}(T) + \overleftarrow{M2}_i^{(n)}(T) \right) \gamma_i(0) \ ,$$

The LST of Eq. (B.8) and (B.9) with respect to w (denoting the transform variable by v) are:

$$\text{(C.1)} \qquad \frac{\partial}{\partial \tau}\underline{\overleftarrow{X1}}^*(\tau, v) + v\underline{\overleftarrow{X1}}^*(\tau, v)\mathbf{R} - \underbrace{\underline{\overleftarrow{X1}}(\tau, 0)\mathbf{R}}_{0} = \underline{\overleftarrow{X1}}^*(\tau, v)\mathbf{Q}_D \ ,$$

and

$$\frac{\partial}{\partial \tau}\underline{\overleftarrow{X2}}^*(\tau, v) + v\underline{\overleftarrow{X2}}^*(\tau, v)\mathbf{R}_\alpha - \underbrace{\underline{\overleftarrow{X2}}(\tau, 0)\mathbf{R}_\alpha}_{0} = \underline{\overleftarrow{X1}}^*(\tau, v)(\mathbf{Q} - \mathbf{Q}_D)^T + \underline{\overleftarrow{X2}}^*(\tau, v)\mathbf{Q}^T$$

(C.2)

The nth derivative of Eq. (C.1) and (C.2) with respect to v are:

$$\text{(C.3)} \frac{\partial}{\partial \tau}\underline{\overleftarrow{X1}^*}^{(n)}(\tau, v) + n\underline{\overleftarrow{X1}^*}^{(n-1)}(\tau, v)\mathbf{R} + v\underline{\overleftarrow{X1}^*}^{(n)}(\tau, v)\mathbf{R} = \underline{\overleftarrow{X1}^*}^{(n)}(\tau, v)\mathbf{Q}_D \ ,$$

and

$$\frac{\partial}{\partial \tau}\overleftarrow{\underline{X2^*}}^{(n)}(\tau, v) + n\overleftarrow{\underline{X2^*}}^{(n-1)}(\tau, v)\mathbf{R}_\alpha + v\overleftarrow{\underline{X2^*}}^{(n)}(\tau, v)\mathbf{R}_\alpha = \overleftarrow{\underline{X1^*}}^{(n)}(\tau, v)(\mathbf{Q} - \mathbf{Q}_D)^T + \overleftarrow{\underline{X2^*}}^{(n)}(\tau, v)\mathbf{Q}^T \ . \tag{C.4}$$

Substituting $v = 0$ provides the theorem.

Appendix D. Corollaries utilized in the proof of Theorem 5.2.

COROLLARY D.1. *The* $\underline{D}^{(n)}(k)$ *terms are upper bounded by*

$$\begin{aligned} &\underline{D}^{(0)}(k) \le \underline{e} && \text{if } n = 0, k > 0 \\ &\underline{D}^{(n)}(k) \le \frac{(k-1)!\ (n+1+nk)}{(n+1)!\ (k-n-1)!}\ \underline{e} && \text{if } n > 0, k > n \end{aligned} \tag{D.1}$$

Proof. Based on their definition $\mathbf{A}_D$, $\mathbf{S}$, $\mathbf{S}_\alpha$ and $\mathbf{A}$ are matrices of non-negative elements whose row-sum are between 0 and 1, hence the following inequalities hold element-wise:

$$\underline{0} \le \underline{e}\ (\mathbf{I} - \mathbf{A}_D^k) \le \underline{e}$$

$$\underline{0} \le \underline{D}^{(n-1)}(k-1)\mathbf{S}_\alpha \le \underline{D}^{(n-1)}(k-1)$$

$$\underline{0} \le \underline{D}^{(n)}(k-1)\mathbf{A} \le \underline{D}^{(n)}(k-1)$$

$$\underline{0} \le\ \underline{e}\ \mathbf{S}^n\mathbf{A}_D^{k-1-n}(\mathbf{A} - \mathbf{A}_D) \le \underline{e}$$

The first inequality shows that the corollary holds for $n = 0$. Assuming that the corollary holds for $\underline{D}^{(n)}(k-1)$ and $\underline{D}^{(n-1)}(k-1)$ we have

$$\begin{aligned} \underline{D}^{(n)}(k) &= \underline{D}^{(n-1)}(k-1)\mathbf{S}_\alpha + \underline{D}^{(n)}(k-1)\mathbf{A} + \binom{k-1}{n}\ \underline{e}\ \mathbf{S}^n\mathbf{A}_D^{k-1-n}(\mathbf{A} - \mathbf{A}_D) \le \\ &\underline{D}^{(n-1)}(k-1) + \underline{D}^{(n)}(k-1) + \binom{k-1}{n}\ \underline{e} \le \\ &\frac{(k-2)!\ (n+(n-1)(k-1))}{n!\ (k-n-1)!}\ \underline{e} + \frac{(k-2)!\ (n+1+n(k-1))}{(n+1)!\ (k-n-2)!}\ \underline{e} + \binom{k-1}{n}\ \underline{e} = \\ &\frac{(k-1)!\ (n+1+nk)}{(n+1)!\ (k-n-1)!}\ \underline{e}. \end{aligned}$$

COROLLARY D.2. *The* $\underline{D}^{(n)}(k)$ *terms for* $k \ge n+1$ *are further upper bounded by*

$$\underline{D}^{(n)}(k) \le \frac{(k-1)!\ (n+1+nk)}{(n+1)!\ (k-n-1)!}\ \underline{e} \le \frac{k!}{n!\ (k-n-1)!}\ \underline{e} \tag{D.2}$$

Proof. Multiplying (D.2) by $\frac{n!\ (k-n-1)!}{(k-1)!}$ we have $1 + \frac{nk}{n+1} \le k$, which holds for $k \ge n+1$.

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